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PARTIALLY PENALIZED IMMERSED FINITE ELEMENT METHODS FOR ELLIPTIC INTERFACE PROBLEMS*

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Abstract. This article presents new immersed finite element (IFE) methods for solving the popular second order elliptic interface problems on structured Cartesian meshes even if the involved interfaces have nontrivial geometries. These IFE methods contain extra stabilization terms introduced only at interface edges for penalizing the discontinuity in IFE functions. With the enhanced stability due to the added penalty, not only can these IFE methods be proven to have the optimal convergence rate in an energy norm provided that the exact solution has sufficient regularity, but also numerical results indicate that their convergence rates in both the H^1-norm and the L^2-norm do not deteriorate when the mesh becomes finer, which is a shortcoming of the classic IFE methods in some situations. Trace inequalities are established for both linear and bilinear IFE functions that are not only critical for the error analysis of these new IFE methods but are also of a great potential to be useful in error analysis for other related IFE methods.

Key words. interface problems, immersed finite element, optimal convergence

AMS subject classifications. 35R05, 65N15, 65N30

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1. Introduction. Without loss of generality, we consider a domain Ω that is a union of rectangular domains in $\mathbb{R}^2$, and we assume that Ω is formed by two different materials separated by a curve Γ. In particular, this means Γ separates Ω into two subdomains Ω^- and Ω^+ such that $\overline{\Omega} = \overline{\Omega^- \cup \Omega^+ \cup \Gamma}$. Consequently, the diffusion coefficient β on Ω is assumed to be a piecewise constant function:

$$\beta(x,y) = \begin{cases} \beta^-, & (x,y) \in \Omega^-, \\ \beta^+, & (x,y) \in \Omega^+, \end{cases}$$

such that $\min\{\beta^-, \beta^+\} > 0$.

The main purpose of this article is to present a group of partially penalized immersed finite element (IFE) methods using Cartesian meshes to solve popular elliptic interface problems appearing in many applications in the form

$$-\nabla \cdot \big(\beta \nabla u(x,y)\big) = f(x,y), \quad (x,y) \in \Omega^- \cup \Omega^+, \tag{1.1}$$

$$u(x,y) = 0, \quad (x,y) \in \partial\Omega, \tag{1.2}$$

together with the jump conditions on the interface $\Gamma \subset \Omega$:

$$[u]\,|_{\Gamma} = 0, \tag{1.3}$$

$$\left[\beta \frac{\partial u}{\partial n}\right]|_{\Gamma} = 0. \tag{1.4}$$

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The homogeneous boundary condition (1.2) is discussed here for simplicity's sake; the method and related analysis can be readily extended to interface problems with a nonhomogeneous boundary condition.

A large number of numerical methods based on Cartesian meshes have been introduced for elliptic interface problems. Since Peskin's pioneering work of the immersed boundary method [49], a variety of methods have been developed in a finite difference formulation, such as the immersed interface method [35], the matched interface and boundary method [61], and the ghost fluid method [24]. We refer to the book [37] for an overview of different numerical methods in a finite difference framework.

In finite element formulation, certain types of modifications need to be executed for elements around the interface. One way is to modify the weak formulation of finite element equations near the interface. We refer to some representative methods such as the penalty finite element method [4, 11], the unfitted finite element method [28], and the discontinuous Galerkin (DG) methods [12, 27]. An alternative approach is to modify the approximating functions around the interface, for instance, the general finite element method [6, 7], the multiscale finite element method [18, 23], the extended finite element method [21], and the partition of unity method [8, 9], to name just a few.

IFE methods are a particular class of finite element (FE) methods belonging to the second approach mentioned above, and they can solve interface problems with meshes independent of the interface [1, 2, 29, 30, 32, 33, 36, 39, 42, 47, 54, 55]. If desired, an IFE method can use a Cartesian mesh to solve a boundary value problem (BVP) whose coefficient is discontinuous across a curve Γ with a nontrivial geometry. The basic idea of an IFE method is to employ standard FE functions in noninterface elements not intersecting with the interface Γ, but on each interface element, it uses IFE functions constructed with piecewise polynomials based on the natural partition of this element formed by the interface and the jump conditions required by the interface problem. The IFE functions are macroelements [13, 20], and each IFE function partially solves the related interface problem because it satisfies the interface jump conditions in a certain sense. Also, the IFE space on an interface element is consistent with the corresponding FE space based on the same polynomial space in the sense that the IFE space becomes the FE space if the discontinuity in the coefficient β disappears in that element; see [29, 30] for more details.

IFE methods have been developed for solving interface problems involving several important types of partial differential equations, such as the second order elliptic equation [1, 2, 25, 29, 30, 33, 34, 36, 39, 41, 54, 55, 59, 60], the biharmonic and beam equations [42], the planar elasticity system [26, 40, 46, 47], the parabolic equation with fixed interfaces [3, 45, 56], and the parabolic equation with a moving interface [32, 43, 44]. When jump conditions are suitably employed in the construction of IFE functions for an interface problem, the resulting IFE space usually has the optimal approximation capability from the point view of polynomials used in this IFE space [1, 15, 16, 30, 38, 52, 60]. Numerical examples [2, 36, 38, 39] demonstrate that methods based on IFE spaces can converge optimally for second order elliptic interface problems. However, the proof for their optimal error bounds is *still elusive* except for the one-dimensional case [2], even though there have been a few attempts [17, 31, 34, 57]. For two-dimensional elliptic interface problems, only a *suboptimal* convergence in the H^1-norm has been rigorously proven [31].

One of the major obstacles is the error estimation on edges between two interface elements where IFE functions have discontinuity. Certain types of trace inequalities are needed and can be established, but it is not clear whether the generic constant factor in these inequalities is actually independent of the interface location. The scaling argument in the standard finite element error estimation is not applicable here because the local IFE spaces on two different interface elements are not affine equivalent in general. Besides, numerical experiments have demonstrated that the classic IFE methods in the literature often have a much larger pointwise error over interface elements which, we believe, is caused by the interelement discontinuity of IFE functions. In some cases, the convergence rates can even deteriorate when the mesh becomes finer. These observations motivate us to apply a certain penalty over interface edges for controlling negative impacts from this discontinuity. Natural candidates are those well-known penalty strategies for handling interelement discontinuity in interior penalty Galerkin methods and DG methods [5, 10, 14, 22, 48, 51, 53, 58]. These considerations lead to the partially penalized IFE methods in this article. Theoretically, thanks to the enhanced stability by the penalty terms, we are able to prove that these new IFE methods do converge optimally in an energy norm. In addition, we have observed through abundant numerical experiments that these partially penalized IFE methods maintain their expected convergence rate in both H^1-norm and L^2-norm when their mesh becomes finer and finer while the classic IFE methods cannot maintain in some situations.

The partial penalty idea has also been used in the unfitted finite element method [28]. In this method, penalty terms are introduced on the interface instead of the interface edges because approximating functions are allowed to be discontinuous inside interface elements but they are continuous on element boundaries within each subdomain. IFE methods reverse this idea by imposing continuity of approximating functions inside each element but allowing discontinuity possibly only across interface edges. In addition, on the same mesh, the unfitted finite element method has a slightly larger number of degrees of freedom than IFE methods. On the other hand, the unfitted finite element method has been proven to have the optimal convergence rate under the usual piecewise H^2 regularity [28] while the analysis in the represent article needs to assume a piecewise H^3 or $W^{2,\infty}$ regular in order to establish the optimal convergence for the partially penalized IFE methods.

Also, we note that these partially penalized IFE methods and their related error analysis can be readily modified to obtain IFE methods based on the DG formulation with advantages such as adaptivity even with Cartesian meshes. However, on the same mesh, the DG IFE methods generally have far more global degrees of freedom. For instance, on a Cartesian triangular mesh, a DG IFE method has about six times more unknowns than the classic IFE method. The partially penalized IFE methods presented here have the same global degrees of freedom as their classic counterparts; hence, they can be more competitive in applications where advantages of DG IFE methods are not needed.

The rest of this article is organized as follows. In section 2, we derive partially penalized IFE methods based on either linear or bilinear IFE functions for the interface problem. In section 3, we show that the well-known trace inequalities on an element are also valid for linear and bilinear IFE functions even though they are not H^2 functions locally in an interface element. In section 4, we show that these IFE schemes do have the optimal convergence rate in an energy norm. In section 5, we will present numerical examples to demonstrate features of these IFE methods.

2. Partially penalized IFE methods. Let $\mathcal{T}_h$, $0 < h < 1$, be a family of Cartesian triangular or rectangular meshes on Ω. For each mesh $\mathcal{T}_h$, we let $\mathcal{N}_h$ be the set of vertices of its elements, $\mathcal{E}_h$ be the set of its edges, and $\mathring{\mathcal{E}}_h$ be the set of interior edges. In addition, we let $\mathcal{T}_h^i$ be the set of interface elements of $\mathcal{T}_h$ and let $\mathcal{T}_h^n$ be the set of noninterface elements. Similarly, we let $\mathring{\mathcal{E}}_h^i$ be the set of interior interface edges and let $\mathring{\mathcal{E}}_h^n$ be the set of interior noninterface edges. For every interior edge $B \in \mathring{\mathcal{E}}_h$, we denote two elements that share the common edge B by $T_{B,1}$ and $T_{B,2}$. For a function u defined on $T_{B,1} \cup T_{B,2}$, we denote its average and jump on B by

$$\{u\}_B = \frac{1}{2}\big((u|_{T_{B,1}})|_B + (u|_{T_{B,2}})|_B\big), \quad [u]_B = (u|_{T_{B,1}})|_B - (u|_{T_{B,2}})|_B.$$

For simplicity's sake, we will often drop the subscript B from these notations if there is no danger of causing confusion. We will also use the function spaces

$$\tilde{W}^{r,p}(\Omega) = \{v \in W^{1,p}(\Omega) \mid u|_{\Omega^s} \in W^{r,p}(\Omega^s), \ s = + \text{ or } -\} \quad \text{for } r \geq 1 \text{ and } 1 \leq p \leq \infty,$$

equipped with the norm

$$\|v\|^p_{\tilde{W}^{r,p}(\Omega)} = \|v\|^p_{W^{r,p}(\Omega^-)} + \|v\|^p_{W^{r,p}(\Omega^+)} \quad \forall v \in \tilde{W}^{r,p}(\Omega).$$

As usual, for $p = 2$, we use $\tilde{H}^r(\Omega) = \tilde{W}^{r,2}(\Omega)$ and denote its corresponding norm by

$$\|v\|_r^2 = \|v\|^2_{\tilde{H}^r(\Omega)} = \|v\|^2_{H^r(\Omega^-)} + \|v\|^2_{H^r(\Omega^+)} \quad \forall v \in \tilde{H}^r(\Omega).$$

With a suitable assumption about the regularity of Γ and f (e.g., [4]), we can assume that the exact solution u to the interface problem is in $\tilde{H}^2(\Omega)$. To derive a weak form of interface problem described by (1.1)–(1.4) for an IFE method, we will use the following space:

$$V_h = \{v \mid v \text{ satisfies conditions (HV1)–(HV4) described as follows}\}$$

(HV1) $v|_K \in H^1(K)$ $\forall K \in \mathcal{T}_h$.
(HV2) v is continuous at every $X \in \mathcal{N}_h$.
(HV3) v is continuous across each $B \in \mathring{\mathcal{E}}_h^n$.
(HV4) $v|_{\partial\Omega} = 0$.

