Approximation Capability of a Bilinear Immersed Finite Element Space

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This article discusses a bilinear immersed finite element (IFE) space for solving second-order elliptic boundary value problems with discontinuous coefficients (interface problem). This is a nonconforming finite element space and its partition can be independent of the interface. The error estimates for the interpolation of a Sobolev function indicate that this IFE space has the usual approximation capability expected from bilinear polynomials. Numerical examples of the related finite element method are provided. Numer Methods Partial Differential Eq 24: 1265–1300, 2008

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I. INTRODUCTION

In this article, we investigate the approximation capability of a bilinear immersed finite element (IFE) space introduced in [1] for solving the following interface problem:

$$-\nabla \cdot (\beta \nabla u) = f, \quad (x, y) \in \Omega, \tag{1.1}$$

$$u|_{\partial\Omega} = g \tag{1.2}$$

together with the jump conditions on the interface Γ:

$$[u]|_{\Gamma} = 0, \tag{1.3}$$

$$\left[\beta \frac{\partial u}{\partial n}\right]|_{\Gamma} = 0. \tag{1.4}$$

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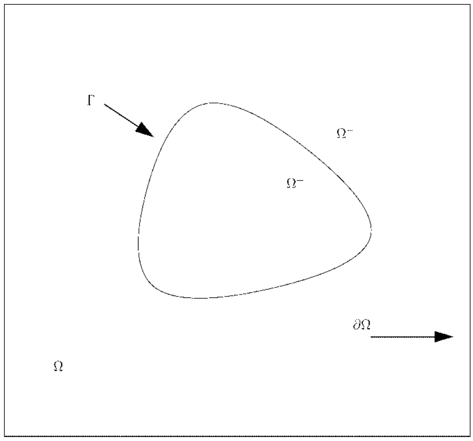

FIG. 1. A sketch of the domain for the interface problem.

Here, see the sketch in Fig. 1, without loss of generality, we assume that $\Omega \subset I\!R^2$ is a rectangular domain, the interface Γ is a curve separating Ω into two sub-domains Ω^-, Ω^+ such that $\overline{\Omega} = \overline{\Omega^-} \cup \overline{\Omega^+} \cup \Gamma$, and the coefficient $\beta(x, y)$ is a piecewise constant function defined by

$$\beta(x, y) = \begin{cases} \beta^-, & (x, y) \in \Omega^-, \\ \beta^+, & (x, y) \in \Omega^+. \end{cases}$$

Our main goal to show that this bilinear IFE space has the usual approximation capability expected from the bilinear polynomials.

It is well known that efficiently solving this interface problem is critical in many applications of engineering and sciences, including flow problems [2–8], electromagnetic problems [9–17], and shape/topology optimization problems [6, 18–25], and the modeling of nonlinear phenomena [26, 27], to name just a few.

Interface problem (1.1)–(1.4) can be solved by conventional numerical methods, including both finite difference (FD) methods, see [28, 29] and references therein, and finite element (FE) methods, see [30–32] and references therein, provided that their meshes are tailored to resolve the interfaces, see the illustration in Fig. 2. Otherwise, the lack of smoothness of the exact solution

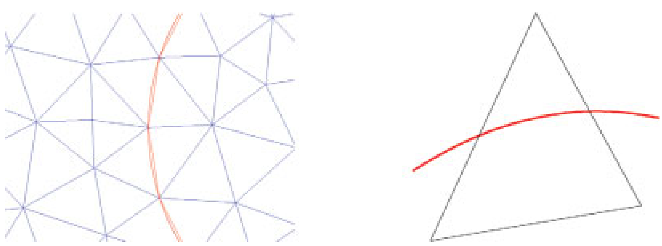

FIG. 2. The plot on the left shows how elements are placed along an interface in a standard FE method. An element not allowed in a standard FE method is illustrated by the plot on the right in which the red curve is the interface. [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

across the interface can make a numerical method not to perform as expected or converge at all [31–33].

Many efforts have been attempted to get rid of this limitation so that interface problems can be solved with meshes independent of the interfaces. In FD formulation, we note the early work of Peskin's immersed boundary method [34, 35]. Since then, FD methods such as the Cartesian grid method, embedded boundary method, cut-cell methods, etc., have been developed, and they have been used to treat Euler flows [36–40], Navier-Stokes flow [41–43], and of course, the interface problems involving the fundamental elliptic operators with discontinuous coefficients [44–50]. In FE formulation, Babuška et al. [51–53] developed the generalized and the partition of unity FE methods in which the local basis functions in an element are formed by solving the interface problem locally. The local basis functions in these methods can capture important features of the exact solution and they can even be non-polynomials. Exemplary methods in this framework are the partition of unity method and the extended finite element methods (X-FEMs) [54–56]. X-FEMs are very versatile and they can be used to handle many problems with discontinuity including the interface problems. The central idea of an X-FEM is to employ appropriate enrichment functions at the places needed. For convection dominant problems, the Eulerian-Lagrangian localized adjoint methods use special test functions designed according to the characteristic lines, see [57, 58] and references therein.

The recently developed IFE methods [1, 59–70] also fall into the general framework of Babuška and Osborn [53, 71] to adapt FE methods for interface problems by employing local basis functions formed according to the interface jump conditions while their meshes do not have to be conformed with the interfaces. However, IFE methods do not locally solve the interface problem and its basis functions are always piecewise polynomials. The main idea in IFE methods is more similar to that used for the Hsieh-Clough-Tocher macro C^1 element [72] where each local basis function in an element is defined piecewisely by cubic polynomials on three subtriangles such that the required continuity can be satisfied. In general, the elements in an IFE method consist of interface elements whose interiors are cut through by the interfaces and the rest called non-interface elements. An IFE method uses standard FE functions in all the non-interface elements, special piecewise finite element functions satisfying interface jump conditions are employed only in interface elements. We note that, when a mesh is fined enough, there are far more non-interface elements than interface elements.

Our goal here is to analyze the approximation capability of the bilinear IFE space introduced in [1], and this is a critical step in errors estimation of a FE (or finite volume-element) method based on this bilinear IFE space for the interface problem which we plan to dress in a forthcoming article. We will basically follow the framework developed in [66] that dealt with a triangular IFE space. However, the local bilinear IFE basis functions have a second degree term involving xy which leads to new difficulties demanding different techniques to analyze the interpolation error. In addition, we note that there are two types of interface elements topologically for a mesh formed by rectangles in contrast with a triangular mesh in which there is basically only one type of interface element, and they need to be discussed separately.

Without loss of generality, we assume in the discussion from now on that the elements in a rectangular mesh of Ω have the following features when the mesh size is small enough:

- (H_1): An interface Γ will not intersect an edge of any element at more than two points unless this edge is part of Γ.
- (H_2): If Γ intersects the boundary of a rectangle at two points, then these two points must be on different edges of this rectangle.

Also, for any subset T of Ω, we let

$$T^s = T \cap \Omega^s, \quad s = -, +.$$

For any function $f(x, y)$ defined in $T \subset \Omega$, we can restrict it to $T^s, s = -, +$ to obtain two functions as

$$f^s(x, y) = f(x, y), \quad \text{if } (x, y) \in T^s, \;\; s = -, +.$$

We use $\overline{DE}$ to denote the line segment between two points $D, E \in \Omega$. For any curve Γ, we use $\mathbf{n}_\Gamma$ to denote its unit normal vector pointing to a particular side of Γ. For any measurable subset Λ of Ω, we use $|\Lambda|$ to denote its measure. In deriving estimates, we often use C to represent a generic constant whose value might be different from line to line. Also, in the discussion later we add assumptions as we progress, and all the assumptions made before any theorem or lemma are assumed to hold for that statement.

The rest of this article is organized as follows. In Section II, we reintroduce the bilinear IFE space and describe basic properties of its local nodal basis functions. In Section III, we use the technique based on the multipoint Taylor expansion to derive error estimates for the bilinear IFE interpolation of the functions in Sobolev spaces. In Section IV, we present several numerical examples generated by the this bilinear IFE space.

II. THE BILINEAR IMMERSED FINITE ELEMENT SPACE

In this section, we first introduce the local bilinear nodal IFE basis functions and then use them to define the IFE space over Ω. We will also describe basic features of these basis functions.

We first consider a typical rectangle element $T \in \mathcal{T}_h$. Here, $\mathcal{T}_h, h > 0$ is a family of rectangular meshes of the solution domain Ω that can be a union of rectangles. Assume that the four vertices of T are $A_i, i = 1, 2, 3, 4$, with $A_i = (x_i, y_i)^t$. If T is an interface element, then we use $D = (x_D, y_D)^T$ and $E = (x_E, y_E)^T$ to denote the interface points on its edges. There are two types of rectangle interface elements. Type I are those for which the interface intersects with two of its adjacent edges; Type II are those for which the interface intersects with two of its opposite edges, see the sketch in Fig. 3.

Our main concern is the FE functions in an interface rectangle $T \in \mathcal{T}_h$. For our interface problems, the interface Γ separates an interface element T into two subsets T^- and T^+, we naturally

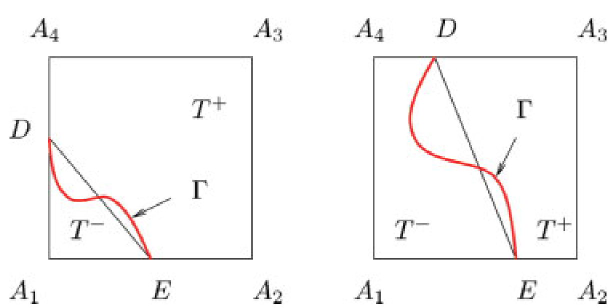

FIG. 3. Two typical interface elements. The element on the left is of Type I while the one on the right is of Type II. [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

can try to form a piecewise function by two bilinear polynomials defined in T^- and T^+, respectively. The challenge is obviously how to put them together so that the jump conditions across the interface are maintained.

Note that each bilinear polynomial has four freedoms (coefficients). The values of the FE function at the vertices of T provide four restrictions. The normal derivative jump condition on $\overline{DE}$ provides another. Then we can have three more restrictions by requiring the continuity of the finite element function at interface points D,E and $\frac{D+E}{2}$. Intuitively, these eight conditions can yield the desired piecewise bilinear polynomial in an interface rectangle. This idea leads us to consider functions defined as follows:

$$\phi(x,y) = \begin{cases} \phi^-(x,y) = a^-x + b^-y + c^- + d^-xy, & (x,y) \in T^-, \\ \phi^+(x,y) = a^+x + b^+y + c^+ + d^+xy, & (x,y) \in T^+, \\ \phi^-(D) = \phi^+(D), \phi^-(E) = \phi^+(E), \\ \phi^-\left(\frac{D+E}{2}\right) = \phi^+\left(\frac{D+E}{2}\right), \\ \int_{\overline{DE}} \left(\beta^- \frac{\partial \phi^-}{\partial \mathbf{n}_{\overline{DE}}} - \beta^+ \frac{\partial \phi^+}{\partial \mathbf{n}_{\overline{DE}}}\right) ds = 0, \end{cases} \tag{2.5}$$

where $\mathbf{n}_{\overline{DE}}$ is the unit vector perpendicular to the line $\overline{DE}$.

Remark 2.1. The last four equations can be modified accordingly to generate the IFE space that can be used to handle problems with inhomogeneous interface jump conditions.

We let $\phi_i(X)$ be the piecewise linear function described by (2.5) such that

$$\phi_i(x_j, y_j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j \end{cases}$$

for $1 \le i, j \le 4$, and we call them the bilinear IFE nodal basis functions on an interface element T.

Now we use the partition $\mathcal{T}_h$ to define the bilinear IFE space $S_h(\Omega)$. First, for every element $T \in \mathcal{T}_h$, we let $S_h(T) = \text{span}\{\phi_i, i = 1,2,3,4\}$, where ϕ_i, $i = 1,2,3,4$ are the standard bilinear nodal basis functions for a non-interface element T; otherwise, ϕ_i, $i = 1,2,3,4$ are the bilinear IFE nodal basis functions defined earlier. Then, we define a piecewise bilinear global nodal basis function $\phi_N(x,y)$ for each node $(x_N, y_N)^t$ of $\mathcal{T}_h$ such that $\phi_N(x_N, y_N) = 1$ but zero at other nodes, and $\phi_N|_T \in S_h(T)$ for any rectangle $T \in \mathcal{T}_h$. Finally, we define $S_h(\Omega)$ as the span of these global nodal basis functions.