We multiply (1.1) by a test function $v \in V_h$, integrate both sides on each element $K \in \mathcal{T}_h$, and apply Green's formula to have

$$\int_K \beta \nabla v \cdot \nabla u dX - \int_{\partial K} \beta \nabla u \cdot \mathbf{n} v ds = \int_K v f dX.$$

Summarizing over all elements leads to

$$\sum_{K\in\mathcal{T}_h} \int_K \beta \nabla v \cdot \nabla u dX - \sum_{B\in\mathring{\mathcal{E}}_h^i} \int_B \{\beta \nabla u \cdot \mathbf{n}_B\} [v] ds = \int_\Omega v f dX. \tag{2.1}$$

Here, we have used the fact that

$$\{\beta \nabla u \cdot \mathbf{n}_B\}_B = (\beta \nabla u \cdot \mathbf{n}_B)|_B \quad \forall B \in \mathring{\mathcal{E}}_h^n.$$

Because of the regularity of u, for arbitrary parameters $\epsilon, \alpha > 0$, and $\sigma_B^0 \geq 0$, we have

$$\epsilon \sum_{B\in\mathring{\mathcal{E}}_h^i} \int_B \{\beta \nabla v \cdot \mathbf{n}_B\} [u] ds = 0, \quad \sum_{B\in\mathring{\mathcal{E}}_h^i} \int_B \frac{\sigma_B^0}{|B|^\alpha} [v][u] ds = 0. \tag{2.2}$$

Therefore, adding (2.2) to (2.1) leads to the following weak form of the interface problem (1.1)–(1.4):

$$
\begin{aligned}
(2.3)\quad & \sum_{K\in\mathcal{T}_h}\int_K \beta\nabla v\cdot\nabla u dX - \sum_{B\in\mathring{\mathcal{E}}_h^i}\int_B \{\beta\nabla u\cdot\mathbf{n}_B\}[v]ds \\
& + \epsilon\sum_{B\in\mathring{\mathcal{E}}_h^i}\int_B \{\beta\nabla v\cdot\mathbf{n}_B\}[u]ds + \sum_{B\in\mathring{\mathcal{E}}_h^i}\int_B \frac{\sigma_B^0}{|B|^\alpha}[v][u]ds = \int_\Omega vfdX \quad \forall v\in V_h.
\end{aligned}
$$

We now recall the linear and bilinear IFE spaces to be used in our partially penalized IFE methods based on the weak form (2.3). On each element $K\in\mathcal{T}_h$, we let

$$
S_h(K) = span\{\phi_j(X), 1\le j\le d_K\}, \quad d_K = \begin{cases} 3 & \text{if } K \text{ is a triangular element,} \\ 4 & \text{if } K \text{ is a rectangular element,} \end{cases}
$$

where $\phi_j, 1\le j\le d_K$ are the standard linear or bilinear nodal basis functions for $K\in\mathcal{T}_h^n$; otherwise, for $K\in\mathcal{T}_h^i$, $\phi_j, 1\le j\le d_K$ are the linear or bilinear IFE basis functions discussed in [38, 39] and [30, 41], respectively. Then, we define the IFE space over the whole solution domain Ω as follows:

$$
S_h(\Omega) = \{v \mid v \text{ satisfies conditions (IFE1)–(IFE3) given below}\}
$$

(IFE1) $v|_K \in S_h(K)\ \forall K\in\mathcal{T}_h$.
(IFE2) v is continuous at every $X\in\mathcal{N}_h$.
(IFE3) $v|_{\partial\Omega} = 0$.

It is easy to see that $S_h(\Omega)\subset V_h(\Omega)$. Now, we describe the partially penalized IFE methods for the interface problem (1.1)–(1.4): find $u_h\in S_h(\Omega)$ such that

$$
(2.4)\qquad a_h(v_h,u_h) = (v_h,f) \quad \forall v_h\in S_h(\Omega),
$$

where the bilinear form $a_h(\cdot,\cdot)$ is defined on $S_h(\Omega)$ by

$$
\begin{aligned}
a_h(v_h,w_h) = & \sum_{K\in\mathcal{T}_h}\int_K \beta\nabla v_h\cdot\nabla w_h dX - \sum_{B\in\mathring{\mathcal{E}}_h^i}\int_B \{\beta\nabla w_h\cdot\mathbf{n}_B\}[v_h]ds \\
& + \epsilon\sum_{B\in\mathring{\mathcal{E}}_h^i}\int_B \{\beta\nabla v_h\cdot\mathbf{n}_B\}[w_h]ds \\
(2.5)\qquad & + \sum_{B\in\mathring{\mathcal{E}}_h^i}\int_B \frac{\sigma_B^0}{|B|^\alpha}[v_h][w_h]ds \quad \forall v_h, w_h\in S_h(\Omega).
\end{aligned}
$$

3. Trace inequalities for IFE functions. Using the standard scaling argument, we can obtain the following well-known trace inequalities [50]: there exists a constant C such that

$$
(3.1)\qquad \|v\|_{L^2(B)} \le C|B|^{1/2}|K|^{-1/2}\left(\|v\|_{L^2(K)} + h\|\nabla v\|_{L^2(K)}\right) \ \forall v\in H^1(K),
$$

$$
(3.2)\quad \|\nabla v\|_{L^2(B)} \le C|B|^{1/2}|K|^{-1/2}\left(\|\nabla v\|_{L^2(K)} + h\left\|\nabla^2 v\right\|_{L^2(K)}\right) \ \forall v\in H^2(K).
$$

where B is an edge of K.

Our goal in this section is to extend these trace inequalities to IFE functions in $S_h(K)$ for $K\in\mathcal{T}_h^i$. First, we recall that $S_h(K)\subset C(K)\cap H^1(K)\ \forall K\in\mathcal{T}_h$ [30, 38]. This implies that inequality (3.1) is also valid for $v\in S_h(K)$ even if $K\in\mathcal{T}_h^i$. However, the second trace inequality (3.2) cannot be applied to $v\in S_h(K)$ with $K\in\mathcal{T}_h^i$ because $v\notin H^2(K)$ in general.

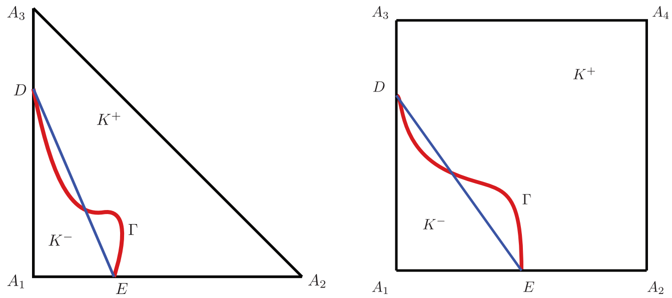

FIG. 1. *A triangular interface element (left) and a rectangular interface element (right).*

3.1. Trace inequalities for linear IFE functions. It is relatively easier to prove that the trace inequality for a linear IFE function in a triangular interface element is true because its gradient is a piecewise constant function. Without loss of generality, we consider the following triangular interface element:

$$K = \triangle A_1A_2A_3, \quad A_1 = (0,0),\ A_2 = (h,0),\ A_3 = (0,h).$$

Assume that the interface Γ intersects the edge of K at points D and E and the straight line $\overline{DE}$ separates K into K^- and K^+; see the illustration on the left in Figure 1. Consider a linear IFE function on K in the form

$$v(x,y) = \begin{cases} v^-(x,y) = c_1^- + c_2^- x + c_3^- y & \text{if } (x,y) \in K^-, \\ v^+(x,y) = c_1^+ + c_2^+ x + c_3^+ y & \text{if } (x,y) \in K^+, \end{cases} \tag{3.3}$$

which satisfies the following jump conditions [38]:

$$v^-(D) = v^+(D), \quad v^-(E) = v^+(E), \quad \beta^- \frac{\partial v^-}{\partial \mathbf{n}_{\overline{DE}}} = \beta^+ \frac{\partial v^+}{\partial \mathbf{n}_{\overline{DE}}}. \tag{3.4}$$

LEMMA 3.1. *There exists a constant $C > 1$ independent of the interface location such that for every linear IFE function v on the interface element K defined in* (3.3)*, the following inequalities hold:*

$$\frac{1}{C} \left\|(c_1^+, c_2^+, c_3^+)\right\| \le \left\|(c_1^-, c_2^-, c_3^-)\right\| \le C \left\|(c_1^+, c_2^+, c_3^+)\right\|. \tag{3.5}$$

Proof. We prove the second inequality in (3.5), and similar arguments can be used to show the first one. Applying the jump conditions (3.4), we can show that coefficients of $v(x,y)$ must satisfy the following equality:

$$M^- \begin{pmatrix} c_1^- \\ c_2^- \\ c_3^- \end{pmatrix} = M^+ \begin{pmatrix} c_1^+ \\ c_2^+ \\ c_3^+ \end{pmatrix},$$

where $M^s,\ s = -,+$ are two matrices. Without loss of generality, we further assume

$$D = (0, dh), E = (eh, 0), \text{ with } 0 \le d \le 1, 0 \le e \le 1.$$

Then

$$M^s = \begin{pmatrix} 1 & 0 & dh \\ 1 & eh & 0 \\ 0 & -\beta^s dh & -\beta^s eh \end{pmatrix}, \quad s = - \text{ or } +,$$

whose determinant is

$$\det(M^s) = -\beta^s(d^2+e^2)h^2, \quad s = - \text{ or } +,$$

which is nonzero because $(d,e) \neq (0,0)$. Hence, we can solve for c_i^+s in terms of c_i^- to have

$$c_i^+ = f_{i1}c_1^- + f_{i2}c_2^- + f_{i3}c_3^-, \quad i = 1,2,3, \tag{3.6}$$

$$f_{ij} = \frac{g_{ij}^-\beta^-}{\beta^+(d^2+e^2)} + \frac{g_{ij}^+\beta^+}{\beta^+(d^2+e^2)}, \quad 1 \le i,j \le 3, \tag{3.7}$$

with

$$\begin{aligned} &g_{11}^- = 0, g_{11}^+ = d^2+e^2,\ g_{12}^- = -d^2eh, g_{12}^+ = d^2eh,\ g_{13}^- = -de^2h, g_{13}^+ = de^2h,\\ &g_{21}^- = 0, g_{21}^+ = 0,\ g_{22}^- = d^2, g_{22}^+ = e^2,\ g_{23}^- = de, g_{23}^+ = -de,\\ &g_{31}^- = 0, g_{31}^+ = 0,\ g_{32}^- = de, g_{32}^+ = -de,\ g_{33}^- = e^2, g_{33}^+ = d^2. \end{aligned}$$

Therefore, there exists a constant C that depends on β^- and β^+ but is independent of d, e, such that

$$|f_{ij}| \le C,\ 1 \le i,j \le 3.$$

Then, the second inequality in (3.5) follows from (3.6) and the above bounds for $|f_{ij}|,\ 1 \le i,j \le 3.$ ∎

Remark 3.1. In the proof of Lemma 3.1, we have shown $f_{i1} = 0, i = 2,3$. Consequently, we can show that there exists a constant $C > 1$ such that

$$\frac{1}{C}\left\|(c_2^+, c_3^+)\right\| \le \left\|(c_2^-, c_3^-)\right\| \le C\left\|(c_2^+, c_3^+)\right\|. \tag{3.8}$$

Now, we establish the trace inequality on a triangular interface element $K = \triangle A_1A_2A_3$.

LEMMA 3.2. *There exists a constant C independent of the interface location such that for every linear IFE function v on K, the following inequalities hold:*

$$\|\beta v_p\|_{L^2(B)} \le Ch^{1/2}|K|^{-1/2}\left\|\sqrt{\beta}\nabla v\right\|_{L^2(K)}, \quad p = x, y, \tag{3.9}$$

$$\|\beta\nabla v \cdot \mathbf{n}_B\|_{L^2(B)} \le Ch^{1/2}|K|^{-1/2}\left\|\sqrt{\beta}\nabla v\right\|_{L^2(K)}. \tag{3.10}$$

Proof. Without loss of generality, we assume again that the interface Γ intersects with the boundary of K at

$$D = (0, dh), E = (eh, 0) \text{ with } 0 \le d \le 1, 0 \le e \le 1,$$

and the line $\overline{DE}$ separates K into two subelement K^- and K^+ with $A_3 \in K^+$ and $|K^+| \ge \frac{1}{2}|K|$. Furthermore, we assume that $B = \overline{A_1A_3}$ is an interface edge with $B = B^- \cup B^+$. Similar arguments can be applied to establish the trace inequality in other cases.

By direct calculations, we have

$$\begin{aligned} \left\|\beta^+ v_x\right\|_{L^2(B^+)}^2 &= (c_2^+)^2\left|B^+\right|\left(\left|\beta^+\right|\right)^2 \le \left((c_2^+)^2 + (c_3^+)^2\right)\left|K^+\right|\frac{|B^+|}{|K^+|}\left(\left|\beta^+\right|\right)^2\\ &= \beta^+\frac{|B^+|}{|K^+|}\left\|\sqrt{\beta^+}\nabla v\right\|_{L^2(K^+)}^2 \le C\frac{|B^+|}{|K^+|}\left\|\sqrt{\beta}\nabla v\right\|_{L^2(K)}^2, \end{aligned}$$

i.e.,

$$\left\|\beta^+ v_x\right\|_{L^2(B^+)} \leq 2Ch^{1/2}\left|K\right|^{-1/2}\left\|\sqrt{\beta}\nabla v\right\|_{L^2(K)}. \tag{3.11}$$

Similarly, we can show that

$$\left\|\beta^+ v_y\right\|_{L^2(B^+)} \leq 2Ch^{1/2}\left|K\right|^{-1/2}\left\|\sqrt{\beta}\nabla v\right\|_{L^2(K)}. \tag{3.12}$$

On B^-, applying the estimates in Remark 3.1, we have

$$\begin{aligned}\left\|\beta^- v_x\right\|^2_{L^2(B^-)} &= (c_2^-)^2\left|B^-\right|\left(\left|\beta^-\right|\right)^2 \leq C\big((c_2^+)^2+(c_3^+)^2\big)\left|K^+\right|\frac{|B^-|}{|K^+|}\left(\left|\beta^-\right|\right)^2\\ &= C\beta^-\frac{|B^-|}{|K^+|}\left\|\sqrt{\beta^+}\nabla v\right\|^2_{L^2(K^+)} \leq C\frac{|B^-|}{|K^+|}\left\|\sqrt{\beta}\nabla v\right\|^2_{L^2(K)}.\end{aligned}$$

Hence,

$$\left\|\beta^- v_x\right\|_{L^2(B^-)} \leq 2Ch^{1/2}\left|K\right|^{-1/2}\left\|\sqrt{\beta}\nabla v\right\|_{L^2(K)}. \tag{3.13}$$

Similarly,

$$\left\|\beta^- v_y\right\|_{L^2(B^-)} \leq 2Ch^{1/2}\left|K\right|^{-1/2}\left\|\sqrt{\beta}\nabla v\right\|_{L^2(K)}. \tag{3.14}$$

Then, the combination of (3.11) and (3.13) yields the inequality (3.9) for $p = x$. Similarly, the inequality (3.9) for $p = y$ follows from combining (3.12) and (3.14). Finally, (3.10) follows directly from (3.9). ∎

3.2. Trace inequalities for bilinear IFE functions. Without loss of generality, we consider a rectangular interface element K with the following vertices:

$$A_1 = (0,0), \quad A_2 = (h,0), \quad A_3 = (0,h), \quad A_4 = (h,h).$$

Again, assume that the interface Γ intersects with ∂K at points D and E and the linear $\overline{DE}$ separates K into two subelements K^- and K^+; see the illustration on the right in Figure 1. We assume that K is one of the two types of rectangular interface elements [29, 30]:

K is a Type I interface element if interface Γ intersects ∂K at

$$D = (0, dh), E = (eh, 0), \quad 0 \leq d \leq 1, 0 \leq e \leq 1.$$

K is a Type II interface element if interface Γ intersects ∂K at

$$D = (dh, h), E = (eh, 0), \quad 0 \leq d \leq 1, 0 \leq e \leq 1.$$

On this interface element K, let v be a bilinear IFE function in the following form:

$$v(x,y) = \begin{cases} v^-(x,y) = c_1^- + c_2^- x + c_3^- y + c_4 xy & \text{if } (x,y) \in K^-,\\ v^+(x,y) = c_1^+ + c_2^+ x + c_3^+ y + c_4 xy & \text{if } (x,y) \in K^+,\end{cases} \tag{3.15}$$

which satisfies pertinent interface jump conditions [29, 31]. First, using similar arguments, we can show that the coefficients of a bilinear IFE function v satisfy inequalities similar to those in Lemma 3.1.

LEMMA 3.3. *There exists a constant $C > 1$ independent of the interface location such that for every bilinear IFE function v on the interface element K defined in* (3.15) *the following inequalities hold:*

$$\frac{1}{C}\left\|(c_1^+, c_2^+, c_3^+, c_4)\right\| \le \left\|(c_1^-, c_2^-, c_3^-, c_4)\right\| \le C\left\|(c_1^+, c_2^+, c_3^+, c_4)\right\|. \tag{3.16}$$

The proof of trace inequalities for a bilinear IFE function is a little more complicated because its gradient is not a constant. The following lemma provides an aid.

LEMMA 3.4. *Assume that K is an interface element such that*

$$|K^s| \ge \frac{1}{2}|K|$$

with $s = -$ or $+$. Then there exists a polygon $\tilde{K} \subset K^s$ and two positive constants C_1 and C_2 independent of the interface location such that

$$\left|\tilde{K}\right| \ge C_1 |K|, \tag{3.17}$$

$$\frac{h}{\left|\tilde{K}\right|}\left\|\sqrt{\beta^s}\nabla v\right\|^2_{L^2(\tilde{K})} \ge C_2\beta^s\big(h(c_2^s)^2 + h(c_3^s)^2 + h^3(c_4)^2\big). \tag{3.18}$$

Proof. Let us partition K into four congruent squares $K_i, i = 1,2,3,4$ by the lines connecting the two pairs of opposite mid points of edges of K such that A_i is a vertex of K_i. Since $|K^s| \ge \frac{1}{2}|K|$, one of these four small squares must be inside K^s. Without loss of generality, we assume that $K_4 \subset K^s$. By direct calculations, we have

$$\begin{aligned}
\|v_x\|^2_{L^2(K_4)} &= \frac{h^2}{48}(12(c_2^s)^2 + 18c_2^s c_4 h + 7c_4^2h^2)\\
&\ge \frac{h^2}{48}\left[\left(12 - \frac{9}{\sigma_1}\right)(c_2^s)^2 + (7 - 9\sigma_1)c_4^2h^2\right],\\
\|v_y\|^2_{L^2(K_4)} &= \frac{h^2}{48}(12(c_3^s)^2 + 18c_3^s c_4 h + 7c_4^2h^2)\\
&\ge \frac{h^2}{48}\left[\left(12 - \frac{9}{\sigma_2}\right)(c_3^s)^2 + (7 - 9\sigma_2)c_4^2h^2\right],
\end{aligned}$$

where σ_1 and σ_2 are arbitrary positive constants. Letting $\sigma_i = \sigma \in (9/12, 7/9), i = 1,2$ in the above inequalities leads to

$$\|\nabla v\|^2_{L^2(K_4)} \ge Ch^2((c_2^s)^2 + (c_3^s)^2 + c_4^2h^2),$$

where

$$C = \min\left\{12 - \frac{9}{\sigma}, 2(7 - 9\sigma)\right\} > 0.$$

Then, (3.17) and (3.18) follow by letting $\tilde{K} = K_4$. ∎

Now, we are ready to establish the trace inequality for bilinear IFE functions on an interface element $K = \square A_1A_2A_3A_4$.