A. Bilinear IFE Basis Functions in the Reference Element

As usual, we only need to define the nodal bilinear IFE basis functions $\hat{\phi}_i(\hat{X})$, $i = 1,2,3,4$ in the reference element $\hat{T}$ with vertices $\hat{A}_i = (\hat{x}_i, \hat{y}_i)^T$, $i = 1,2,3,4$:

$$\hat{A}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \hat{A}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \hat{A}_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \hat{A}_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The interface element T is related to the reference element by the usual affine mapping:

$$X = F(\hat{X}) = B + M\hat{X}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \hat{X} = \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}. \tag{2.6}$$

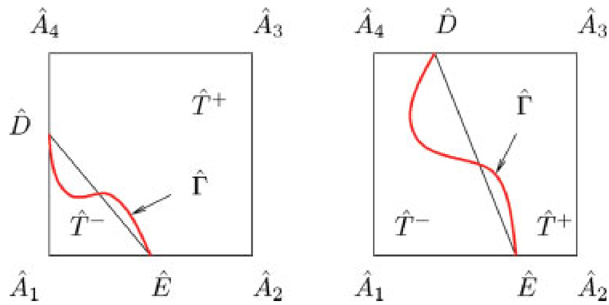

FIG. 4. Two types of reference interface elements. The element on the left is of Type I while the one on the right is of Type II. [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

This affine mapping and $\hat{\phi}_i(\hat{X}), i = 1, 2, 3, 4$ can be used to defined $\phi_i(X), i = 1, 2, 3, 4$ through the standard procedure.

Under the affine mapping, we assume that $\Gamma \cap T$ becomes $\hat{\Gamma}$, D becomes $\hat{D}$, E becomes $\hat{E}$, and $\hat{\Gamma}$ separates $\hat{T}$ into $\hat{T}^+$ and $\hat{T}^-$. Accordingly, there are two types of reference interface elements, see the sketch in Fig. 4.

By choosing proper B and M, i.e., a proper affine mapping, we can assume

$$\hat{D} = \begin{pmatrix} 0 \\ \hat{b} \end{pmatrix}, \quad \hat{E} = \begin{pmatrix} \hat{a} \\ 0 \end{pmatrix} \tag{2.7}$$

for Type I reference element, and

$$\hat{D} = \begin{pmatrix} \hat{b} \\ 1 \end{pmatrix}, \quad \hat{E} = \begin{pmatrix} \hat{a} \\ 0 \end{pmatrix} \tag{2.8}$$

for Type II element. Obviously, we can assume $0 < \hat{a}, \hat{b} \leq 1$ for Type I reference element, and $0 \leq \hat{a}, \hat{b} \leq 1$ for Type II reference element.

Assume $\hat{\phi}_i(\hat{X}), i = 1, 2, 3, 4$ are the bilinear IFE nodal basis on the reference element $\hat{T}$ such that

$$\hat{\phi}_i(\hat{x}, \hat{y}) = \begin{cases} \hat{a}_i^- + \hat{b}_i^- \hat{x} + \hat{c}_i^- \hat{y} + \hat{d}_i^- \hat{x}\hat{y}, & \text{if } (\hat{x}, \hat{y}) \in \hat{T}^-, \\ \hat{a}_i^+ + \hat{b}_i^+ \hat{x} + \hat{c}_i^+ \hat{y} + \hat{d}_i^+ \hat{x}\hat{y}, & \text{if } (\hat{x}, \hat{y}) \in \hat{T}^+. \end{cases}$$

Then $\hat{\phi}_i(\hat{X}), i = 1, 2, 3, 4$ should satisfy

$$\begin{cases} \hat{\phi}_i(\hat{A}_j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases} \\ \hat{\phi}_i^-(\hat{D}) = \hat{\phi}_i^+(\hat{D}), \quad \hat{\phi}_i^-(\hat{E}) = \hat{\phi}_i^+(\hat{E}), \quad \hat{\phi}_i^-\left(\dfrac{\hat{D}+\hat{E}}{2}\right) = \hat{\phi}_i^+\left(\dfrac{\hat{D}+\hat{E}}{2}\right), \\ \int_{\overline{\hat{D}\hat{E}}} \left(\beta^+ \dfrac{\partial \hat{\phi}_i^-}{\partial \mathbf{n}_{\overline{\hat{D}\hat{E}}}} - \beta^- \dfrac{\partial \hat{\phi}_i^+}{\partial \mathbf{n}_{\overline{\hat{D}\hat{E}}}}\right) ds = 0. \end{cases} \tag{2.9}$$

Basis Functions in Type I Elements. For Type I element, we first note that the nodal value constraints at $\hat{A}_1$ imply that

$$\hat{a}_i^- = \begin{cases} 1, & i = 1, \\ 0, & i = 2, 3, 4. \end{cases}$$

Also, the nodal value constraints at $\hat{A}_i$, $i = 2, 3, 4$ allow us to express $\hat{b}_i^+, \hat{c}_i^+$ and $\hat{d}_i^+$ as linear functions of $\hat{a}_i^+$. Then, conditions across the interface leads to a linear system about $\hat{b}_i^-, \hat{c}_i^-, \hat{d}_i^-, \hat{a}_i^+$. Solving this linear system, we can see that

$$\begin{aligned} &\hat{b}_i^- = \frac{P_{i,1}(\hat{a},\hat{b})}{W}, \quad \hat{c}_i^- = \frac{P_{i,2}(\hat{a},\hat{b})}{W}, \quad \hat{d}_i^- = \frac{P_{i,3}(\hat{a},\hat{b})}{W}, \quad \hat{a}_i^+ = \frac{P_{i,4}(\hat{a},\hat{b})}{W}, \\ &W = \begin{cases} [\hat{a}\hat{b}^2(2-\hat{b}) + \hat{a}^2\hat{b}(2-\hat{a})] + R[2\hat{b}^2(1-\hat{a}) + 2\hat{a}^2(1-\hat{b}) + \hat{a}^3\hat{b} + \hat{a}\hat{b}^3], \\ \qquad \text{if } R = \beta^-/\beta^+ \geq 1, \\ R[\hat{a}\hat{b}^2(2-\hat{b}) + \hat{a}^2\hat{b}(2-\hat{a})] + [2\hat{b}^2(1-\hat{a}) + 2\hat{a}^2(1-\hat{b}) + \hat{a}^3\hat{b} + \hat{a}\hat{b}^3], \\ \qquad \text{if } R = \beta^+/\beta^- \geq 1 \end{cases} \end{aligned} \tag{2.10}$$

where $P_{i,j}(\hat{a},\hat{b})$, $j = 1, 2, 3, 4$ are polynomials of $\hat{a}$ and $\hat{b}$. Moreover, $P_{i,j}(\hat{a},\hat{b})$, $j = 1, 2, 3, 4$ are linear combinations of the following terms:

$$\hat{a}^3, \hat{b}^3, \hat{a}^2, \hat{b}^2, \hat{a}\hat{b}, \hat{a}^2\hat{b}, \hat{a}^3\hat{b}, \hat{a}\hat{b}^2, \hat{a}\hat{b}^3. \tag{2.11}$$

Basis Functions in Type II Elements. Similarly, the nodal value constraints require that

$$\hat{a}_i^- = \begin{cases} 1, & i = 1, \\ 0, & i = 2, 3, 4. \end{cases} \qquad \hat{c}_i^- = \begin{cases} -1, & i = 1, \\ 0, & i = 2, 3, \\ 1, & i = 4. \end{cases}$$

Also, the nodal value constraints imply that $\hat{b}_i^+, \hat{c}_i^+$ are linear functions of $\hat{a}_i^+$ and $\hat{d}_i^+$. Then, the conditions across the interface lead to a linear system about $\hat{b}_i^-, \hat{d}_i^-, \hat{a}_i^+$, and $\hat{d}_i^+$. Solving this linear system, we have

$$\begin{aligned} &\hat{b}_i^- = \frac{P_{i,1}(\hat{a},\hat{b})}{W}, \quad \hat{c}_i^- = \frac{P_{i,2}(\hat{a},\hat{b})}{W}, \quad \hat{d}_i^- = \frac{P_{i,3}(\hat{a},\hat{b})}{W}, \quad \hat{a}_i^+ = \frac{P_{i,4}(\hat{a},\hat{b})}{W}, \\ &W = \begin{cases} \text{if } R = \beta^-/\beta^+ \geq 1: \\ [\hat{a}(1-\hat{a})^2 + \hat{b}(1-\hat{b}^2) + 2(\hat{a}-\hat{b})^2 + \hat{a}^2\hat{b} + \hat{a}\hat{b}^2] + R[(2-\hat{a}-\hat{b}) + (\hat{a}+\hat{b})(\hat{a}-\hat{b})^2] \\ \text{if } R = \beta^+/\beta^- \geq 1: \\ R[\hat{a}(1-\hat{a})^2 + \hat{b}(1-\hat{b}^2) + 2(\hat{a}-\hat{b})^2 + \hat{a}^2\hat{b} + \hat{a}\hat{b}^2] + [(2-\hat{a}-\hat{b}) + (\hat{a}+\hat{b})(\hat{a}-\hat{b})^2] \end{cases} \end{aligned} \tag{2.12}$$

where $P_{i,j}(\hat{a},\hat{b})$, $j = 1, 2, 3, 4$ are polynomials of $\hat{a}$ and $\hat{b}$.

B. Basic Properties of the Bilinear IFE Space

First, it is easy to see that $S_h(\Omega)$ has the following properties:

- The IFE space $S_h(\Omega)$ has the same number of nodal basis functions as that formed by the usual bilinear polynomials on the same partition of Ω.
- For a partition $\mathcal{T}_h$ fine enough, most of its rectangles are non-interface rectangles, and most of the nodal basis functions of the IFE space $S_h(\Omega)$ are just the usual bilinear nodal basis functions except for few nodes in the vicinity of the interface Γ.
- For any $\phi \in S_h(\Omega)$, we have

$$\phi|_{\Omega\backslash\Omega'} \in H^1(\Omega\backslash\Omega'),$$

where Ω' is the union of interface rectangles.

In the discussion from now on, we denote any $v(x,y) \in S_h(T)$ as follows

$$v(x,y) = \begin{cases} v^-(x,y) = a^- + b^-x + c^-y + d^-xy, & (x,y) \in T^-, \\ v^+(x,y) = a^+ + b^+x + c^+y + d^+xy, & (x,y) \in T^+. \end{cases} \tag{2.13}$$

The results in the following two lemmas are related to the continuity of functions in $S_h(\Omega)$ across the interface. First, the following lemma shows that every function $v \in S_h(T)$ where T is an interface element is continuous across $\overline{DE}$, and the mixed second derivative of $v \in S_h(T)$ is also continuous.

Lemma 2.1. *For any $v \in S_h(T)$ written as (2.13), we have*

$$v^- \equiv v^+, \quad \text{on } \overline{DE}, \tag{2.14}$$

$$d^+ = d^-. \tag{2.15}$$

Proof. Without loss of generality, we can assume that v is one of four basis functions of $S_h(T)$, and let $\hat{v}(\hat{X}) = v(B + M\hat{X})$ be the piecewise bilinear function on the reference element defined by v and the affine mapping. First, we note that

$$d^s = \frac{\partial^2 v^s}{\partial x \partial y}, \quad \hat{d}^s = \frac{\partial^2 \hat{v}^s}{\partial \hat{x} \partial \hat{y}}, \quad s = \pm.$$

By direct verification, we can further show that

$$\hat{d}^- = \hat{d}^+.$$

Then, (2.15) follows from the relationship between v and $\hat{v}$ defined by the affine mapping.

We now prove (2.14) for a Type I element T, the proof for a Type II element is similar. Without loss of generality, we assume the slope of $\overline{DE}$ is nonzero and finite, and the equation of $\overline{DE}$ is $y = kx + p$. Hence, on $\overline{DE}$, v becomes a piecewise quadratic polynomial in terms of x:

$$v(x,y) = \begin{cases} v^-(x,y) = (a^- + pc^-) + (b^- + kc^- + pd^-)x + kd^-x^2, & (x,y) \in T^- \\ v^+(x,y) = (a^+ + pc^+) + (b^+ + kc^+ + pd^+)x + kd^+x^2, & (x,y) \in T^+ \end{cases}$$

Because $v(x, y) \in S_h(T)$, we have

$$v^-(D) = v^+(D), \quad v^-(E) = v^+(E), \quad v^-\left(\frac{D+E}{2}\right) = v^+\left(\frac{D+E}{2}\right)$$

Hence $v^- \equiv v^+$ on $\overline{DE}$. ■

Lemma 2.2. *Assume $T \in \mathcal{T}_h$ is an interface element.*

1. *If $\Gamma \cap T$ is a line segment, then*

$$\phi^-|_{\Gamma\cap T} = \phi^+|_{\Gamma\cap T}, \quad \forall \phi \in S_h(\Omega).$$

2. *Every function $\phi \in S_h(T)$ satisfies the flux jump condition on $\Gamma \cap T$ exactly in the following weak sense:*

$$\int_{\Gamma\cap T} (\beta^- \nabla\phi^- - \beta^+ \nabla\phi^+) \cdot \mathbf{n}_\Gamma ds = 0.$$

Proof. Property 1 follows directly from (2.14). For any $\phi \in S_h(T)$, it is obvious that $\phi^s \in H^2(T^s)$, $s = -, +$. Also, because ϕ is a piecewise bilinear polynomial satisfying (2.5), Green's formula leads to

$$\int_{\Gamma\cap T} (\beta^- \nabla\phi^- - \beta^+ \nabla\phi^+) \cdot \mathbf{n}_\Gamma ds = -\int_{\overline{DE}} (\beta^- \nabla\phi^- - \beta^+ \nabla\phi^+) \cdot \mathbf{n}_{\overline{DE}} ds = 0.$$

■

The local basis functions of this bilinear IFE space has the property of partition of unity.