LEMMA 3.5. *There exists a constant C independent of the interface location such that for every bilinear IFE function $v(x,y)$ on K, the following inequalities hold:*

$$\|\beta v_p\|_{L^2(B)} \le Ch^{1/2}|K|^{-1/2}\left\|\sqrt{\beta}\nabla v\right\|_{L^2(K)}, \quad p = x, y, \tag{3.19}$$

$$\|\beta \nabla v\cdot \mathbf{n}_B\|_{L^2(B)} \le Ch^{1/2}|K|^{-1/2}\left\|\sqrt{\beta}\nabla v\right\|_{L^2(K)}. \tag{3.20}$$

Proof. Without loss of generality, we assume that K is a Type I interface element and $B = \overline{A_1A_3}$ is an interface edge. To be more specific, we also assume that $A_4 \in K^+$ and $|K^+| \ge \frac{1}{2}|K|$. Then

$$B = B^- \cup B^+, \quad B^- = \overline{A_1 D}, \ B^+ = \overline{DA_3}.$$

Direct calculations lead to

$$\left\|\beta^- v_x\right\|^2_{L^2(B^-)} = (\beta^-)^2\left(dh(c_2^-)^2 + d^2h^2c_2^-c_4 + \frac{1}{3}d^3h^3c_4^2\right), \tag{3.21}$$

$$\left\|\beta^- v_y\right\|^2_{L^2(B^-)} = (\beta^-)^2 dh(c_3^-)^2, \tag{3.22}$$

$$\left\|\beta^+ v_x\right\|^2_{L^2(B^+)} = \left(\beta^+\right)^2\left[(1-d)h(c_2^+)^2 + (1-d^2)h^2(c_2^+)c_4 + \frac{1}{3}(1-d^3)h^3c_4^2\right], \tag{3.23}$$

$$\left\|\beta^+ v_y\right\|^2_{L^2(B^+)} = \left(\beta^+\right)^2(1-d)h(c_3^+)^2. \tag{3.24}$$

Applying (3.17) and (3.18) to (3.23) and (3.24) yields

$$\left\|\beta^+ v_p\right\|^2_{L^2(B^+)} \le C\frac{h}{|K|}\left\|\sqrt{\beta}\nabla v\right\|^2_{L^2(K^+)} \le C\frac{h}{|K|}\left\|\sqrt{\beta}\nabla v\right\|^2_{L^2(K)}, \quad p = x, y. \tag{3.25}$$

Moreover, applying (3.16), (3.17), and (3.18) to (3.21) and (3.22) leads to

$$\left\|\beta^- v_p\right\|^2_{L^2(B^-)} \le C\frac{h}{|K|}\left\|\sqrt{\beta}\nabla v\right\|^2_{L^2(K^+)} \le C\frac{h}{|K|}\left\|\sqrt{\beta}\nabla v\right\|^2_{L^2(K)}, \quad p = x, y. \tag{3.26}$$

Then, (3.19) follows from combining (3.25) and (3.26). Finally, (3.20) obviously follows from (3.19). ∎

4. Error estimation for partially penalized IFE methods. We show that the IFE solution to the interface problem solved from (2.4) has an optimal convergence from the point of the polynomials used in the involved IFE spaces. Unless otherwise specified, we always assume that $\mathcal{T}_h, 0 < h < 1$ is a family of regular Cartesian triangular or rectangular meshes [19].

We start by proving the coercivity of the bilinear form $a_h(\cdot,\cdot)$ defined in (2.5) on the IFE space $S_h(\Omega)$ with respect to the following energy norm:

$$\|v_h\|_h = \left(\sum_{K\in\mathcal{T}_h}\int_K \beta\nabla v_h\cdot\nabla v_h dX + \sum_{B\in\mathring{\mathcal{E}}_h^i}\int_B \frac{\sigma_B^0}{|B|^\alpha}[v_h][v_h]ds\right)^{1/2}. \tag{4.1}$$

LEMMA 4.1. *There exists a constant $\kappa > 0$ such that*

$$\kappa\|v_h\|_h^2 \le a_h(v_h, v_h) \quad \forall v_h \in S_h(\Omega) \tag{4.2}$$

is true for $\epsilon = 1$ unconditionally and is true for $\epsilon = 0$ or $\epsilon = -1$ under the condition that the stabilization parameter σ_B^0 in $a_h(\cdot,\cdot)$ is large enough.

Proof. First, for $\epsilon = 1$, we note that the coercivity follows directly from the definitions of $a_h(\cdot,\cdot)$ and $\|\cdot\|_h$.

For $\epsilon = -1, 0$, note that

$$a_h(v_h, v_h) = \sum_{K\in\mathcal{T}_h}\int_K \beta\nabla v_h\cdot\nabla v_h dX + (\epsilon-1)\sum_{B\in\mathring{\mathcal{E}}_h^i}\int_B \{\beta\nabla v_h\cdot\mathbf{n}_B\}[v_h]ds + \sum_{B\in\mathring{\mathcal{E}}_h^i}\int_B \frac{\sigma_B^0}{|B|^\alpha}[v_h][v_h]ds, \tag{4.3}$$

and the main concern is the second term on the right-hand side. For each interface edge $B\in\mathring{\mathcal{E}}_h^i$, we let $K_{B,i}\in\mathcal{T}_h, i = 1, 2$ be the two elements sharing B as their common edge. Then, by the trace inequality (3.10) or (3.20) and using $\alpha\geq 1$, we have

$$\begin{aligned}
&\int_B \{\beta\nabla v_h\cdot\mathbf{n}_B\}[v_h]ds\\
&\leq \|\{\beta\nabla v_h\cdot\mathbf{n}_B\}\|_{L^2(B)}\|[v_h]\|_{L^2(B)}\\
&\leq \left(\frac{1}{2}\left\|(\beta\nabla v_h\cdot\mathbf{n}_B)|_{K_{B,1}}\right\|_{L^2(B)} + \frac{1}{2}\left\|(\beta\nabla v_h\cdot\mathbf{n}_B)|_{K_{B,2}}\right\|_{L^2(B)}\right)\|[v_h]\|_{L^2(B)}\\
&\leq \left(\frac{C}{2}h_{K_{B,1}}^{-1/2}\left\|\sqrt{\beta}\nabla v_h\right\|_{L^2(K_{B,1})} + \frac{C}{2}h_{K_{B,2}}^{-1/2}\left\|\sqrt{\beta}\nabla v_h\right\|_{L^2(K_{B,2})}\right)\|[v_h]\|_{L^2(B)}\\
&= \frac{C}{2}|B|^{\alpha/2}\left(h_{K_{B,1}}^{-1/2}\left\|\sqrt{\beta}\nabla v_h\right\|_{L^2(K_{B,1})}\right.\\
&\qquad\left. + h_{K_{B,2}}^{-1/2}\left\|\sqrt{\beta}\nabla v_h\right\|_{L^2(K_{B,2})}\right)\frac{1}{|B|^{\alpha/2}}\|[v_h]\|_{L^2(B)}\\
&\leq C\left(\left\|\sqrt{\beta}\nabla v_h\right\|^2_{L^2(K_{B,1})} + \left\|\sqrt{\beta}\nabla v_h\right\|^2_{L^2(K_{B,2})}\right)^{1/2}\frac{1}{|B|^{\alpha/2}}\|[v_h]\|_{L^2(B)}.
\end{aligned}$$

Therefore, for any $\delta > 0$, we have

$$\begin{aligned}
&\sum_{B\in\mathring{\mathcal{E}}_h^i}\int_B \{\beta\nabla v_h\cdot\mathbf{n}_B\}[v_h]ds\\
&\quad\leq \sum_{B\in\mathring{\mathcal{E}}_h^i} C\left(\left\|\sqrt{\beta}\nabla v_h\right\|^2_{L^2(K_{B,1})} + \left\|\sqrt{\beta}\nabla v_h\right\|^2_{L^2(K_{B,2})}\right)^{1/2}\frac{1}{|B|^{\alpha/2}}\|[v_h]\|_{L^2(B)}\\
&\quad\leq \frac{\delta}{2}\sum_{K\in\mathcal{T}_h}\left\|\sqrt{\beta}\nabla v_h\right\|^2_{L^2(K)} + \frac{C}{2\delta}\sum_{B\in\mathring{\mathcal{E}}_h^i}\frac{1}{|B|^\alpha}\|[v_h]\|^2_{L^2(B)}.
\end{aligned} \tag{4.4}$$

Then for $\epsilon = 0$ we let $\delta = 1$ and $\sigma_B^0 = C$, and for $\epsilon = -1$ we let $\delta = 1/2$ and $\sigma_B^0 = 5C/2$, where C is in the above inequality. The coercivity result (4.2) follows from using these parameters in (4.4) and putting it in (4.3). ∎

In the error estimation for the IFE solution, we need to use the fact that both linear and bilinear IFE spaces have the optimal approximation capability [29, 30, 38]. In particular, for every $u\in\tilde{H}_0^2(\Omega)$ satisfying the interface jump conditions (1.3) and (1.4), there exists a constant C such that the interpolation I_hu in the (either linear or bilinear) IFE space $S_h(\Omega)$ has the following error bound:

$$\|u - I_h u\|_{L^2(\Omega)} + h \left(\sum_{T\in\mathcal{T}_h} \|u - I_h u\|^2_{H^1(T)} \right)^{\frac{1}{2}} \le Ch^2 \|u\|_{\tilde{H}^2(\Omega)}. \tag{4.5}$$

In addition, we also need the error bound for $I_h u$ on interface edges, which is given in the following lemma.

LEMMA 4.2. *For every $u \in \tilde{H}^3(\Omega)$ satisfying the interface jump conditions* (1.3) *and* (1.4)*, there exists a constant C independent of the interface such that its interpolation $I_h u$ in the IFE space $S_h(\Omega)$ has the following error bound:*

$$\|\beta(\nabla(u - I_h u))|_K \cdot \mathbf{n}_B\|^2_{L^2(B)} \le C\big(h^2 \|u\|^2_{\tilde{H}^3(\Omega)} + h \|u\|^2_{\tilde{H}^2(K)}\big), \tag{4.6}$$

where K is an interface element and B is one of its interface edge.

Proof. We give a proof for linear IFEs, and the arguments can be used to establish this error bound for bilinear IFEs.

Without loss of generality, let $K = \triangle A_1A_2A_3$ be an interface triangle such that

$$A_1 = (0, h), A_2 = (0, 0), A_3 = (h, 0) \tag{4.7}$$

and assume that the interface points on the edge of K are

$$D = (0, d), E = (e, h - e) \tag{4.8}$$

with $A_1 \in K^+$. Also, we only discuss $B = \overline{A_1A_2}$; the estimate on the other interface edge can be established similarly.

By Lemmas 3.3 and 3.4 in [38], for every $X \in \overline{DA_2}$, we have

$$\begin{aligned} (I_h u(X) - u(X))_p &= \big(N^-(D) - N_{\overline{DE}}\big)\nabla u^-(X)(A_1 - D)\frac{\partial \phi_1(X)}{\partial p} \\ &\quad + I_1(X)\frac{\partial \phi_1(X)}{\partial p} + I_2(X)\frac{\partial \phi_2(X)}{\partial p} + I_3(X)\frac{\partial \phi_3(X)}{\partial p}, \quad p = x, y, \end{aligned} \tag{4.9}$$

where

$$N^-(D) = \begin{pmatrix} n_y(D)^2 + \rho n_x(D)^2 & (\rho - 1)n_x(D)n_y(D) \\ (\rho - 1)n_x(D)n_y(D) & n_y(D)^2 + \rho n_x(D)^2 \end{pmatrix},$$

$$N_{\overline{DE}} = \begin{pmatrix} \bar{n}_y^2 + \rho\bar{n}_x^2 & (\rho - 1)\bar{n}_x\bar{n}_y \\ (\rho - 1)\bar{n}_x\bar{n}_y & \bar{n}_y^2 + \rho\bar{n}_x^2 \end{pmatrix}$$

$\rho = \beta^-/\beta^+$, $\mathbf{n}(X) = (n_x(X), n_y(X))^T$ is the normal of Γ at X, $\mathbf{n}(\overline{DE}) = (\bar{n}_x, \bar{n}_y)^T$ is the normal of $\overline{DE}$, and

$$\begin{aligned} I_1(X) &= (1 - t_d)\big(N^-(D) - I\big) \int_0^1 \frac{d\nabla u^-}{dt}(tD + (1 - t)X) \cdot (A_1 - X)dt \\ &\quad + \int_0^{t_d} (1 - t)\frac{d^2 u^-}{dt^2}(tA_1 + (1 - t)X)dt \\ &\quad + \int_{t_d}^1 (1 - t)\frac{d^2 u^+}{dt^2}(tA_1 + (1 - t)X)dt, \end{aligned} \tag{4.10}$$

$$I_i(X) = \int_0^1 (1 - t)\frac{d^2 u^-}{dt^2}(tA_2 + (1 - t)X)dt, \qquad i = 2, 3, \tag{4.11}$$

where $D = t_d A_1 + (1-t_d)X = X + t_d(A_1 - X)$. By Lemma 3.1 and Theorem 2.4 of [38], we have

$$\int_{\overline{DA_2}} \left(\left(N^-(D) - N_{\overline{DE}}\right) \nabla u^-(X)(A_1 - D) \frac{\partial \phi_1(X)}{\partial p} \right)^2 dX \le Ch^3 \|u\|^2_{H^3(\Omega^-)} \tag{4.12}$$

for $p = x, y$. By direct calculations, we have

$$\begin{aligned}
&\left| \frac{d\nabla u^-}{dt}(tD + (1-t)X) \cdot (A_1 - X) \right| \\
&\quad \le \Big(\left|u^-_{xx}(tD + (1-t)X)(x_d - x)(x_1 - x)\right| + \left|u^-_{xy}(tD + (1-t)X)(y_d - y)(x_1 - x)\right| \\
&\qquad + \left|u^-_{yx}(tD + (1-t)X)(x_d - x)(y_1 - y)\right| + \left|u^-_{yy}(tD + (1-t)X)(y_d - y)(y_1 - y)\right| \Big)
\end{aligned}$$

and

$$\begin{aligned}
&\left| \frac{d^2 u^s}{dt^2}(tA_i + (1-t)X) \right| \\
&\le \Big(\left|u^s_{xx}(tA_i + (1-t)X)(x_i - x)(x_i - x)\right| + \left|u^s_{xy}(tA_i + (1-t)X)(y_i - y)(x_i - x)\right| \\
&\quad + \left|u^s_{yx}(tA_i + (1-t)X)(x_i - x)(y_i - y)\right| + \left|u^s_{yy}(tA_i + (1-t)X)(y_i - y)(y_i - y)\right| \Big),
\end{aligned}$$

where $s = \pm, i = 1,2,3$. Let $I_{1,i}(X), i = 1,2,3$ be three integrals in $I_1(X)$, respectively. Then, by Theorem 2.4 of [38], we have

$$\begin{aligned}
&\int_{\overline{DA_2}} \left(I_{1,1}(X) \frac{\partial \phi_1(X)}{\partial p} \right)^2 dX \\
&\quad \le \frac{C}{h^2} \int_0^d (1-t_d)^2 \int_0^1 \left|u^-_{yy}(0, ty_d + (1-t)y)(y_d - y)(h - y)\right|^2 dt dy \\
&\quad \le Ch^2 \int_0^d \left|u^-_{yy}(0,z)\right|^2 dz \le Ch^2 \|u\|^2_{H^3(\Omega^-)},
\end{aligned} \tag{4.13}$$

$$\begin{aligned}
&\int_{\overline{DA_2}} \left(I_{1,2}(X) \frac{\partial \phi_1(X)}{\partial p} \right)^2 dX \\
&\quad \le C \int_0^d \int_0^{t_d} \left|u^-_{yy}(0, y + t(h-y))\right|^2 (h-y)^2 (1-t)^2 dt dy \\
&\quad \le Ch^2 \int_0^d \left|u^-_{yy}(0,z)\right|^2 dz \le Ch^2 \|u\|^2_{H^3(\Omega^-)},
\end{aligned} \tag{4.14}$$

$$\begin{aligned}
&\int_{\overline{DA_2}} \left(I_{1,3}(X) \frac{\partial \phi_1(X)}{\partial p} \right)^2 dX \\
&\quad \le C \int_0^d \int_{t_d}^1 \left|u^+_{yy}(0, y + t(h-y))\right|^2 (h-y)^2 (1-t)^2 dt dy \\
&\quad \le Ch^2 \int_d^h \left|u^+_{yy}(0,z)\right|^2 dz \le Ch^2 \|u\|^2_{H^3(\Omega^+)}.
\end{aligned} \tag{4.15}$$

Similarly, we can show that

$$\begin{aligned}
\int_{\overline{DA_2}} \left(I_2(X) \frac{\partial \phi_2(X)}{\partial p} \right)^2 dX &\le C \int_0^d \int_0^1 \left|u^-_{yy}(0, (1-t)y)\right|^2 y^2 (1-t)^2 dt dy \\
&\le Ch^2 \int_0^d \left|u^-_{yy}(0,z)\right|^2 dz \le Ch^2 \|u\|^2_{H^3(\Omega^-)}.
\end{aligned} \tag{4.16}$$

For the term involving $I_3(X)$, we have

$$\begin{aligned}\int_{\overline{DA_2}} \left(I_3(X)\frac{\partial\phi_3(X)}{\partial p}\right)^2 dX \le C\Bigg(h^2\int_0^d\int_0^1 &\left|u^-_{xx}(th,(1-t)y)\right|^2(1-t)^2dtdy\\ &+\int_0^d\int_0^1\left|u^-_{xy}(th,(1-t)y)\right|^2y^2(1-t)^2dtdy\\ &+\int_0^d\int_0^1\left|u^-_{yx}(th,(1-t)y)\right|^2y^2(1-t)^2dtdy\\ &+\int_0^d\int_0^1\left|u^-_{yy}(th,(1-t)y)\right|^2y^2(1-t)^2dtdy\Bigg).\end{aligned}\tag{4.17}$$

Let $th=p,(1-t)y=q$, and then we have

$$t=\frac{p}{h}, y=\frac{q}{1-t}=\frac{q}{1-p/h}$$

and

$$\frac{q^2}{h-p}=\frac{(1-t)^2y^2}{h-th}=(1-t)y\frac{y}{h}, |(1-t)y|\le h, \frac{y}{h}\le 1.$$

Hence,

$$\begin{aligned}&h^2\int_0^d\int_0^1\left|u^-_{xx}(th,(1-t)y)\right|^2(1-t)^2dtdy\\ &\quad=\iint\limits_{\triangle_{DA_2A_3}}\left|u^-_{xx}(p,q)\right|^2\frac{h^2(1-p/h)^2}{h-p}dpdq\\ &\quad\le\iint\limits_{\triangle_{DA_2A_3}}\left|u^-_{xx}(p,q)\right|^2(h-p)dpdq\le Ch\|u\|^2_{\tilde{H}^2(K)},\\ &\int_0^d\int_0^1\left|u^-_{xy}(th,(1-t)y)\right|^2y^2(1-t)^2dtdy\\ &\quad=\iint\limits_{\triangle_{DA_2A_3}}\left|u^-_{xy}(p,q)\right|^2\frac{q^2}{h-p}dpdq\le Ch\|u\|^2_{\tilde{H}^2(K)},\\ &\int_0^d\int_0^1\left|u^-_{yy}(th,(1-t)y)\right|^2y^2(1-t)^2dtdy\\ &\quad=\iint\limits_{\triangle_{DA_2A_3}}\left|u^-_{yy}(p,q)\right|^2\frac{q^2}{h-p}dpdq\le Ch\|u\|^2_{\tilde{H}^2(K)}.\end{aligned}$$

Using these estimates in (4.16), we have

$$\int_{\overline{DA_2}}\left(I_3(X)\frac{\partial\phi_3(X)}{\partial p}\right)^2dX\le Ch\|u\|^2_{\tilde{H}^2(K)}.\tag{4.18}$$

Finally, the inequality (4.6) follows from putting estimates (4.12)–(4.16) and (4.18) into (4.9). ∎

Remark 4.1. For every $u\in\tilde{W}^{2,\infty}(\Omega)$ satisfying the interface jump conditions (1.3) and (1.4), we can also show that there exists a constant C independent of the interface such that the interpolation $I_h u$ in the IFE space $S_h(\Omega)$ fulfills

$$\|\beta(\nabla(u - I_h u))|_K \cdot \mathbf{n}_B\|^2_{L^2(B)} \le Ch^3 \|u\|^2_{\tilde{W}^{2,\infty}(\Omega)}, \tag{4.19}$$

where K is an interface element and B is one of its interface edges. A proof for (4.19) is given in Appendix A, which uses arguments similar to those for proving Lemma 4.2.

Now, we are ready to derive the error bound for IFE solutions generated by the partially penalized IFE method (2.4).