Theorem 2.1. *Let $T \in \mathcal{T}_h$ be an interface element and let $\phi_i(X) \in S_h(T)$, $i = 1, 2, 3, 4$ be the bilinear IFE nodal basis functions defined above. Then,*

$$\phi_1(X) + \phi_2(X) + \phi_3(X) + \phi_4(X) = 1, \forall X \in T$$

Proof. We only need to verify this for the corresponding basis functions on the reference element $\hat{T}$. For either Type I or Type II element, by direct calculations, we can see that

$$\sum_{i=1}^{4} \hat{a}_i^s = 1, \quad \sum_{i=1}^{4} \hat{b}_i^s = 0, \; \sum_{i=1}^{4} \hat{c}_i^s = 0, \; \sum_{i=1}^{4} \hat{d}_i^s = 0, \; s = \pm.$$

These imply that the partition of unity holds for the basis functions on the reference element and the result of this theorem follows. ■

The following lemma suggests that the bilinear IFE functions are consistent with standard bilinear FE functions.

Lemma 2.3. *Consider an interface element $T \in \mathcal{T}_h$ and a function $\phi \in S_h(T)$. If $\beta^- = \beta^+$, then*

$$\phi^- = \phi^+$$

and ϕ becomes a bilinear polynomial.

Proof. By direct calculations we can see that the result is true for $\hat{\phi}_i$, $i = 1, 2, 3, 4$ and then for ϕ_i, $i = 1, 2, 3, 4$. Since $\phi \in S_h(T)$ is a linear combination of ϕ_i, $i = 1, 2, 3, 4$, we know that the result of this lemma is also true for every $\phi \in S_h(T)$. ∎

In the discussion later, we need another assumption on the partition $\mathcal{T}_h$.

(H_3): The family of partitions $\mathcal{T}_h$ with $h > 0$ is regular. (See Definition 3.4.1 of [73])

The following theorem establishes bounds for the bilinear IFE basis functions.

Theorem 2.2. *Let $T \in \mathcal{T}_h$ be an interface element and let $\phi_i(X) \in S_h(T)$, $i = 1, 2, 3, 4$, be the bilinear IFE nodal basis functions defined above. Then, there exists constants C such that*

$$|\phi_i(X)| \leq C, \quad i = 1, 2, 3, 4, \tag{2.16}$$

$$\|\nabla\phi_i(X)\| \leq Ch^{-1}, \quad i = 1, 2, 3, 4. \tag{2.17}$$

Proof. Without loss of generality, we assume that $R = \beta^-/\beta^+ \geq 1$, the similar arguments hold for $R = \beta^+/\beta^- \geq 1$. Also, we consider the case in which $T \in \mathcal{T}_h$ is a Type I element, similar arguments can be applied to the case in which $T \in \mathcal{T}_h$ is a Type II element.

First, we show that the coefficients $\hat{a}_i^s, \hat{b}_i^s, \hat{c}_i^s, \hat{d}_i^s$, $s = \pm$, $i = 1, 2, 3, 4$ of $\hat{\phi}_i$s are bounded. Note that these coefficients are linear combinations of $\frac{\hat{a}^k\hat{b}^l}{W}$ with the values of k and l listed in (2.11). By direct calculations, we can see that there exists a constant C such that $0 \leq |\frac{\hat{a}^k\hat{b}^l}{W}| \leq C$ for the values of k and l listed in (2.11). These inequalities lead to the boundedness of $\hat{a}_i, \hat{b}_i, \hat{c}_i, \hat{d}_i$, $i = 1, 2, 3, 4$ which imply the boundedness of $\hat{\phi}_i$, $i = 1, 2, 3, 4$. Then (2.16) follows because the affine transformation (2.6) is used to define ϕ_i, $i = 1, 2, 3, 4$ from $\hat{\phi}_i$s.

Since the partition is regular, we have $\|M^{-T}\| \leq Ch^{-1}$. Then, (2.17) follows from $\nabla\phi_i = M^{-T}\nabla\hat{\phi}_i$ and the boundedness of the coefficients of $\nabla\hat{\phi}_i$s. ∎

III. ERROR ESTIMATES FOR INTERPOLATION APPROXIMATIONS

We now discuss the approximation capability of the bilinear IFE space introduced in the last section. We focus on the bilinear IFE interpolation of a function from a suitable Sobolev space, and will derive error estimates in the corresponding Sobolev norms.

For an element T in Ω, we let

$$PW_p^m(T) = \left\{u | u|_{T^s} \in W_p^m(T^s), s = -, +\right\}, p \geq 1, m = 0, 1, 2,$$

$$PH_{\text{int}}^2(T) = \left\{u \in C(T), u|_{T^s} \in H^2(T^s), s = -, +, \left[\beta\frac{\partial u}{\partial \mathbf{n}_\Gamma}\right] = 0 \text{ on } \Gamma \cap T\right\},$$

$$PC_{\text{int}}^m(T) = \left\{u \in C(T), u|_{T^s} \in C^m(T^s), s = -, +, \left[\beta\frac{\partial u}{\partial \mathbf{n}_\Gamma}\right] = 0 \text{ on } \Gamma\right\},$$

where $W_p^m(\Lambda)$ is the standard Soblev space defined on a set Λ equipped with the norm $\|\cdot\|_{m,p,\Lambda}$ and seminorm $|\cdot|_{m,p,\Lambda}$. As usual, we let $PH^m(T) = PW_2^m(T)$. Obviously, we have $PC_{\text{int}}^2(T) \subset PH_{\text{int}}^2(T)$. Also, for any function $u \in PW_p^m(T)$, we let

$$\|u\|_{m,p,T}^2 = \|u\|_{m,p,T^-}^2 + \|u\|_{m,p,T^+}^2, \tag{3.18}$$

and the seminorm of $PW_p^m(T)$ can be defined accordingly by

$$|u|_{m,p,T}^2 = |u|_{m,p,T^-}^2 + |u|_{m,p,T^+}^2. \quad (3.19)$$

When $p = 2$, we will drop p from the notation of the norms, e.g., we will use $\|u\|_{m,T} = \|u\|_{m,2,T}$. Similar definitions can be introduced for $PH_{\text{int}}^2(T), PC_{\text{int}}^m(T)$ for any $T \in \mathcal{T}_h$ and $PH^m(\Omega), PH_{\text{int}}^2(\Omega), PC_{\text{int}}^m(\Omega)$.

In this section, we assume that the interface curve Γ and the partition $\mathcal{T}_h$ satisfy the following assumptions:

(H_4): The interface curve Γ is defined by a piecewise C^2 function, and the partition $\mathcal{T}_h$ is formed such that the subset of Γ in any interface element is C^2.

(H_5): The interface Γ is smooth enough so that $PC_{\text{int}}^3(T)$ is dense in $PH_{\text{int}}^2(T)$ for any interface element T of $\mathcal{T}_h$.

We note that (H_5) will hold if Γ is sufficiently smooth, see the results of [74, 75] on the transmission problems.

For a function $u \in PH_{\text{int}}^2(T)$, $T \in \mathcal{T}_h$, we let $I_{h,T}u \in S_h(T)$ be its interpolant such that $I_{h,T}u(X) = u(X)$ when X is a vertex of T. For an element T with vertices A_1, A_2, A_3, A_4, we have

$$I_{h,T}u(X) = u(A_1)\phi_1(X) + u(A_2)\phi_2(X) + u(A_3)\phi_3(X) + u(A_4)\phi_4(X).$$

Accordingly, for a function $u \in PH_{\text{int}}^2(\Omega)$, we let $I_h u \in S_h(\Omega)$ be its interpolation such that $I_h u|_T = I_{h,T}(u|_T)$ for any $T \in \mathcal{T}_h$.

The purpose of this section is to derive error estimates for the interpolation of $u \in PH_{\text{int}}^2(\Omega)$, and we will carry the discussion piecewisely for each element T in the partition $\mathcal{T}_h$. Recall that the error estimate of $I_h u$ in any non-interface rectangular element T is well known, see for example [73]:

$$\|I_h u - u\|_{0,T} + h\|I_h u - u\|_{1,T} \le Ch^2\|u\|_{2,T}.$$

Therefore, in the discussion from now on, we focus on interface elements of $\mathcal{T}_h$.

We call a point $X = (x, y)^T$ in an interface element T an obscure point if one of the four line segments connecting X and the vertices of T intersects the interface more than once. Without loss of generality, we discuss interface elements that do not contain any obscure point because the arguments used below can be readily extended to handle the interface elements with obscure points.

Now we discuss the IFE interpolation error estimates for the two types of interface elements, i.e, Type I elements and Type II elements.

A. Interpolation Error on a Type I Interface Element

Without loss of generality, we assume $T \in \mathcal{T}_h$ is a Type I interface element with vertices $A_i = (x_i, y_i)$, $i = 1, 2, 3, 4$, such that $A_1 \in T^+$ and $A_i \in T^-$, $i = 2, 3, 4$, see Fig. 5.

We start with the estimation on T^-. Consider a point $X = (x, y)^T \in T^-$ and assume that line segments $\overline{XA_i}$, $i = 2, 3, 4$ do not intersect with the interface and $\overline{DE}$, while line segment $\overline{XA_1}$ meets Γ at $\widetilde{A}_1$ (see Fig. 5) with

$$\widetilde{A}_1 = \tilde{t}A_1 + (1 - \tilde{t})X = (\widetilde{x}_1, \widetilde{y}_1)^T$$

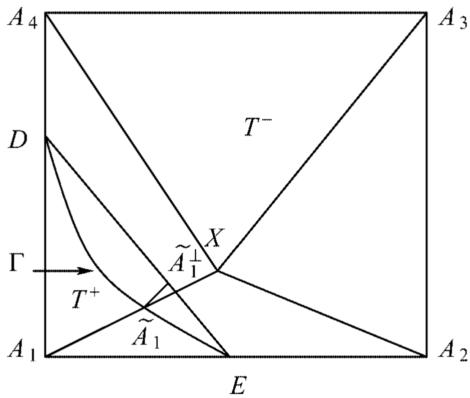

FIG. 5. An interface rectangle element with no obscure point. A point X $\in T^-$ is connected to the four vertices by line segments in a Type I interface element.

for a certain $\tilde{t}$. For any point $\widetilde{A} \in \Gamma$, let $\widetilde{A}^{\perp}$ be the orthogonal projection of $\tilde{A}$ onto $\overline{DE}$ (see Fig. 5). We will also use the following notations: $\tilde{\rho} = \frac{\beta^+}{\beta^-}$, $\rho = \frac{\beta^-}{\beta^+}$.

First, let us recall two lemmas from [66].

Lemma 3.1. *Assume* $\mathbf{n}(\widetilde{A}) = (n_x(\widetilde{A}), n_y(\widetilde{A}))^T$ *is the unit normal vector of* Γ *at* $\widetilde{A}$, $\mathbf{n}(\overline{DE}) = (\overline{n}_x, \overline{n}_y)^T$ *is the unit normal vector of* $\overline{DE}$, *and* $X_{\overline{DE}}$ *is a point on* $\overline{DE}$. *Then, for every function* $u(x,y)$ *satisfying the interface jump conditions (1.3) and (1.4), we have*

$$\nabla u^+(\widetilde{A}) = N^-(\widetilde{A})\nabla u^-(\widetilde{A}), \quad N^-(\widetilde{A}) = \begin{pmatrix} n_y(\widetilde{A})^2 + \rho n_x(\widetilde{A})^2 & (\rho - 1)n_x(\widetilde{A})n_y(\widetilde{A}) \\ (\rho - 1)n_x(\widetilde{A})n_y(\widetilde{A}) & n_x(\widetilde{A})^2 + \rho n_y(\widetilde{A})^2 \end{pmatrix}, \tag{3.20}$$

$$\nabla u^-(\widetilde{A}) = N^+(\widetilde{A})\nabla u^+(\widetilde{A}), \quad N^+(\widetilde{A}) = \begin{pmatrix} n_y(\widetilde{A})^2 + \tilde{\rho} n_x(\widetilde{A})^2 & (\tilde{\rho} - 1)n_x(\widetilde{A})n_y(\widetilde{A}) \\ (\tilde{\rho} - 1)n_x(\widetilde{A})n_y(\widetilde{A}) & n_x(\widetilde{A})^2 + \tilde{\rho} n_y(\widetilde{A})^2 \end{pmatrix}, \tag{3.21}$$

and for every $u \in S_h(T)$ *we have*

$$\nabla u^+(X_{\overline{DE}}) = N^-_{\overline{DE}}\nabla u^-(X_{\overline{DE}}), \quad N^-_{\overline{DE}} = \begin{pmatrix} \overline{n}_y^2 + \rho\overline{n}_x^2 & (\rho - 1)\overline{n}_x\overline{n}_y \\ (\rho - 1)\overline{n}_x\overline{n}_y & \overline{n}_x^2 + \rho\overline{n}_y^2 \end{pmatrix}, \tag{3.22}$$

$$\nabla u^-(X_{\overline{DE}}) = N^+_{\overline{DE}}\nabla u^+(X_{\overline{DE}}), \quad N^+_{\overline{DE}} = \begin{pmatrix} \overline{n}_y^2 + \tilde{\rho}\overline{n}_x^2 & (\tilde{\rho} - 1)\overline{n}_x\overline{n}_y \\ (\tilde{\rho} - 1)\overline{n}_x\overline{n}_y & \overline{n}_x^2 + \tilde{\rho}\overline{n}_y^2 \end{pmatrix}. \tag{3.23}$$

Lemma 3.2. *There exist constants* $C > 0$ *such that for any point* $\widetilde{A} \in \Gamma$, *we have*

$$\|\widetilde{A} - \widetilde{A}^{\perp}\| \le Ch^2. \tag{3.24}$$

$$\left\| N^s_{\overline{DE}} - N^s(\widetilde{A}) \right\| \le Ch, \quad s = -, +. \tag{3.25}$$

The following lemma gives the straight forward Taylor expansion of a bilinear function.