THEOREM 4.3. *Assume that the exact solution u to the interface problem* (1.1)–(1.4) *is in $\tilde{H}^3(\Omega)$ and u_h is its IFE solution generated with $\alpha = 1$ on a Cartesian (either triangular or rectangular) mesh $\mathcal{T}_h$. Then there exists a constant C such that*

$$\|u - u_h\|_h \le Ch \|u\|_{\tilde{H}^3(\Omega)}. \tag{4.20}$$

Proof. From the weak form (2.3) and the IFE equation (2.4), we have

$$a_h(v_h, u_h - w_h) = a_h(v_h, u - w_h) \quad \forall v_h, w_h \in S_h(\Omega). \tag{4.21}$$

Letting $v_h = u_h - w_h$ in (4.21) and using the coercivity of $a_h(\cdot,\cdot)$, we have

$$\begin{aligned}\kappa \|u_h - w_h\|^2_h &\le |a_h(u_h - w_h, u_h - w_h)| = |a_h(u_h - w_h, u - w_h)| \\ &\le \left|\sum_{K\in\mathcal{T}_h}\int_K \beta\nabla(u_h - w_h)\cdot\nabla(u - w_h)dX\right| \\ &\quad + \left|\sum_{B\in\mathring{\mathcal{E}}^i_h}\int_B \{\beta\nabla(u - w_h)\cdot\mathbf{n}_B\}[u_h - w_h]ds\right| \\ &\quad + \left|\epsilon\sum_{B\in\mathring{\mathcal{E}}^i_h}\int_B \{\beta\nabla(u_h - w_h)\cdot\mathbf{n}_B\}[u - w_h]ds\right| \\ &\quad + \left|\sum_{B\in\mathring{\mathcal{E}}^i_h}\int_B \frac{\sigma^0_B}{|B|^\alpha}[u_h - w_h][u - w_h]ds\right|.\end{aligned} \tag{4.22}$$

We denote the four terms on the right of (4.22) by $Q_i, i = 1,2,3,4$. Then,

$$\begin{aligned}Q_1 &\le \left(\sum_{K\in\mathcal{T}_h}\left\|\beta^{1/2}\nabla(u - w_h)\right\|^2_{L^2(K)}\right)^{1/2}\left(\sum_{K\in\mathcal{T}_h}\left\|\beta^{1/2}\nabla(u_h - w_h)\right\|^2_{L^2(K)}\right)^{1/2} \\ &\le \frac{3}{2\kappa}\max(\beta^-,\beta^+)\|\nabla(u - w_h)\|^2_{L^2(\Omega)} + \frac{\kappa}{6}\sum_{K\in\mathcal{T}_h}\left\|\beta^{1/2}\nabla(u_h - w_h)\right\|^2_{L^2(K)} \\ &\le C\|\nabla(u - w_h)\|^2_{L^2(\Omega)} + \frac{\kappa}{6}\|u_h - w_h\|^2_h,\end{aligned} \tag{4.23}$$

$$\begin{aligned}Q_2 &\le \frac{\kappa}{6}\sum_{B\in\mathring{\mathcal{E}}^i_h}\frac{\sigma^0_B}{|B|^\alpha}\|[u_h - w_h]\|^2_{L^2(B)} + C\sum_{B\in\mathring{\mathcal{E}}^i_h}\frac{|B|^\alpha}{\sigma^0_B}\|\{\beta\nabla(u - w_h)\cdot\mathbf{n}_B\}\|^2_{L^2(B)} \\ &\le \frac{\kappa}{6}\|u_h - w_h\|^2_h + C\sum_{B\in\mathring{\mathcal{E}}^i_h}\frac{|B|^\alpha}{\sigma^0_B}\|\{\beta\nabla(u - w_h)\cdot\mathbf{n}_B\}\|^2_{L^2(B)}.\end{aligned} \tag{4.24}$$

To bound Q_3, for each $B \in \mathring{\mathcal{E}}_h^i$ we let $K_{B,i} \in \mathcal{T}_h, i = 1,2$ be such that $B = K_{B,1} \cap K_{B,2}$. First, by the standard trace inequality on elements for H^1 functions, we have

$$
\begin{aligned}
\|[u - w_h]\|_{L^2(B)} &\leq \left\|(u - w_h)|_{K_{B,1}}\right\|_{L^2(B)} + \left\|(u - w_h)|_{K_{B,2}}\right\|_{L^2(B)} \\
&\leq Ch^{-1/2}\left(\|u - w_h\|_{L^2(K_{B,1})} + h\|\nabla(u - w_h)\|_{L^2(K_{B,1})}\right) \\
&\quad + Ch^{-1/2}\left(\|u - w_h\|_{L^2(K_{B,2})} + h\|\nabla(u - w_h)\|_{L^2(K_{B,2})}\right).
\end{aligned}
$$

Then, applying the trace inequalities established in Lemma 3.2 or Lemma 3.5 depending on whether linear IFEs or bilinear IFE are considered, we have

$$
\begin{aligned}
&\|\{\beta\nabla(u_h - w_h)\cdot \mathbf{n}_B\}\|_{L^2(B)} \\
&\leq \frac{C}{2}h^{-1/2}\left(\left\|\sqrt{\beta}\nabla(u_h - w_h)\right\|_{L^2(K_{B,1})} + \left\|\sqrt{\beta}\nabla(u_h - w_h)\right\|_{L^2(K_{B,2})}\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
Q_3 &\leq |\epsilon| \sum_{B\in\mathring{\mathcal{E}}_h^i} \|\{\beta\nabla(u_h - w_h)\cdot \mathbf{n}_B\}\|_{L^2(B)} \|[u - w_h]\|_{L^2(B)} \\
&\leq Ch^{-2}\left(\|u - w_h\|^2_{L^2(\Omega)} + h^2\|\nabla(u - w_h)\|^2_{L^2(\Omega)}\right) + \frac{\kappa}{6}\|u_h - w_h\|_h^2. \qquad (4.25)
\end{aligned}
$$

To bound Q_4, by the standard trace inequality we have

$$
\begin{aligned}
&\int_B \frac{\sigma_B^0}{|B|^\alpha}[u - w_h][u - w_h]ds = \frac{\sigma_B^0}{|B|^\alpha}\|[u - w_h]\|^2_{L^2(B)} \\
&\leq \frac{\sigma_B^0}{|B|^\alpha}\left(\left\|(u - w_h)|_{K_{B,1}}\right\|_{L^2(B)} + \left\|(u - w_h)|_{K_{B,2}}\right\|_{L^2(B)}\right)^2 \\
&\leq \frac{\sigma_B^0}{|B|^\alpha}C|B||K_{B,1}|^{-1}\left(\|u - w_h\|_{L^2(K_{B,1})} + h\|\nabla(u - w_h)\|_{L^2(K_{B,1})}\right)^2 \\
&\quad + \frac{\sigma_B^0}{|B|^\alpha}C|B||K_{B,2}|^{-1}\left(\|u - w_h\|_{L^2(K_{B,2})} + h\|\nabla(u - w_h)\|_{L^2(K_{B,2})}\right)^2 \\
&\leq Ch^{-(\alpha+1)}\left(\|u - w_h\|_{L^2(K_{B,1})} + h\|\nabla(u - w_h)\|_{L^2(K_{B,1})}\right)^2 \\
&\quad + Ch^{-(\alpha+1)}\left(\|u - w_h\|_{L^2(K_{B,2})} + h\|\nabla(u - w_h)\|_{L^2(K_{B,2})}\right)^2, \qquad (4.26)
\end{aligned}
$$

where we have used the facts that

$$
|B| = h \quad \text{or} \quad |B| = \sqrt{2}h, \quad |K_{B,i}| = \frac{h^2}{2}, \quad \text{or} \quad |K_{B,i}| = h^2, i = 1,2.
$$

Then,

$$
\begin{aligned}
Q_4 &\leq \sum_{B\in\mathring{\mathcal{E}}_h^i}\left(\frac{\kappa}{6}\int_B \frac{\sigma_B^0}{|B|^\alpha}[u_h - w_h][u_h - w_h]ds + \frac{3}{2\kappa}\int_B \frac{\sigma_B^0}{|B|^\alpha}[u - w_h][u - w_h]ds\right) \\
&\leq \frac{\kappa}{6}\|u_h - w_h\|_h^2 + \frac{3}{2\kappa}\sum_{B\in\mathring{\mathcal{E}}_h^i}\int_B \frac{\sigma_B^0}{|B|^\alpha}[u - w_h][u - w_h]ds \\
&\leq \frac{\kappa}{6}\|u_h - w_h\|_h^2 + Ch^{-(\alpha+1)}\left(\|u - w_h\|^2_{L^2(\Omega)} + h^2\|\nabla(u - w_h)\|^2_{L^2(\Omega)}\right). \qquad (4.27)
\end{aligned}
$$

Then, we put all these bounds for $Q_i, i = 1, 2, 3, 4$ in (4.22) to have

$$\begin{aligned}\|u_h - w_h\|_h^2 \le C\,\|\nabla(u - w_h)\|_{L^2(\Omega)}^2 + C \sum_{B\in\mathring{\mathcal{E}}_h^i} \frac{|B|^\alpha}{\sigma_B^0} \|\{\beta\nabla(u - w_h)\cdot\mathbf{n}_B\}\|_{L^2(B)}^2 \\ + Ch^{-2}\left(\|u - w_h\|_{L^2(\Omega)}^2 + h^2\,\|\nabla(u - w_h)\|_{L^2(\Omega)}^2\right) \\ + Ch^{-(\alpha+1)}\big(\|u - w_h\|_{L^2(\Omega)}^2 + h^2\,\|\nabla(u - w_h)\|_{L^2(\Omega)}^2\big).\end{aligned}$$

Hence, we let $w_h = I_h u$ in the above and use the optimal approximation capability of linear and bilinear IFE spaces (4.5) to have

$$\begin{aligned}\|u_h - I_h u\|_h^2 \le C\big(h^2 + h^{4-(\alpha+1)}\big)\,\|u\|_{\tilde{H}^2(\Omega)}^2 \\ + C \sum_{B\in\mathring{\mathcal{E}}_h^i} \frac{|B|^\alpha}{\sigma_B^0} \|\{\beta\nabla(u - I_h u)\cdot\mathbf{n}_B\}\|_{L^2(B)}^2 .\end{aligned} \tag{4.28}$$

For the second term on the right in (4.28), we use Lemma 4.2 to bound it:

$$\begin{aligned}&\sum_{B\in\mathring{\mathcal{E}}_h^i} \frac{|B|^\alpha}{\sigma_B^0} \|\{\beta\nabla(u - I_h u)\cdot\mathbf{n}_B\}\|_{L^2(B)}^2 \\ &\quad\le \sum_{B\in\mathring{\mathcal{E}}_h^i} \frac{|B|^\alpha}{\sigma_B^0} C\big(h^2\,\|u\|_{3,\Omega}^2 + h\,\|u\|_{2,K_{B,1}} + h\,\|u\|_{2,K_{B,1}}\big) \\ &\quad\le Ch^{1+\alpha}\,\|u\|_{\tilde{H}^3(\Omega)}^2 .\end{aligned} \tag{4.29}$$

Here, we have used the fact that the number of interface elements is of $O(h^{-1})$. Hence, for $\alpha = 1$, we can combine (4.28) and (4.29) to have

$$\|u_h - I_h u\|_h \le Ch\,\|u\|_{\tilde{H}^3(\Omega)} . \tag{4.30}$$

In addition, using the optimal approximation capability of linear and bilinear IFE spaces (4.5) and (4.26), we can show that

$$\|u - I_h u\|_h \le Ch\,\|u\|_{\tilde{H}^2(\Omega)} . \tag{4.31}$$

Finally, the error estimate (4.20) follows from applying (4.30) and (4.31) to the following standard inequality:

$$\|u - u_h\|_h \le \|u - I_h u\|_h + \|u_h - I_h u\|_h . \qquad \square$$

Remark 4.2. If the exact solution u to the interface problem (1.1)–(1.4) is in the function space $\tilde{W}^{2,\infty}(\Omega)$, then, using Remark 4.1 and arguments similar to those for the proof of Theorem 4.3, we can show that the IFE solution u_h generated with $\alpha = 1$ on a Cartesian (either triangular or rectangular) mesh $\mathcal{T}_h$ has the following error estimate:

$$\|u - u_h\|_h \le C\big(h\,\|u\|_{\tilde{H}^2(\Omega)} + h^{3/2}\,\|u\|_{\tilde{W}^{2,\infty}(\Omega)}\big). \tag{4.32}$$

5. Numerical examples. In this section, we present a couple of numerical examples to demonstrate features of the partially penalized IFE methods for elliptic interface problems.