Lemma 3.3. *If $f(x, y) = a + bx + cy + dxy$, $X = (x, y)$, $Z = (x_z, y_z)$, then*

$$f(Z) = f(X) + \nabla f(X) \cdot (Z - X) + d(x_z - x)(y_z - y).$$

In all the discussion from now on, for a given point $X = (x, y)^T$, we let $X^s = (y, x)^T$. $\forall X = (x, y)^T \in T^-$, $A = (x_A, y_A)^T \in T^+$, we let $\widetilde{A} = (\widetilde{x}, \widetilde{y})^T$ be the intersection point of Γ and $\overline{AX}$.

The lemma below establishes an expansion of a bilinear IFE function across the interface.

Lemma 3.4. *Assume that $v \in S_h(T)$, $X = (x, y)^T \in T^-$, $A = (x_A, y_A)^T \in T^+$. Then*

$$\begin{aligned} v(A) = v(X) &+ \nabla v(X) \cdot (A - X) + \big(N^-_{\overline{DE}} - I\big)\nabla v(X) \cdot (A - \widetilde{A}) \\ &+ \big(N^-_{\overline{DE}} - I\big)\nabla v(X) \cdot (\widetilde{A} - X_{\overline{DE}}) + d^- N^-_{\overline{DE}}\big(X^s_{\overline{DE}} - X^s\big) \cdot (A - X_{\overline{DE}}) \\ &+ d^-(x_A - \bar{x})(y_A - \bar{y}) + d^-(\bar{x} - x)(\bar{y} - y) \end{aligned}$$

where $X_{\overline{DE}} = (\bar{x}, \bar{y})^T$ is an arbitrary point on $\overline{DE}$.

Proof. By Lemma 3.3, Lemma 2.1 and (3.22), we have

$$\begin{aligned} v(A) = v^+(A) &= v^+(X_{\overline{DE}}) + \nabla v^+(X_{\overline{DE}}) \cdot (A - X_{\overline{DE}}) + d^+(x_A - \bar{x})(y_A - \bar{y}) \\ &= v^-(X_{\overline{DE}}) + N^-_{\overline{DE}}\nabla v^-(X_{\overline{DE}}) \cdot (A - X_{\overline{DE}}) + d^-(x_A - \bar{x})(y_A - \bar{y}) \\ &= v^-(X) + \nabla v^-(X) \cdot (X_{\overline{DE}} - X) + d^-(\bar{x} - x)(\bar{y} - y) \\ &\quad + N^-_{\overline{DE}}\nabla v^-(X_{\overline{DE}}) \cdot (A - X_{\overline{DE}}) + d^-(x_A - \bar{x})(y_A - \bar{y}) \\ &= v^-(X) + \nabla v^-(X) \cdot (A - X) + \big(N^-_{\overline{DE}} - I\big)\nabla v^-(X) \cdot (A - X_{\overline{DE}}) \\ &\quad + N^-_{\overline{DE}}[\nabla v^-(X_{\overline{DE}}) - \nabla v^-(X)] \cdot (A - X_{\overline{DE}}) + d^-(x_A - \bar{x})(y_A - \bar{y}) \\ &\quad + d^-(\bar{x} - x)(\bar{y} - y). \end{aligned}$$

Because

$$\nabla v^-(X_{\overline{DE}}) = \begin{pmatrix} b^- + d^-\bar{y} \\ c^- + d^-\bar{x} \end{pmatrix}, \quad \nabla v^-(X) = \begin{pmatrix} b^- + d^- y \\ c^- + d^- x \end{pmatrix},$$

we have $\nabla v^-(X_{\overline{DE}}) - \nabla v^-(X) = d^-(X^s_{\overline{DE}} - X^s)$. Hence,

$$\begin{aligned} v(A) &= v^-(X) + \nabla v^-(X) \cdot (A - X) + \big(N^-_{\overline{DE}} - I\big)\nabla v^-(X) \cdot (A - X_{\overline{DE}}) \\ &\quad + N^-_{\overline{DE}} d^-\big(X^s_{\overline{DE}} - X^s\big) \cdot (A - X_{\overline{DE}}) + d^-(x_A - \bar{x})(y_A - \bar{y}) + d^-(\bar{x} - x)(\bar{y} - y) \\ &= v(X) + \nabla v(X) \cdot (A - X) + \big(N^-_{\overline{DE}} - I\big)\nabla v(X) \cdot (A - \widetilde{A}) \\ &\quad + \big(N^-_{\overline{DE}} - I\big)\nabla v(X) \cdot (\widetilde{A} - X_{\overline{DE}}) + d^- N^-_{\overline{DE}}\big(X^s_{\overline{DE}} - X^s\big) \cdot (A - X_{\overline{DE}}) \\ &\quad + d^-(x_A - \bar{x})(y_A - \bar{y}) + d^-(\bar{x} - x)(\bar{y} - y). \end{aligned}$$

∎

Lemma 3.5. *Assume that $v \in S_h(T), X = (x, y)^T \in T^-$. Then we have*

$$\nabla v(X) \cdot \sum_{i=1}^{4}(A_i - X)\phi_i(X) = -\big(N^-_{\overline{DE}} - I\big)\nabla v(X) \cdot (A_1 - \widetilde{A}_1)\phi_1(X) - \big(N^-_{\overline{DE}} - I\big)\nabla v(X) \cdot (\widetilde{A}_1 - X_{\overline{DE}})\phi_1(X) - d^-\Bigg[N^-_{\overline{DE}}\big(X^s_{\overline{DE}} - X^s\big) \cdot (A_1 - X_{\overline{DE}})\phi_1(X) + (x_1 - \bar{x})(y_1 - \bar{y})\phi_1(X) + (\bar{x} - x)(\bar{y} - y)\phi_1(X) + \sum_{i=2}^{4}[(x_i - x)(y_i - y)\phi_i(X)]\Bigg].$$

Proof. By using Lemma 3.3 and Lemma 3.4, we can get

$$\begin{aligned}
v(A_i) &= v(X) + \nabla v(X) \cdot (A_i - X) + d^-(x_i - x)(y_i - y), i = 2, 3, 4,\\
v(A_1) &= v(X) + \nabla v(X) \cdot (A_1 - X) + \big(N^-_{\overline{DE}} - I\big)\nabla v(X) \cdot (A_1 - \widetilde{A}_1)\\
&\quad + \big(N^-_{\overline{DE}} - I\big)\nabla v(X) \cdot (\widetilde{A}_1 - X_{\overline{DE}}) + d^- N^-_{\overline{DE}}\big(X^s_{\overline{DE}} - X^s\big) \cdot (A_1 - X_{\overline{DE}})\\
&\quad + d^-(x_1 - \bar{x})(y_1 - \bar{y}) + d^-(\bar{x} - x_1)(\bar{y} - y_1).
\end{aligned}$$

Because $v \in S_h(T)$,

$$\begin{aligned}
v(X) = I_{h,T}v(X) &= \sum_{i=1}^{4} v(A_i)\phi_i(X) = v(X)\sum_{i=1}^{4}\phi_i(X) + \nabla v(X) \cdot \sum_{i=1}^{4}(A_i - X)\phi_i(X)\\
&\quad + \big(N^-_{\overline{DE}} - I\big)\nabla v(X) \cdot (A_1 - \widetilde{A}_1)\phi_1(X) + \big(N^-_{\overline{DE}} - I\big)\nabla v(X) \cdot (\widetilde{A}_1 - X_{\overline{DE}})\phi_1(X)\\
&\quad + d^-\Bigg[N^-_{\overline{DE}}\big(X^s_{\overline{DE}} - X^s\big) \cdot (A_1 - X_{\overline{DE}})\phi_1(X) + (x_1 - \bar{x})(y_1 - \bar{y})\phi_1(X)\\
&\quad + (\bar{x} - x)(\bar{y} - y)\phi_1(X) + \sum_{i=2}^{4}(x_i - x)(y_i - y)\phi_i(X)\Bigg]
\end{aligned}$$

which leads to the result of this lemma because of Theorem 2.1. ■

Lemma 3.6. *Given a two-dimensional vector* **q**, *a point $X \in T^-$ and two real numbers r, d^-, then there exists a function $v \in S_h(T)$ such that $\nabla v(X) = \mathbf{q}, v(X) = r, \frac{\partial^2 v^-(X)}{\partial x \partial y} = d^-$ and*

$$\mathbf{q} \cdot \sum_{i=1}^{4}(A_i - X)\phi_i(X) = -\big(N^-_{\overline{DE}} - I\big)\mathbf{q} \cdot (A_1 - \widetilde{A}_1)\phi_1(X) - \big(N^-_{\overline{DE}} - I\big)\mathbf{q} \cdot (\widetilde{A}_1 - X_{\overline{DE}})\phi_1(X) - d^-\Bigg[N^-_{\overline{DE}}\big(X^s_{\overline{DE}} - X^s\big) \cdot (A_1 - X_{\overline{DE}})\phi_1(X) + (x_1 - \bar{x})(y_1 - \bar{y})\phi_1(X) + (\bar{x} - x)(\bar{y} - y)\phi_1(X) + \sum_{i=2}^{4}(x_i - x)(y_i - y)\phi_i(X)\Bigg]$$

where $X_{\overline{DE}}$ is an arbitrary point on $\overline{DE}$.

Proof. Let $v(\hat{Y})$ be a piecewise bilinear function in term of $\hat{Y}$. First, $\nabla v(X) = \mathbf{q}$, $v(X) = r$, $\frac{\partial^2 v^-(X)}{\partial x \partial y} = d^-$ uniquely determine $v^-(\hat{Y})$. Then the interface conditions $\int_{\overline{DE}} \left(\beta^+ \frac{\partial v^+}{\partial \mathbf{n}_{\overline{DE}}} - \beta^- \frac{\partial v^-}{\partial \mathbf{n}_{\overline{DE}}}\right) ds = 0$ and $v^-(D) = v^+(D)$, $v^-(E) = v^+(E)$, $v^-(E) = v^+(E)$ uniquely determine $v^+(\hat{Y})$. These conditions also imply that $v(\hat{Y})$ is a function in the local bilinear IFE space $S_h(T)$. The proof is finished by replacing $\nabla v(X)$ by $\mathbf{q}$ in Lemma 3.5. ■

We can now derive an expansion of the bilinear IFE interpolation error.

Theorem 3.1. *For any $u \in PC^2_{\text{int}}(T)$ and $X = (x, y)^T \in T^-$, we have*

$$\begin{aligned}
I_{h,T}u(X) - u(X) &= \left(N^-(\widetilde{A}_1) - N^-_{\overline{DE}}\right)\nabla u(X) \cdot (A_1 - \widetilde{A}_1)\phi_1(X) - \left(N^-_{\overline{DE}} - I\right)\nabla u(X) \\
&\quad \times \cdot(\widetilde{A}_1 - X_{\overline{DE}})\phi_1(X) - \frac{\partial^2 u^-(X)}{\partial x \partial y}\Bigg[N^-_{\overline{DE}}\left(X^s_{\overline{DE}} - X^s\right) \cdot (A_1 - X_{\overline{DE}})\phi_1(X) \\
&\quad + (x_1 - \bar{x})(y_1 - \bar{y})\phi_1(X) + (\bar{x} - x)(\bar{y} - y)\phi_1(X) \\
&\quad + \sum_{i=2}^{4}(x_i - x)(y_i - y)\phi_i(X)\Bigg] \\
&\quad + (N^-(\widetilde{A}_1) - I)\int_0^1 \frac{d\nabla u^-(t\widetilde{A}_1 + (1-t)X)}{dt} \cdot (A_1 - \widetilde{A}_1)dt\phi_1(X) \\
&\quad + \int_0^{\tilde{t}} (1-t)\frac{d^2 u(tA_1 + (1-t)X)}{dt^2}dt\phi_1(X) \\
&\quad + \int_{\tilde{t}}^1 (1-t)\frac{d^2 u(tA_1 + (1-t)X)}{dt^2}dt\phi_1(X) \\
&\quad + \sum_{i=2}^{4}\int_0^1 (1-t)\frac{d^2 u(tA_i + (1-t)X)}{dt^2}dt\phi_i(X).
\end{aligned} \tag{3.26}$$

where $X_{\overline{DE}}$ is an arbitrary point on $\overline{DE}$.

Proof. Since $t \mapsto u(tA_i + (1-t)X))$, $i = 2, 3, 4$ are C^2 functions in terms of t, we have

$$\begin{aligned}
u(A_i) &= u(X) + \int_0^1 \frac{du(tA_i + (1-t)X)}{dt}dt \\
&= u(X) + \nabla u(X) \cdot (A_i - X) + \int_0^1 (1-t)\frac{d^2 u(tA_i + (1-t)X)}{dt^2}dt.
\end{aligned} \tag{3.27}$$

By using (3.20), we have

$$
\begin{aligned}
u(A_1) &= u(X) + \int_0^1 \frac{du(tA_1 + (1-t)X)}{\mathrm{d}t} dt \\
&= u(X) + \int_0^{\tilde{t}} \frac{du(tA_1 + (1-t)X)}{\mathrm{d}t} dt + \int_{\tilde{t}}^1 \frac{du(tA_1 + (1-t)X)}{\mathrm{d}t} dt \\
&= u(X) - \nabla u^-(\widetilde{A}_1) \cdot (A_1 - \widetilde{A}_1) + \nabla u(X) \cdot (A_1 - X) \\
&\quad + \int_0^{\tilde{t}} (1-t) \frac{d^2u(tA_1 + (1-t)X)}{dt^2} dt + \nabla u^+(\widetilde{A}_1) \cdot (A_1 - \widetilde{A}_1) \\
&\quad + \int_{\tilde{t}}^1 (1-t) \frac{d^2u(tA_1 + (1-t)X)}{dt^2} dt \\
&= u(X) + \nabla u(X) \cdot (A_1 - X) + (N^-(\widetilde{A}_1) - I) \nabla u^-(\widetilde{A}_1) \cdot (A_1 - \widetilde{A}_1) \\
&\quad + \int_0^{\tilde{t}} (1-t) \frac{d^2u(tA_1 + (1-t)X)}{dt^2} dt + \int_{\tilde{t}}^1 (1-t) \frac{d^2u(tA_1 + (1-t)X)}{dt^2} dt \\
&= u(X) + \nabla u(X) \cdot (A_1 - X) + (N^-(\widetilde{A}_1) - I) \nabla u(X) \cdot (A_1 - \widetilde{A}_1) \\
&\quad + (N^-(\widetilde{A}_1) - I) \int_0^1 \frac{d\nabla u^-(t\widetilde{A}_1 + (1-t)X)}{dt} \cdot (A_1 - \widetilde{A}_1) dt \\
&\quad + \int_0^{\tilde{t}} (1-t) \frac{d^2u(tA_1 + (1-t)X)}{dt^2} dt + \int_{\tilde{t}}^1 (1-t) \frac{d^2u(tA_1 + (1-t)X)}{dt^2} dt.
\end{aligned} \tag{3.28}
$$

Then

$$
\begin{aligned}
I_{h,T}u(X) &= \sum_{i=1}^4 u(A_i)\phi_i(X) = u(X) \sum_{i=1}^4 \phi_i(X) + \nabla u(X) \cdot \sum_{i=1}^4 (A_i - X)\phi_i(X) \\
&\quad + (N^-(\widetilde{A}_1) - I) \nabla u(X) \cdot (A_1 - \widetilde{A}_1)\phi_1(X) \\
&\quad + (N^-(\widetilde{A}_1) - I) \int_0^1 \frac{d\nabla u^-(t\widetilde{A}_1 + (1-t)X)}{dt} \cdot (A_1 - \widetilde{A}_1) dt \phi_1(X) \\
&\quad + \int_0^{\tilde{t}} (1-t) \frac{d^2u(tA_1 + (1-t)X)}{dt^2} dt \phi_1(X) \\
&\quad + \int_{\tilde{t}}^1 (1-t) \frac{d^2u(tA_1 + (1-t)X)}{dt^2} dt \phi_1(X) \\
&\quad + \sum_{i=2}^4 \int_0^1 (1-t) \frac{d^2u(tA_i + (1-t)X)}{dt^2} dt \phi_i(X).
\end{aligned} \tag{3.29}
$$

Now letting $\mathbf{q} = \nabla u(X), r = u(X), d^- = \dfrac{\partial^2 u(X)}{\partial x \partial y}$ in Lemma 3.6, we have

$$\nabla u(X) \cdot \sum_{i=1}^{4} (A_i - X)\phi_i(X) = -\big(N^-_{\overline{DE}} - I\big)\nabla u(X) \cdot (A_1 - \widetilde{A}_1)\phi_1(X)$$
$$- \big(N^-_{\overline{DE}} - I\big)\nabla u(X) \cdot (\widetilde{A}_1 - X_{\overline{DE}})\phi_1(X) - \frac{\partial^2 u(X)}{\partial x \partial y}\Bigg[N^-_{\overline{DE}}\big(X^s_{\overline{DE}} - X^s\big) \cdot (A_1 - X_{\overline{DE}})\phi_1(X)$$
$$+ (x_1 - \bar{x})(y_1 - \bar{y})\phi_1(X) + (\bar{x} - x)(\bar{y} - y)\phi_1(X) + \sum_{i=2}^{4}(x_i - x)(y_i - y)\phi_i(X)\Bigg]. \quad (3.30)$$

Finally, (3.26) follows from (3.29), (3.30) and Theorem 2.1. ■

The following theorem establish a bound in L^2 norm for the bilinear IFE interpolation error.

Theorem 3.2. *There exits a constant C such that*

$$\|I_{h,T}u - u\|_{0,T^-} \leq Ch^2\|u\|_{2,T} \quad (3.31)$$

for any $u \in PH^2_{\text{int}}(T)$ where $T \in \mathcal{T}_h$ is an interface element.

Proof. Let $Q_i, i = 1, 2, \cdots, 9$ be the 9 terms on the right hand side of (3.26), and we proceed by estimating their L^2 norms. By Lemma 3.2, Theorem 2.2, and by letting $X_{\overline{DE}} = \widetilde{A}_1^{\perp}$ in (3.26), we have the following estimate for the L^2 norms of the first 3 terms:

$$\|Q_1\|_{0,T^-} + \|Q_2\|_{0,T^-} + \|Q_3\|_{0,T^-} = \big\|\big(N^-(\widetilde{A}_1) - N^-_{\overline{DE}}\big)\nabla u(X) \cdot (A_1 - \widetilde{A}_1)\phi_1(X)\big\|_{0,T^-}$$
$$+ \big\|\big(N^-_{\overline{DE}} - I\big)\nabla u^-(X) \cdot \big(\widetilde{A}_1 - \widetilde{A}_1^{\perp}\big)\phi_1(X)\big\|_{0,T^-}$$
$$+ \Bigg\|\frac{\partial^2 u^-(X)}{\partial x \partial y}\Bigg[N^-_{\overline{DE}}\big(X^s_{\overline{DE}} - X^s\big) \cdot (A_1 - X_{\overline{DE}})\phi_1(X)$$
$$+ (x_1 - \bar{x})(y_1 - \bar{y})\phi_1(X) + (\bar{x} - x)(\bar{y} - y)\phi_1(X)$$
$$+ \sum_{i=2}^{4}(x_i - x)(y_i - y)\phi_i(X)\Bigg]\Bigg\|_{0,T^-}$$
$$\leq Ch^2\|u\|_{1,T^-} + Ch^2\|u\|_{2,T^-} \leq Ch^2\|u\|_{2,T^-}.$$

For the fourth term, we first note that

$$\frac{d\nabla u^-}{dt}(t\widetilde{A}_1 + (1-t)X) \cdot (A_1 - \widetilde{A}_1) = u_{\xi\xi}(\xi, \eta)(\widetilde{x}_1 - x)(x_1 - \widetilde{x}_1) + 2u_{\xi\eta}(\xi, \eta)[\widetilde{y}_1 - y)(x_1 - \widetilde{x}_1)$$
$$+ (\widetilde{x}_1 - x)(y_1 - \widetilde{y}_1)] + u_{\eta\eta}(\xi, \eta)(\widetilde{y}_1 - y)(y_1 - \widetilde{y}_1)$$

with $\xi = t\widetilde{x}_1 + (1-t)x, \eta = t\widetilde{y}_1 + (1-t)y$. Then,

$$Q_4^2 \le C\left(\int_0^1 [u_{\xi\xi}(\xi,\eta)(\widetilde{x}_1 - x)(x_1 - \widetilde{x}_1) + 2u_{\xi\eta}(\xi,\eta)((\widetilde{y}_1 - y)(x_1 - \widetilde{x}_1)\right.$$

$$\left. + (\widetilde{x}_1 - x)(y_1 - \widetilde{y}_1)) + u_{\eta\eta}(\xi,\eta)(\widetilde{y}_1 - y)(y_1 - \widetilde{y}_1)]dt\right)^2$$

$$\le Ch^4\left(\int_0^1 [u_{\xi\xi}(\xi,\eta) + 2u_{\xi\eta}(\xi,\eta) + u_{\eta\eta}(\xi,\eta)]dt\right)^2$$

$$\le Ch^4 \int_0^1 \left[u_{\xi\xi}^2(\xi,\eta) + u_{\xi\eta}^2(\xi,\eta) + u_{\eta\eta}^2(\xi,\eta)\right]dt,$$

here C stands for a generic constant. Since $X\widetilde{A}_1$ must be in T^-, then $(\xi,\eta) \in T^-$. Therefore,

$$\|Q_4\|_{0,T^-}^2 = \int_{T^-} Q_4^2 d\xi d\eta \le Ch^4 \int_{T^-}\int_0^1 \left[u_{\xi\xi}^2(\xi,\eta) + u_{\xi\eta}^2(\xi,\eta) + u_{\eta\eta}^2(\xi,\eta)\right]dtd\xi d\eta$$

$$\le Ch^4 \int_{T^-} \left[u_{\xi\xi}^2(\xi,\eta) + u_{\xi\eta}^2(\xi,\eta) + u_{\eta\eta}^2(\xi,\eta)\right]d\xi d\eta \le Ch^4\|u\|_{2,T^-}^2,$$

or

$$\|Q_4\|_{0,T^-} \le Ch^2\|u\|_{2,T^-} \le Ch^2\|u\|_{2,T}.$$

For the fifth term, we have

$$Q_5^2 \le C\left(\int_0^{\widetilde{t}} (1-t)[u_{\xi\xi}(\xi,\eta)(x_1 - x)^2 + 2u_{\xi\eta}(\xi,\eta)(x_1 - x)(y_1 - y) + u_{\eta\eta}(\xi,\eta)(y_1 - y)^2]dt\right)^2$$

$$\le Ch^4\left(\int_0^{\widetilde{t}} (1-t)[u_{\xi\xi}(\xi,\eta) + 2u_{\xi\eta}(\xi,\eta) + u_{\eta\eta}(\xi,\eta)]dt\right)^2$$

$$\le Ch^4 \int_0^{\widetilde{t}} (1-t)^2\left[u_{\xi\xi}^2(\xi,\eta) + u_{\xi\eta}^2(\xi,\eta) + u_{\eta\eta}^2(\xi,\eta)\right]dt$$

with $\xi = tx_1 + (1-t)x, \eta = ty_1 + (1-t)y$. Therefore

$$\|Q_5\|_{T^-}^2 \le Ch^4 \int_{T^-}\int_0^{\widetilde{t}} (1-t)^2\left[u_{\xi\xi}^2(\xi,\eta) + u_{\xi\eta}^2(\xi,\eta) + u_{\eta\eta}^2(\xi,\eta)\right]dtd\xi d\eta$$

$$\le Ch^4 \int_{T^-} \left[u_{\xi\xi}^2(\xi,\eta) + u_{\xi\eta}^2(\xi,\eta) + u_{\eta\eta}^2(\xi,\eta)\right]d\xi d\eta \le Ch^4\|u\|_{2,T^-}$$

or

$$\|Q_5\|_{T^-} \le Ch^2\|u\|_{2,T^-} \le Ch^2\|u\|_{2,T}.$$

Similarly, we can show that $\|Q_i\|_{T^-} \le Ch^2\|u\|_{2,T}, i = 6,7,8,9$. Finally, (3.31) follows from the estimates for $Q_i, i = 1,2,\cdots,9$ above. ■

We now turn to the estimate of bilinear IFE interpolation error in H^1 norm on the subelement T^-. In the discussion later, we let I_i, $i = 1, 2, 3, 4$ be the integral terms involving the vertices A_i, $i = 1, 2, 3, 4$ in (3.26).