For comparison, the interface problem to be solved in the numerical examples is the same as the one in [30]. Specifically, we consider the interface problem (1.1)–(1.4) except that (1.2) is replaced by the nonhomogeneous boundary condition $u|_{\partial\Omega} = g$. Let the solution domain Ω be the open rectangle $(-1,1) \times (-1,1)$ and the interface Γ be the circle centered at origin point with a radius $r_0 = \pi/6.28$ which separates Ω into two subdomains, denoted by Ω^- and Ω^+, i.e.,

$$\Omega^- = \{(x,y) : x^2 + y^2 < r_0^2\} \quad \text{and} \quad \Omega^+ = \{(x,y) : x^2 + y^2 > r_0^2\}.$$

The exact solution u to the interface problem is chosen as follows:

$$u(x,y) = \begin{cases} \frac{r^\alpha}{\beta^-} & \text{if } r \le r_0, \\ \frac{r^\alpha}{\beta^+} + \left(\frac{1}{\beta^-} - \frac{1}{\beta^+}\right) & \text{otherwise,} \end{cases} \tag{5.1}$$

where $\alpha = 5$ and $r = \sqrt{x^2 + y^2}$. The functions f and g in this interface problem are consequently determined by u. The Cartesian meshes $\mathcal{T}_h, h > 0$ are formed by partitioning Ω into $N \times N$ congruent squares of size $h = 2/N$ for a set of values of integer N.

To describe our numerical results, we rewrite the bilinear form in the partially penalized IFE methods (2.4) as follows:

$$\begin{aligned} a_h(u_h, v_h) = & \sum_{T \in \mathcal{T}_h} \int_T \beta \nabla u_h \cdot \nabla v_h dX + \delta \sum_{B \in \mathring{\mathcal{E}}_h^i} \int_B \{\beta \nabla u_h \cdot \mathbf{n}_B\}[v_h] ds \\ & + \epsilon \sum_{B \in \mathring{\mathcal{E}}_h^i} \int_B \{\beta \nabla v_h \cdot \mathbf{n}_B\}[u_h] ds + \sum_{B \in \mathring{\mathcal{E}}_h^i} \frac{\sigma_B^0}{|B|} \int_B [u_h][v_h] ds. \end{aligned} \tag{5.2}$$

When $\delta = 0$, $\epsilon = 0$, and $\sigma_B^0 = 0\ \forall B \in \mathring{\mathcal{E}}_h^i$, the partially penalized bilinear IFE method becomes the *classic* bilinear IFE method proposed in [29, 30]. When $\delta = -1$ and $\sigma_B^0 > 0$, we call the partially penalized IFE methods corresponding to $\epsilon = -1, 0, 1$, respectively, the symmetric, incomplete, and nonsymmetric PPIFE methods because of their similarity to the corresponding DG methods [50]. In our numerical experiment, we use the parameter $\alpha = 1$ as suggested by our error estimation. Also, we use $\sigma_B^0 = 10\max(\beta^-, \beta^+)$ in the SPPIFE and IPPIFE schemes and $\sigma_B^0 = 1$ in the NPPIFE scheme.

In the first example, we test these IFE methods with the above interface problem whose diffusion coefficient has a large jump $(\beta^-, \beta^+) = (1, 10000)$. Because of the large value of β^+, the exact solution $u(x,y)$ varies little in Ω^+, which might be one of the reasons this example is not overly difficult for all of the IFE methods, and they indeed perform comparably well; see Tables 1–3. Specifically, all the partially penalized IFE methods converge optimally in the H^1-norm as predicted by the error analysis in the previous section. The classic IFE method also converges optimally in the H^1-norm even though the related error bound has not been rigorously established yet. The data in Table 2 demonstrate that all the IFE methods converge in the L^2-norm at the expected optimal rate. Even though they all seem to converge in the L^∞-norm, the data in Table 3 do not reveal any definite rates for them.

TABLE 1
Comparison of $|u_h - u|_{H^1(\Omega)}$ for different IFE methods with $\beta^- = 1$, $\beta^+ = 10000$.

	Classic IFE		SPP IFE		IPP IFE		NPP IFE	
N	$\lvert\cdot\rvert_{H^1}$	rate	$\lvert\cdot\rvert_{H^1}$	rate	$\lvert\cdot\rvert_{H^1}$	rate	$\lvert\cdot\rvert_{H^1}$	rate
20	$1.9495E{-}2$		$1.9538E{-}2$		$1.9538E{-}2$		$1.9482E{-}2$	
40	$1.0539E{-}2$	0.8873	$1.0601E{-}2$	0.8821	$1.0602E{-}2$	0.8820	$1.0562E{-}2$	0.8832
80	$5.4219E{-}3$	0.9590	$5.5838E{-}3$	0.9249	$5.5844E{-}3$	0.9248	$5.4945E{-}3$	0.9428
160	$2.7157E{-}3$	0.9975	$2.7720E{-}3$	1.0103	$2.7724E{-}3$	1.0103	$2.7291E{-}3$	1.0096
320	$1.3535E{-}3$	1.0046	$1.3887E{-}3$	0.9973	$1.3890E{-}3$	0.9970	$1.3613E{-}3$	1.0035
640	$6.7876E{-}4$	0.9957	$6.9915E{-}4$	0.9900	$6.9923E{-}4$	0.9902	$6.7915E{-}4$	1.0032
1280	$3.4023E{-}4$	0.9964	$3.4439E{-}4$	1.0216	$3.4447E{-}4$	1.0214	$3.3979E{-}4$	0.9991
2560	$1.7079E{-}4$	0.9942	$1.7036E{-}4$	1.0155	$1.7038E{-}4$	1.0156	$1.6994E{-}4$	0.9996

TABLE 2
Comparison of $\|u_h - u\|_{L^2(\Omega)}$ for different IFE methods with $\beta^- = 1$, $\beta^+ = 10000$.

	Classic IFE		SPP IFE		IPP IFE		NPP IFE	
N	$\lVert\cdot\rVert_{L^2}$	rate	$\lVert\cdot\rVert_{L^2}$	rate	$\lVert\cdot\rVert_{L^2}$	rate	$\lVert\cdot\rVert_{L^2}$	rate
20	$1.1175E{-}3$		$1.1273E{-}3$		$1.1276E{-}3$		$1.1179E{-}3$	
40	$2.8572E{-}4$	1.9676	$2.9171E{-}4$	1.9503	$2.9181E{-}4$	1.9501	$2.8567E{-}4$	1.9683
80	$7.5990E{-}5$	1.9107	$8.5150E{-}5$	1.7764	$8.5223E{-}5$	1.7757	$7.6342E{-}5$	1.9038
160	$1.8116E{-}5$	2.0685	$2.2589E{-}5$	1.9144	$2.2645E{-}5$	1.9120	$1.8098E{-}5$	2.0766
320	$4.4753E{-}6$	2.0172	$6.1332E{-}6$	1.8809	$6.1629E{-}6$	1.8775	$4.5193E{-}6$	2.0017
640	$1.0969E{-}6$	2.0286	$1.6502E{-}6$	1.8940	$1.6502E{-}6$	1.9010	$1.1235E{-}6$	2.0081
1280	$2.6689E{-}7$	2.0391	$3.7104E{-}7$	2.1530	$3.7275E{-}7$	2.1464	$2.7680E{-}7$	2.0211
2560	$6.3940E{-}8$	2.0615	$7.3251E{-}8$	2.3407	$7.3657E{-}8$	2.3393	$6.8809E{-}8$	2.0082

TABLE 3
Comparison of $\|u_h - u\|_{L^\infty(\Omega)}$ for different IFE methods with $\beta^- = 1$, $\beta^+ = 10000$.

	Classic IFE		SPP IFE		IPP IFE		NPP IFE	
N	$\lVert\cdot\rVert_{L^\infty}$	rate	$\lVert\cdot\rVert_{L^\infty}$	rate	$\lVert\cdot\rVert_{L^\infty}$	rate	$\lVert\cdot\rVert_{L^\infty}$	rate
20	$8.8830E{-}4$		$9.9803E{-}4$		$9.9820E{-}4$		$8.4663E{-}4$	
40	$4.3525E{-}4$	1.0292	$5.2256E{-}4$	0.9335	$5.2185E{-}4$	0.9357	$4.2059E{-}4$	1.0093
80	$1.6536E{-}4$	1.3962	$2.2692E{-}4$	1.2034	$2.2745E{-}4$	1.1981	$1.6816E{-}4$	1.3225
160	$7.4603E{-}5$	1.1483	$9.8441E{-}5$	1.2049	$9.8619E{-}5$	1.2056	$7.2067E{-}5$	1.2225
320	$1.2972E{-}5$	2.5238	$3.6045E{-}5$	1.4494	$3.6421E{-}5$	1.4371	$1.4362E{-}5$	2.3271
640	$6.1332E{-}6$	1.0807	$1.7607E{-}5$	1.0337	$1.7531E{-}5$	1.0548	$6.4702E{-}5$	1.1504
1280	$2.5136E{-}6$	1.2869	$5.1961E{-}6$	1.7606	$5.1799E{-}6$	1.7589	$1.6993E{-}6$	1.9289
2560	$1.2810E{-}6$	0.9725	$1.0242E{-}6$	2.3430	$1.0274E{-}6$	2.3339	$9.1534E{-}7$	0.8926

However, we have observed that the partially penalized IFE methods outperform the classic IFE method in many situations. We demonstrate this by numerical results generated by these IFE methods for the interface problem whose diffusion coefficient has a typical moderate jump $(\beta^-, \beta^+) = (1, 10)$. IFE solution errors are listed in Tables 4–6. From the data in Table 4, we can see that all the partially penalized IFE methods maintain their predicted $O(h)$ convergence rate in the H^1-norm over all the meshes up to the finest one, while the classic IFE method slightly looses its convergence rate in the H^1-norm when the mesh becomes very fine. The effects of the penalization are more prominent when the errors are gauged in the L^2-norm and L^∞-norm. According to the data in Table 5, all the partially penalized IFE methods converge at the optimal rate in the L^2-norm but the L^2-norm rate of the classic IFE method clearly degenerates when the mesh becomes finer. Similar phenomenon for the L^∞-norm convergence can be observed from those data in Table 6.