Theorem 3.3. *For any $u \in PC^3_{\text{int}}(T)$ and $X = (x, y)^T \in T^-$, we have*

$$\begin{aligned}\frac{\partial(I_{h,T}u(X) - u(X))}{\partial x} &= \big(N^-(\widetilde{A}_1) - N^-_{\overline{DE}}\big)\nabla u(X)\cdot(A_1 - \widetilde{A}_1)\frac{\partial\phi_1(X)}{\partial x} \\ &- \big(N^-_{\overline{DE}} - I\big)\nabla u(X)\cdot(\widetilde{A}_1 - X_{\overline{DE}})\frac{\phi_1(X)}{\partial x} \\ &- \frac{\partial^2 u^-(X)}{\partial x\partial y}\Bigg[N^-_{\overline{DE}}\big(X^s_{\overline{DE}} - X^s\big)\cdot(A_1 - X_{\overline{DE}})\frac{\phi_1(X)}{\partial x} \\ &+ N^-_{\overline{DE}}(0, -1)^T\cdot(A_1 - X_{\overline{DE}})\phi_1(X) + (x_1 - \bar{x})(y_1 - \bar{y})\frac{\phi_1(X)}{\partial x} \\ &- (\bar{y} - y)\phi_1(X) + (\bar{x} - x)(\bar{y} - y)\frac{\phi_1(X)}{\partial x} \\ &+ \sum_{i=2}^{4}\Bigg[-(y_i - y)\phi_i(X) + (x_i - x)(y_i - y)\frac{\phi_i(X)}{\partial x}\Bigg]\Bigg] + \sum_{i=1}^{4} I_i\frac{\phi_i(X)}{\partial x}.\end{aligned} \tag{3.32}$$

$$\begin{aligned}\frac{\partial(I_{h,T}u(X) - u(X))}{\partial y} &= \big(N^-(\widetilde{A}_1) - N^-_{\overline{DE}}\big)\nabla u(X)\cdot(A_1 - \widetilde{A}_1)\frac{\partial\phi_1(X)}{\partial y} \\ &- \big(N^-_{\overline{DE}} - I\big)\nabla u(X)\cdot(\widetilde{A}_1 - X_{\overline{DE}})\frac{\phi_1(X)}{\partial y} \\ &- \frac{\partial^2 u^-(X)}{\partial x\partial y}\Bigg[N^-_{\overline{DE}}\big(X^s_{\overline{DE}} - X^s\big)\cdot(A_1 - X_{\overline{DE}})\frac{\phi_1(X)}{\partial y} \\ &+ N^-_{\overline{DE}}(-1, 0)^T\cdot(A_1 - X_{\overline{DE}})\phi_1(X) + (x_1 - \bar{x})(y_1 - \bar{y})\frac{\phi_1(X)}{\partial y} \\ &- (\bar{x} - x)\phi_1(X) + (\bar{x} - x)(\bar{y} - y)\frac{\phi_1(X)}{\partial y} \\ &+ \sum_{i=2}^{4}\Bigg[-(x_i - x)\phi_i(X) + (x_i - x)(y_i - y)\frac{\phi_i(X)}{\partial y}\Bigg]\Bigg] + \sum_{i=1}^{4} I_i\frac{\phi_i(X)}{\partial y}.\end{aligned} \tag{3.33}$$

where $X_{\overline{DE}}$ is an arbitrary point on $\overline{DE}$.

Proof. We give a proof only for (3.32), similar arguments can be used to show (3.33). From (3.26), we can get

$$\begin{aligned}\frac{\partial(I_{h,T}u(X) - u(X))}{\partial x} &= \frac{\partial}{\partial x}\big[\big(N^-(\widetilde{A}_1) - N^-_{\overline{DE}}\big)\nabla u(X)\cdot(A_1 - \widetilde{A}_1)\big]\phi_1(X) \\ &+ \big(N^-(\widetilde{A}_1) - N^-_{\overline{DE}}\big)\nabla u(X)\cdot(A_1 - \widetilde{A}_1)\frac{\partial\phi_1(X)}{\partial x}\end{aligned}$$

$$
\begin{aligned}
&- \frac{\partial}{\partial x}\left[\left(N^-_{\overline{DE}} - I\right)\nabla u(X) \cdot (\widetilde{A}_1 - X_{\overline{DE}})\right]\phi_1(X) \\
&- \left(N^-_{\overline{DE}} - I\right)\nabla u(X) \cdot (\widetilde{A}_1 - X_{\overline{DE}})\frac{\phi_1(X)}{\partial x} \\
&- \frac{\partial^3 u^-(X)}{\partial x^2 \partial y}\left[N^-_{\overline{DE}}\left(X^s_{\overline{DE}} - X^s\right) \cdot (A_1 - X_{\overline{DE}})\phi_1(X)\right. \\
&+ (x_1 - \bar{x})(y_1 - \bar{y})\phi_1(X) + (\bar{x} - x)(\bar{y} - y)\phi_1(X) \\
&\left.+ \sum_{i=2}^{4}(x_i - x)(y_i - y)\phi_i(X)\right] \\
&- \frac{\partial^2 u^-(X)}{\partial x \partial y}\left[N^-_{\overline{DE}}\left(X^s_{\overline{DE}} - X^s\right) \cdot (A_1 - X_{\overline{DE}})\frac{\phi_1(X)}{\partial x}\right. \\
&+ N^-_{\overline{DE}}(0, -1)^T \cdot (A_1 - X_{\overline{DE}})\phi_1(X) + (x_1 - \bar{x})(y_1 - \bar{y})\frac{\phi_1(X)}{\partial x} \\
&- (\bar{y} - y)\phi_1(X) + (\bar{x} - x)(\bar{y} - y)\frac{\phi_1(X)}{\partial x} \\
&\left.+ \sum_{i=2}^{4}\left[-(y_i - y)\phi_i(X) + (x_i - x)(y_i - y)\frac{\phi_i(X)}{\partial x}\right]\right] \\
&+ \sum_{i=1}^{4} I_i \frac{\phi_i(X)}{\partial x} + \sum_{i=1}^{4}\frac{\partial I_i}{\partial x}\phi_i(X). \qquad (3.34)
\end{aligned}
$$

Taking the first derivative with respect to x on both sides of (3.27) and (3.28), we can get

$$
\begin{aligned}
\frac{\partial I_i}{\partial x} &= -P \cdot (A_i - X), i = 2, 3, 4 \text{ with } P = \frac{\partial}{\partial x}\nabla u(X), \\
\frac{\partial I_1}{\partial x} &= -P \cdot (A_1 - X) - \frac{\partial}{\partial x}[(N^-(\widetilde{A}_1) - I)\nabla u(X)(A_1 - \widetilde{A}_1)].
\end{aligned}
$$

Hence,

$$
\sum_{i=1}^{4}\frac{\partial I_i}{\partial x}\phi_i(X) = -P \cdot \sum_{i=1}^{4}(A_i - X)\phi_i(X) - \frac{\partial}{\partial x}[(N^-(\widetilde{A}_1) - I)\nabla u(X)(A_1 - \widetilde{A}_1)]\phi_1(X).
$$

Applying Lemma 3.6 to the first term on the right hand side above, letting $\mathbf{q} = P, d^- = \dfrac{\partial^3 u^-(X)}{\partial x^2 \partial y}$, we have

$$
\begin{aligned}
\sum_{i=1}^{4}\frac{\partial I_i}{\partial x}\phi_i(X) = &-\frac{\partial}{\partial x}[(N^-(\widetilde{A}_1) - I)\nabla u(X)(A_1 - \widetilde{A}_1)]\phi_1(X) + \left(N^-_{\overline{DE}} - I\right)P \cdot (A_1 - \widetilde{A}_1)\phi_1(X) \\
&+ \left(N^-_{\overline{DE}} - I\right)P \cdot (\widetilde{A}_1 - X_{\overline{DE}})\phi_1(X) + \frac{\partial^3 u^-(X)}{\partial x^2 \partial y}
\end{aligned}
$$

$$\times\left[N^-_{\overline{DE}}\left(X^s_{\overline{DE}}-X^s\right)\cdot(A_1-X_{\overline{DE}})\phi_1(X)+(\bar{x}-x)(\bar{y}-y)\phi_1(X)\right.$$
$$\left.+(\bar{x}-x)(\bar{y}-y)\phi_1(X)+\sum_{i=2}^{4}(x_i-x)(y_i-y)\phi_i(X)\right]. \tag{3.35}$$

By direct calculations, we also have

$$\frac{\partial}{\partial x}\left(N^-(\widetilde{A}_1)-N^-_{\overline{DE}}\right)\nabla u(X)\cdot(A_1-\widetilde{A}_1)\phi_1(X)-\frac{\partial}{\partial x}\left[\left(N^-_{\overline{DE}}-I\right)\nabla u(X)\cdot(\widetilde{A}_1-X_{\overline{DE}})\right]\phi_1(X)$$
$$-\frac{\partial}{\partial x}[(N^-(\widetilde{A}_1)-I)\nabla u(X)(A_1-\widetilde{A}_1)]\phi_1(X)+\left(N^-_{\overline{DE}}-I\right)P\cdot(A_1-\widetilde{A}_1)\phi_1(X)$$
$$+\left(N^-_{\overline{DE}}-I\right)P\cdot(\widetilde{A}_1-X_{\overline{DE}})\phi_1(X)=0. \tag{3.36}$$

Putting (3.35) and (3.36) into (3.34), we finish the proof of (3.32). ∎

Theorem 3.4. *There exits a constant C such that*

$$\left\|\frac{\partial(I_{h,T}u-u)}{\partial v}\right\|_{0,T^-}\le Ch\|u\|_{2,T},\quad v=x,y, \tag{3.37}$$

for any $u\in PH^2_{\text{int}}(T)$ where T is a Type I interface element.

Proof. The result follows by letting $X_{\overline{DE}}=\widetilde{A}_1^{\perp}$ in (3.32) and (3.33), applying Theorem 2.2, and applying arguments similar to those used in the proof of Theorem 3.2. ∎

The estimation on T^+ is rather similar. We state the results in the following Lemmas and Theorems without proof in order to reduce page usage. Let $X=(x,y)^T$ be a point in T^+. Without loss of generality, we can assume that line segments $\overline{XA_1}$ does not intersect with the interface and $\overline{DE}$, while line segment $\overline{XA_i}, i=2,3,4$ meet Γ at $\widetilde{A}_i$, $i=2,3,4$, see Fig. 6. Also, we assume that $A_i=(x_i,y_i)^T$, $i=1,2,3,4$ and

$$\widetilde{A}_i=\tilde{t}_iA_i+(1-\tilde{t}_i)X=(\widetilde{x}_i,\widetilde{y}_i)^T, i=2,3,4.$$

Lemma 3.7. *Given a two-dimensional vector* $\mathbf{q}$*, a point $X\in T^+$ and two real numbers r,d^+, then there exists a $v\in S_h(T)$ such that $\nabla v(X)=\mathbf{q}$, $v(X)=r$, $\dfrac{\partial^2 v^+(X)}{\partial x\partial y}=d^+$ and*

$$\mathbf{q}\cdot\sum_{i=1}^{4}[(A_i-X)\phi_i(X)]=\sum_{i=2}^{4}\Big[-\left(N^+_{\overline{DE}}-I\right)\mathbf{q}\cdot(A_i-\widetilde{A}_i)\phi_1(X)$$
$$-\left(N^+_{\overline{DE}}-I\right)\mathbf{q}\cdot\left(\widetilde{A}_i-X^{(i)}_{\overline{DE}}\right)\phi_1(X)\Big]$$

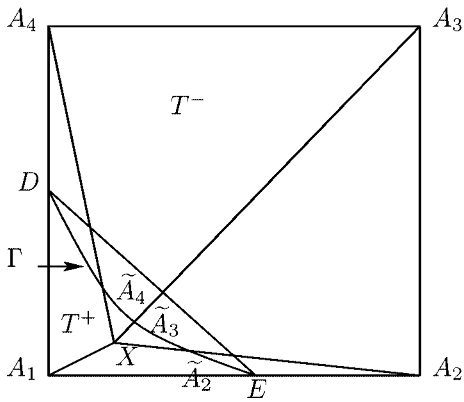

FIG. 6. A point X $\in T^+$ is connected to the four vertices by line segments in a Type I interface element.

$$
\begin{aligned}
&- d^+ \Bigg[\sum_{i=2}^{4} \big[N^+_{\overline{DE}} \big(X^{(i)s}_{\overline{DE}} - X^s \big) \cdot \big(A_i - X^{(i)}_{\overline{DE}} \big) \phi_i(X) \\
&+ (x_i - \bar{x}_i)(y_i - \bar{y}_i)\phi_i(X) + (\bar{x}_i - x)(\bar{y}_i - y)\phi_1(X) \big] \\
&+ (x_1 - x)(y_1 - y)\phi_1(X) \Bigg]
\end{aligned}
$$

where $X^{(i)}_{\overline{DE}} = (\bar{x}_i, \bar{y}_i)^T$, $i = 2, 3, 4$ *are arbitrary points on* $\overline{DE}$.

Theorem 3.5. *For any* $u \in PC^2_{\text{int}}(T)$, $X = (x, y)^T \in T^+$, *we have*

$$
\begin{aligned}
I_{h,T}u(X) - u(X) = &\sum_{i=2}^{4} \big[\big(N^+(\widetilde{A}_i) - N^+_{\overline{DE}} \big) \nabla u(X) \cdot (A_i - \widetilde{A}_i)\phi_i(X) \\
&- \big(N^+_{\overline{DE}} - I \big) \nabla u(X) \cdot \big(\widetilde{A}_i - X^{(i)}_{\overline{DE}} \big) \phi_i(X) \big] - \frac{\partial^2 u^+(X)}{\partial x \partial y} \\
&\times \Bigg[\sum_{i=2}^{4} \big[N^+_{\overline{DE}} \big(X^{(i)s}_{\overline{DE}} - X^s \big) \cdot \big(A_i - X^{(i)}_{\overline{DE}} \big) \phi_i(X) + (x_i - \bar{x}_i)(y_i - \bar{y}_i)\phi_i(X) \\
&+ (\bar{x}_i - x)(\bar{y}_i - y)\phi_i(X) \big] + (x_1 - x)(y_1 - y)\phi_1(X) \big] \\
&+ \sum_{i=2}^{4} \Bigg[(N^+(\widetilde{A}_i) - I) \int_0^1 \frac{d\nabla u^+(t\widetilde{A}_i + (1-t)X)}{dt} \cdot (A_i - \widetilde{A}_i) dt \phi_i(X) \\
&+ \int_0^{\tilde{t}_i} (1-t) \frac{d^2 u(tA_i + (1-t)X)}{dt^2} dt \phi_i(X)
\end{aligned}
$$

$$+ \int_{\tilde{t}_i}^{1} (1-t)\frac{d^2 u(tA_i + (1-t)X)}{dt^2} dt\phi_i(X)\Bigg]$$
$$+ \int_{0}^{1} (1-t)\frac{d^2 u(tA_1 + (1-t)X)}{dt^2} dt\phi_1(X). \tag{3.38}$$

where $X_{\overline{DE}}^{(i)} = (\bar{x}_i, \bar{y}_i)^T$, $i = 2,3,4$ *are arbitrary points on* $\overline{DE}$.

We now let I_i, $i = 1,2,3,4$ be the integral involving the vertices A_i, $i = 1,2,3,4$ in (3.38).

Theorem 3.6. *For any* $u \in PC^3_{\text{int}}(T)$, $X = (x,y)^T \in T^+$, *we have*

$$\begin{aligned}
\frac{\partial(I_{h,T}u(X) - u(X))}{\partial x} &= \sum_{i=2}^{4}\Bigg[\big(N^+(\widetilde{A}_i) - N^+_{\overline{DE}}\big)\nabla u(X)\cdot(A_i - \widetilde{A}_i)\frac{\partial \phi_i(X)}{\partial x} \\
&\quad - \big(N^+_{\overline{DE}} - I\big)\nabla u(X)\cdot\big(\widetilde{A}_i - X^{(i)}_{\overline{DE}}\big)\frac{\phi_i(X)}{\partial x}\Bigg] \\
&\quad - \frac{\partial^2 u^+(X)}{\partial x \partial y}\Bigg[\sum_{i=2}^{4}\Bigg[N^+_{\overline{DE}}\big(X^{(i)s}_{\overline{DE}} - X^s\big)\cdot\big(A_i - X^{(i)}_{\overline{DE}}\big)\frac{\phi_i(X)}{\partial x} \\
&\quad + N^+_{\overline{DE}}(0,-1)^T\cdot\big(A_i - X^{(i)}_{\overline{DE}}\big)\phi_i(X) + (x_i - \bar{x}_i)(y_i - \bar{y}_i)\frac{\phi_i(X)}{\partial x} \\
&\quad - (\bar{y}_i - y)\phi_i(X) + (\bar{x}_i - x)(\bar{y}_i - y)\frac{\phi_i(X)}{\partial x}\Bigg] - (y_1 - y)\phi_1(X) \\
&\quad + (x_1 - x)(y_1 - y)\frac{\phi_1(X)}{\partial x}\Bigg] + \sum_{i=1}^{4} I_i\frac{\phi_i(X)}{\partial x}.
\end{aligned} \tag{3.39}$$

$$\begin{aligned}
\frac{\partial(I_{h,T}u(X) - u(X))}{\partial y} &= \sum_{i=2}^{4}\Bigg[\big(N^+(\widetilde{A}_i) - N^+_{\overline{DE}}\big)\nabla u(X)\cdot(A_i - \widetilde{A}_i)\frac{\partial \phi_i(X)}{\partial y} \\
&\quad - \big(N^+_{\overline{DE}} - I\big)\nabla u(X)\cdot\big(\widetilde{A}_i - X^{(i)}_{\overline{DE}}\big)\frac{\phi_i(X)}{\partial y}\Bigg] \\
&\quad - \frac{\partial^2 u^+(X)}{\partial x \partial y}\Bigg[\sum_{i=2}^{4}\Bigg[N^+_{\overline{DE}}\big(X^{(i)s}_{\overline{DE}} - X^s\big)\cdot\big(A_i - X^{(i)}_{\overline{DE}}\big)\frac{\phi_i(X)}{\partial y} \\
&\quad + N^+_{\overline{DE}}(-1,0)^T\cdot\big(A_i - X^{(i)}_{\overline{DE}}\big)\phi_i(X) + (x_i - \bar{x}_i)(y_i - \bar{y}_i)\frac{\phi_i(X)}{\partial y} \\
&\quad - (\bar{x}_i - x)\phi_i(X) + (\bar{x}_i - x)(\bar{y}_i - y)\frac{\phi_i(X)}{\partial y}\Bigg] - (x_1 - x)\phi_1(X) \\
&\quad + (x_1 - x)(y_1 - y)\frac{\phi_1(X)}{\partial y}\Bigg] + \sum_{i=1}^{4} I_i\frac{\phi_i(X)}{\partial y}.
\end{aligned} \tag{3.40}$$

where $X_{\overline{DE}}^{(i)} = (\bar{x}_i, \bar{y}_i)^T$, $i = 2,3,4$ *are arbitrary points on* $\overline{DE}$.

Theorem 3.7. *There exits a constant C such that*

$$\|I_{h,T}u - u\|_{0,T^+} \leq Ch^2\|u\|_{2,T} \tag{3.41}$$

and

$$\left\|\frac{\partial(I_{h,T}u - u)}{\partial v}\right\|_{0,T^+} \leq Ch\|u\|_{2,T}, \quad v = x, y \tag{3.42}$$

for any $u \in PH^2_{\text{int}}(T)$, where T is a Type I interface element.

B. Interpolation Error on a Type II Interface Element

The estimate for Type II interface elements is similar to that for Type I interface elements, so we only state the results in this section. Without loss of generality, we assume $T \in \mathcal{T}_h$ is a Type II interface element with vertices $A_i = (x_i, y_i)$, $i = 1, 2, 3, 4$, such that $A_1, A_2 \in T^+$ and $A_3, A_4 \in T^-$, see Fig. 7.

We start with the estimation on T^-. Let $X = (x, y)^T$ be a point in T^-. Without loss of generality, we can assume that line segments $\overline{XA_i}, i = 3, 4$ do not intersect with the interface Γ and $\overline{DE}$, while line segment $\overline{XA_i}, i = 1, 2$ meet Γ at $\widetilde{A}_i, i = 1, 2$, see Fig. 7. Also, we assume that

$$\widetilde{A}_i = \tilde{t}_i A_i + (1 - \tilde{t}_i)X = (\widetilde{x}_i, \widetilde{y}_i)^T, \quad i = 1, 2$$

Lemma 3.8. *For any $v \in S_h(T)$,*

$$\nabla v(X) \cdot \sum_{i=1}^{4}(A_i - X)\phi_i(X) = \sum_{i=1}^{2}\Big[-\big(N^-_{\overline{DE}} - I\big)\nabla v(X) \cdot (A_i - \widetilde{A}_i)\phi_i(X) - \big(N^-_{\overline{DE}} - I\big)\nabla v(X) \cdot \big(\widetilde{A}_i - X^{(i)}_{\overline{DE}}\big)\phi_i(X)\Big]$$

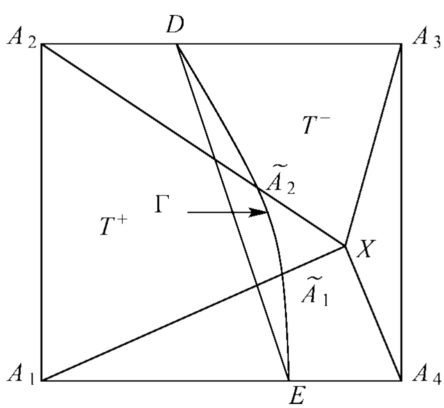

FIG. 7. A point X $\in T^-$ is connected to the four vertices by line segments in a Type II interface element.

$$
\begin{aligned}
&- d^- \Bigg[\sum_{i=1}^{2} \big[N_{\overline{DE}}^- \big(X_{\overline{DE}}^{(i)s} - X^s \big) \cdot \big(A_i - X_{\overline{DE}}^{(i)} \big) \phi_i(X) \\
&+ (x_i - \bar{x}_i)(y_i - \bar{y}_i)\phi_i(X) + (\bar{x}_i - x)(\bar{y}_i - y)\phi_i(X) \big] \\
&+ \sum_{i=3}^{4} [(x_i - x)(y_i - y)\phi_i(X)] \Bigg]
\end{aligned}
$$

where $X_{\overline{DE}}^{(i)} = (\bar{x}_i, \bar{y}_i)^T$, $i = 1, 2$ *are arbitrary points on* $\overline{DE}$.

Lemma 3.9. *Given a two-dimensional vector* $\mathbf{q}$, *a point* $X \in T^-$ *and two real numbers* r, d^-, *then there exists a function* $v \in S_h(T)$ *such that* $\nabla v(X) = \mathbf{q}$, $v(X) = r$, $\frac{\partial^2 v^-(X)}{\partial x \partial y} = d^-$ *and*

$$
\begin{aligned}
\mathbf{q}(X) \cdot \sum_{i=1}^{4} (A_i - X)\phi_i(X) = \sum_{i=1}^{2} \big[&- \big(N_{\overline{DE}}^- - I \big) \mathbf{q} \cdot (A_i - \widetilde{A}_i)\phi_i(X) \\
&- \big(N_{\overline{DE}}^- - I \big) \mathbf{q} \cdot \big(\widetilde{A}_i - X_{\overline{DE}}^{(i)} \big) \phi_i(X) \big] \\
&- d^- \Bigg[\sum_{i=1}^{2} \big[N_{\overline{DE}}^- \big(X_{\overline{DE}}^{(i)s} - X^s \big) \cdot \big(A_i - X_{\overline{DE}}^{(i)} \big) \phi_i(X) \\
&+ (x_i - \bar{x}_i)(y_i - \bar{y}_i)\phi_i(X) + (\bar{x}_i - x)(\bar{y}_i - y)\phi_i(X) \big] \\
&+ \sum_{i=3}^{4} [(x_i - x)(y_i - y)\phi_i(X)] \Bigg]
\end{aligned}
$$

where $X_{\overline{DE}}^{(i)} = (\bar{x}_i, \bar{y}_i)^T$, $i = 1, 2$ *are arbitrary points on* $\overline{DE}$.