It is known that the pointwise accuracy of the classic IFE methods in the literature is usually quite poor around the interface, and we suspect the discontinuity of the

Table 4
Comparison of $|u_h - u|_{H^1(\Omega)}$ for different IFE methods with $\beta^- = 1$, $\beta^+ = 10$.

	Classic IFE		SPP IFE		IPP IFE		NPP IFE	
N	$\lvert\cdot\rvert_{H^1}$	rate	$\lvert\cdot\rvert_{H^1}$	rate	$\lvert\cdot\rvert_{H^1}$	rate	$\lvert\cdot\rvert_{H^1}$	rate
20	6.4758E−2		6.4751E−2		6.4753E−2		6.4719E−2	
40	3.2779E−2	0.9823	3.2650E−2	0.9878	3.2651E−2	0.9878	3.2637E−2	0.9877
80	1.6596E−2	0.9723	1.6386E−2	0.9947	1.6386E−2	0.9947	1.6382E−2	0.9944
160	8.4072E−3	0.9811	8.2081E−3	0.9974	8.2082E−3	0.9973	8.2063E−3	0.9973
320	4.2566E−3	0.9819	4.1071E−3	0.9988	4.1071E−3	0.9989	4.1068E−3	0.9987
640	2.2267E−3	0.9348	2.0544E−3	0.9994	2.0545E−3	0.9994	2.0544E−3	0.9994
1280	1.1795E−3	0.9167	1.0274E−3	0.9997	1.0274E−3	0.9997	1.0274E−3	0.9996
2560	6.6893E−4	0.8182	5.1377E−4	0.9998	5.1377E−4	0.9998	5.1377E−4	0.9998

Table 5
Comparison of $\|u_h - u\|_{L^2(\Omega)}$ for different IFE methods with $\beta^- = 1$, $\beta^+ = 10$.

	Classic IFE		SPP IFE		IPP IFE		NPP IFE	
N	$\lVert\cdot\rVert_{L^2}$	rate	$\lVert\cdot\rVert_{L^2}$	rate	$\lVert\cdot\rVert_{L^2}$	rate	$\lVert\cdot\rVert_{L^2}$	rate
20	4.3003E−3		4.2945E−3		4.2989E−3		4.2869E−3	
40	1.0622E−3	2.0174	1.0749E−3	1.9983	1.0745E−3	2.0003	1.0626E−3	2.0124
80	2.6196E−4	2.0196	2.6833E−4	2.0021	2.6797E−4	2.0036	2.6440E−4	2.0067
160	6.4952E−5	2.0119	6.7047E−5	2.0008	6.6872E−5	2.0026	6.5876E−5	2.0049
320	1.6311E−5	1.9935	1.6829E−5	1.9942	1.6794E−5	1.9935	1.6594E−5	1.9891
640	4.4482E−6	1.8746	4.2038E−6	2.0012	4.1934E−6	2.0017	4.1383E−6	2.0035
1280	1.4445E−6	1.6226	1.0501E−6	2.0012	1.0472E−6	2.0015	1.0336E−6	2.0014
2560	6.7593E−7	1.0956	2.6254E−7	1.9999	2.6149E−7	2.0017	2.5821E−7	2.0357

Table 6
Comparison of $\|u_h - u\|_{L^\infty(\Omega)}$ for different IFE methods with $\beta^- = 1$, $\beta^+ = 10$.

	Classic IFE		SPP IFE		IPP IFE		NPP IFE	
N	$\lVert\cdot\rVert_{L^\infty}$	rate	$\lVert\cdot\rVert_{L^\infty}$	rate	$\lVert\cdot\rVert_{L^\infty}$	rate	$\lVert\cdot\rVert_{L^\infty}$	rate
20	1.0969E−3		1.3680E−3		1.3785E−3		1.0082E−3	
40	5.4748E−4	1.0026	3.9775E−4	1.7822	3.9769E−4	1.7934	1.9172E−4	2.3947
80	5.0812E−4	0.1077	1.0601E−4	1.9077	1.0582E−4	1.9100	5.4491E−5	1.8149
160	2.2635E−4	1.1667	3.1598E−5	1.7463	3.1217E−5	1.7612	1.4045E−5	1.9559
320	1.2290E−4	0.8811	7.0324E−6	2.1677	6.9364E−6	2.1701	3.5092E−6	2.0009
640	7.0810E−5	0.7954	1.9288E−6	1.8664	1.9213E−6	1.8521	9.1942E−7	1.9324
1280	3.4111E−5	1.0537	5.0505E−7	1.9332	4.9869E−7	1.9459	2.2932E−7	2.0034
2560	1.7815E−5	0.9371	1.0199E−7	2.3080	1.0075E−7	2.3074	5.9784E−8	1.9395

IFE functions across interface edges is the main cause for this shortcoming. With the penalty to control the discontinuity in IFE functions, a partially penalized IFE method has the potential to produce better pointwise approximations. To demonstrate this, we plot errors of a classic bilinear IFE solution and a NPP IFE solution in Figure 2. The IFE solutions in these plots are generated on the mesh with 80×80 rectangles. From the plot on the left, we can easily see that the classic IFE solution has much larger errors in the vicinity of the interface. The plot on the right shows that the magnitude of the error in the NPP IFE solution is much smaller uniformly over the whole solution domain. This advantage is also observed for other partially penalized IFE solutions.

A. A proof of Remark 4.1. We give a proof for the linear IFEs, and the same arguments can be used to show this error bound for bilinear IFEs.

Without loss of generality, let $K = \triangle A_1A_2A_3$ be an interface triangle whose vertices and interface intersection points are given in (4.7) and (4.8) with $A_1 \in K^+$.

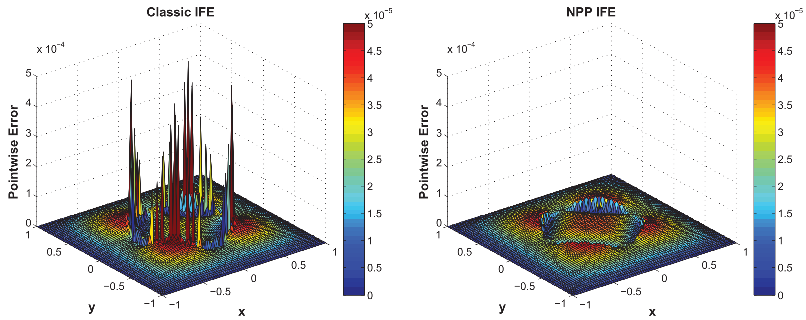

FIG. 2. *A comparison of pointwise errors of a classic bilinear IFE solution and a nonsymmetric partially penalized IFE solution. The coefficient in the interface problem has a moderate jump:* $(\beta^-, \beta^+) = (1, 10)$.

Also, we only discuss $B = \overline{A_1A_2}$; the estimate on the other interface edge can be established similarly.

By Lemmas 3.3 and 3.4 in [38], for every $X \in \overline{DA_2}$, we have

$$(I_hu(X) - u(X))_p = \big(N^-(D) - N_{\overline{DE}}\big)\nabla u^-(X)(A_1 - D)\frac{\partial \phi_1(X)}{\partial p}$$

$$+ I_1(X)\frac{\partial \phi_1(X)}{\partial p} + I_2(X)\frac{\partial \phi_2(X)}{\partial p} + I_3(X)\frac{\partial \phi_3(X)}{\partial p}, \quad p = x, y, \tag{A.1}$$

where $N^-(D)$, $N_{\overline{DE}}$, ρ, $\mathbf{n}(X)$, and $\mathbf{n}(\overline{DE})$ are defined the same as in the proof of Lemma 4.2. The quantities $I_i(X)$, $i = 1, 2, 3$, are given in (4.10) and (4.11). By Lemma 3.1 and Theorem 2.4 of [38], we have

(A.2)
$$\int_{\overline{DA_2}} \left(\big(N^-(D) - N_{\overline{DE}}\big)\nabla u^-(X)(A_1 - D)\frac{\partial \phi_1(X)}{\partial p}\right)^2 dX \le Ch^3 \|u\|^2_{W^{2,\infty}(\Omega^-)}$$

for $p = x, y$. By direct calculations, we have for $i = 1, 2, 3, p = x, y$

$$\begin{aligned}
&\left|\frac{d\nabla u^-}{dt}(tD + (1-t)X)\cdot(A_1 - X)\right| \\
&\quad \le \big(\left|u^-_{xx}(tD + (1-t)X)\right| + \left|u^-_{xy}(tD + (1-t)X)\right| \\
&\qquad + \left|u^-_{yx}(tD + (1-t)X)\right| + \left|u^-_{yy}(tD + (1-t)X)\right| \big)h^2, \\
&\left|\frac{d^2u^s}{dt^2}(tA_i + (1-t)X)\right| \\
&\quad \le \big(\left|u^s_{xx}(tA_i + (1-t)X)\right| + \left|u^s_{xy}(tA_i + (1-t)X)\right| \\
&\qquad + \left|u^s_{yx}(tA_i + (1-t)X)\right| + \left|u^s_{yy}(tA_i + (1-t)X)\right| \big)h^2.
\end{aligned}$$

By these estimates and Theorem 2.4 of [38], we have

$$\int_{\overline{DA_2}} \left(I_i(X)\frac{\partial \phi_i(X)}{\partial p}\right)^2 dX \le Ch^3 \|u\|^2_{\tilde{W}^{2,\infty}(\Omega)}. \tag{A.3}$$

Then, using (A.1) and applying (A.2) and (A.3) we have

$$\int_{\overline{DA_2}} \big((I_hu(X) - u(X))_p\big)^2 dX \le Ch^3 \|u\|^2_{\tilde{W}^{2,\infty}(\Omega)}, \quad p = x, y. \tag{A.4}$$

Similar arguments can be used to show

$$\int_{\overline{A_1 D}} \left((I_h u(X) - u(X))_p\right)^2 dX \le Ch^3 \|u\|^2_{\tilde{W}^{2,\infty}(\Omega)}, \quad p = x, y. \tag{A.5}$$

Finally, the estimate (4.19) on the interface edge $B = \overline{A_1 A_2}$ follows from (A.4) and (A.5).

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