Theorem 3.8. *For any* $u \in PC_{\text{int}}^2(T)$ *and* $X = (x, y)^T \in T^-$, *we have*

$$
\begin{aligned}
I_{h,T}u(X) - u(X) = \sum_{i=1}^{2} \big[&\big(N^-(\widetilde{A}_i) - N_{\overline{DE}}^- \big) \nabla u(X) \cdot (A_i - \widetilde{A}_i)\phi_i(X) \\
&- \big(N_{\overline{DE}}^- - I \big) \nabla u(X) \cdot \big(\widetilde{A}_i - X_{\overline{DE}}^{(i)} \big) \phi_i(X) \big] \\
&- \frac{\partial^2 u^-(X)}{\partial x \partial y} \Bigg[\sum_{i=1}^{2} \big[N_{\overline{DE}}^- \big(X_{\overline{DE}}^{(i)s} - X^s \big) \cdot \big(A_i - X_{\overline{DE}}^{(i)} \big) \phi_i(X) \\
&+ (x_i - \bar{x}_i)(y_i - \bar{y}_i)\phi_i(X) + (\bar{x}_i - x)(\bar{y}_i - y)\phi_i(X) \big] \\
&+ \sum_{i=3}^{4} (x_i - x)(y_i - y)\phi_i(X) \Bigg] \\
&+ \sum_{i=1}^{2} \Bigg[(N^-(\widetilde{A}_i) - I) \int_0^1 \frac{d\nabla u^-(t\widetilde{A}_i + (1-t)X)}{dt} \cdot (A_i - \widetilde{A}_i) dt \phi_i(X)
\end{aligned}
$$

$$+\int_0^{\tilde{t}_i}(1-t)\frac{d^2u(tA_i+(1-t)X)}{dt^2}dt\phi_i(X)$$
$$+\int_{\tilde{t}_i}^1(1-t)\frac{d^2u(tA_i+(1-t)X)}{dt^2}dt\phi_i(X)\Bigg]$$
$$+\sum_{i=3}^4\int_0^1(1-t)\frac{d^2u(tA_i+(1-t)X)}{dt^2}dt\phi_i(X) \tag{3.43}$$

where $X_{\overline{DE}}^{(i)}=(\bar{x}_i,\bar{y}_i)^T$, $i=1,2$ *are arbitrary points on* $\overline{DE}$.

Let $I_i, i=1,2,3,4$ be the integral terms involving vertices $A_i, i=1,2,3,4$ in (3.43).

Theorem 3.9. *For any* $u\in PC_{\text{int}}^3(T)$ *and* $X=(x,y)^T\in T^-$, *we have*

$$\frac{\partial(I_{h,T}u(X)-u(X))}{\partial x}=\sum_{i=1}^2\Bigg[\Big(N^-(\widetilde{A}_i)-N_{\overline{DE}}^-\Big)\nabla u(X)\cdot(A_i-\widetilde{A}_i)\frac{\partial\phi_i(X)}{\partial x}$$
$$-\Big(N_{\overline{DE}}^--I\Big)\nabla u(X)\cdot\Big(\widetilde{A}_i-X_{\overline{DE}}^{(i)}\Big)\frac{\phi_i(X)}{\partial x}\Bigg]$$
$$-\frac{\partial^2u^-(X)}{\partial x\partial y}\Bigg[\sum_{i=1}^2\Bigg[N_{\overline{DE}}^-\Big(X_{\overline{DE}}^{(i)s}-X^s\Big)\cdot\Big(A_i-X_{\overline{DE}}^{(i)}\Big)\frac{\phi_i(X)}{\partial x}$$
$$+N_{\overline{DE}}^-(0,-1)^T\cdot\Big(A_i-X_{\overline{DE}}^{(i)}\Big)\phi_i(X)+(x_i-\bar{x}_i)(y_i-\bar{y}_i)\frac{\phi_i(X)}{\partial x}$$
$$-(\bar{y}_i-y)\phi_i(X)+(\bar{x}_i-x)(\bar{y}_i-y)\frac{\phi_i(X)}{\partial x}\Bigg]$$
$$+\sum_{i=3}^4\Bigg[-(y_i-y)\phi_i(X)+(x_i-x)(y_i-y)\frac{\phi_i(X)}{\partial x}\Bigg]\Bigg]+\sum_{i=1}^4I_i\frac{\phi_i(X)}{\partial x} \tag{3.44}$$

$$\frac{\partial(I_{h,T}u(X)-u(X))}{\partial y}=\sum_{i=1}^2\Bigg[\Big(N^-(\widetilde{A}_i)-N_{\overline{DE}}^-\Big)\nabla u(X)\cdot(A_i-\widetilde{A}_i)\frac{\partial\phi_i(X)}{\partial y}$$
$$-\Big(N_{\overline{DE}}^--I\Big)\nabla u(X)\cdot\Big(\widetilde{A}_i-X_{\overline{DE}}^{(i)}\Big)\frac{\phi_i(X)}{\partial y}\Bigg]$$
$$-\frac{\partial^2u^-(X)}{\partial x\partial y}\Bigg[\sum_{i=1}^2\Bigg[N_{\overline{DE}}^-\Big(X_{\overline{DE}}^{(i)s}-X^s\Big)\cdot\Big(A_i-X_{\overline{DE}}^{(i)}\Big)\frac{\phi_i(X)}{\partial y}$$
$$+N_{\overline{DE}}^-(-1,0)^T\cdot\Big(A_i-X_{\overline{DE}}^{(i)}\Big)\phi_i(X)+(x_i-\bar{x}_i)(y_i-\bar{y}_i)\frac{\phi_i(X)}{\partial y}$$

$$- (\bar{x}_i - x)\phi_i(X) + (\bar{x}_i - x)(\bar{y}_i - y)\frac{\phi_i(X)}{\partial y}\Bigg]$$
$$+ \sum_{i=3}^{4}\left[-(x_i - x)\phi_i(X) + (x_i - x)(y_i - y)\frac{\phi_i(X)}{\partial y}\right]\Bigg] + \sum_{i=1}^{4} I_i \frac{\phi_i(X)}{\partial y} \tag{3.45}$$

where $X_{\overline{DE}}^{(i)} = (\bar{x}_i, \bar{y}_i)^T$, $i = 1, 2$ *are arbitrary points on* $\overline{DE}$.

Theorem 3.10. *There exits a constant* C *such that*

$$\|I_{h,T}u - u\|_{0,T^-} \le Ch^2\|u\|_{2,T} \tag{3.46}$$

and

$$\left\|\frac{\partial(I_{h,T}u - u)}{\partial v}\right\|_{0,T^-} \le Ch\|u\|_{2,T}, \quad v = x, y, \tag{3.47}$$

for any $u \in PH^2_{\text{int}}(T)$ *where* T *is an arbitrary interface triangle.*

As for the estimates on T^+, we let $X = (x, y)^T$ be a point in T^+. Without loss of generality, we assume that line segments $\overline{XA_i}, i = 1, 2$ do not intersect with the interface and $\overline{DE}$, while line segment $\overline{XA_i}, i = 3, 4$ meet Γ at $\widetilde{A}_i, i = 3, 4$, see Fig. 8. Also, we assume that $A_i = (x_i, y_i)^T, i = 1, 2, 3, 4$ and

$$\widetilde{A}_i = \tilde{t}_i A_i + (1 - \tilde{t}_i)X = (\widetilde{x}_i, \widetilde{y}_i)^T, i = 3, 4$$

Lemma 3.10. *Given a two-dimensional vector* $\mathbf{q}$*, a point* $X \in T^+$ *and two real numbers* r, d^+*, then there exists a function* $v \in S_h(T)$ *such that* $\nabla v(X) = \mathbf{q}, v(X) = r, \dfrac{\partial^2 v^+(X)}{\partial x \partial y} = d^+$ *and*

$$\mathbf{q}(X) \cdot \sum_{i=1}^{4}(A_i - X)\phi_i(X) = \sum_{i=3}^{4}\Big[-\big(N_{\overline{DE}}^+ - I\big)\mathbf{q}\cdot(A_i - \widetilde{A}_i)\phi_i(X) - \big(N_{\overline{DE}}^+ - I\big)\mathbf{q}\cdot\big(\widetilde{A}_i - X_{\overline{DE}}^{(i)}\big)\phi_i(X)\Big]$$

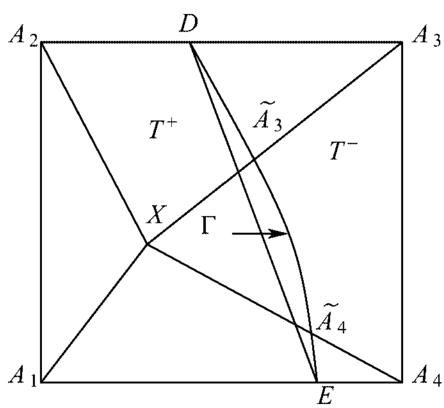

FIG. 8. A point X ∈ T^+ is connected to the four vertices by line segments in a Type II interface element.

$$
\begin{aligned}
&- d^+ \Bigg[\sum_{i=3}^{4} \big[N^+_{\overline{DE}} \big(X^{(i)s}_{\overline{DE}} - X^s \big) \cdot \big(A_i - X^{(i)}_{\overline{DE}} \big) \phi_i(X) \\
&+ (x_i - \bar{x}_i)(y_i - \bar{y}_i)\phi_i(X) + (\bar{x}_i - x)(\bar{y}_i - y)\phi_i(X) \big] \\
&+ \sum_{i=1}^{2} [(x_i - x)(y_i - y)\phi_i(X)] \Bigg]
\end{aligned}
$$

where $X^{(i)}_{\overline{DE}} = (\bar{x}_i, \bar{y}_i)^T$, $i = 3, 4$ *are arbitrary points on* $\overline{DE}$.

Theorem 3.11. *For any* $u \in PC^2_{\text{int}}(T)$ *and* $X = (x, y)^T \in T^+$, *we have*

$$
\begin{aligned}
I_{h,T}u(X) - u(X) = &\sum_{i=3}^{4} \Big[\big(N^+(\widetilde{A}_i) - N^+_{\overline{DE}} \big) \nabla u(X) \cdot (A_i - \widetilde{A}_i)\phi_i(X) \\
&- \big(N^+_{\overline{DE}} - I \big) \nabla u(X) \cdot \big(\widetilde{A}_i - X^{(i)}_{\overline{DE}} \big) \phi_i(X) \Big] \\
&- \frac{\partial^2 u^+(X)}{\partial x \partial y} \Bigg[\sum_{i=3}^{4} \big[N^+_{\overline{DE}} \big(X^{(i)s}_{\overline{DE}} - X^s \big) \cdot \big(A_i - X^{(i)}_{\overline{DE}} \big) \phi_i(X) \\
&+ (x_i - \bar{x}_i)(y_i - \bar{y}_i)\phi_i(X) + (\bar{x}_i - x)(\bar{y}_i - y)\phi_i(X) \big] \\
&+ \sum_{i=1}^{2} (x_i - x)(y_i - y)\phi_i(X) \Bigg] \\
&+ \sum_{i=3}^{4} \Bigg[(N^+(\widetilde{A}_i) - I) \int_0^1 \frac{d\nabla u^+(t\widetilde{A}_i + (1-t)X)}{dt} \cdot (A_i - \widetilde{A}_i) dt \phi_i(X) \\
&+ \int_0^{\tilde{t}_i} (1-t) \frac{d^2 u(tA_i + (1-t)X)}{dt^2} dt \phi_i(X) \\
&+ \int_{\tilde{t}_i}^1 (1-t) \frac{d^2 u(tA_i + (1-t)X)}{dt^2} dt \phi_i(X) \Bigg] \\
&+ \sum_{i=1}^{2} \int_0^1 (1-t) \frac{d^2 u(tA_i + (1-t)X)}{dt^2} dt \phi_i(X)
\end{aligned} \tag{3.48}
$$

where $X^{(i)}_{\overline{DE}} = (\bar{x}_i, \bar{y}_i)^T$, $i = 3, 4$ *are arbitrary points on* $\overline{DE}$.

Let I_i, $i = 1, 2, 3, 4$ be the integral terms involving vertices A_i, $i = 1, 2, 3, 4$ in (3.48).

Theorem 3.12. *For any* $u \in PC^3_{\text{int}}(T)$ *and* $X = (x, y)^T \in T^+$, *we have*

$$
\begin{aligned}
\frac{\partial (I_{h,T}u(X) - u(X))}{\partial x} = &\sum_{i=3}^{4} \Bigg[\big(N^+(\widetilde{A}_i) - N^+_{\overline{DE}} \big) \nabla u(X) \cdot (A_i - \widetilde{A}_i) \frac{\partial \phi_i(X)}{\partial x} \\
&- \big(N^+_{\overline{DE}} - I \big) \nabla u(X) \cdot \big(\widetilde{A}_i - X^{(i)}_{\overline{DE}} \big) \frac{\phi_i(X)}{\partial x} \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\partial^2 u^+(X)}{\partial x \partial y} \left[\sum_{i=3}^{4} \left[N^+_{\overline{DE}} \big(X^{(i)s}_{\overline{DE}} - X^s \big) \cdot \big(A_i - X^{(i)}_{\overline{DE}} \big) \frac{\phi_i(X)}{\partial x} \right.\right. \\
&+ N^+_{\overline{DE}} (0,-1)^T \cdot \big(A_i - X^{(i)}_{\overline{DE}} \big) \phi_i(X) + (x_i - \bar{x}_i)(y_i - \bar{y}_i) \frac{\phi_i(X)}{\partial x} \\
&\left. - (\bar{y}_i - y)\phi_i(X) + (\bar{x}_i - x)(\bar{y}_i - y) \frac{\phi_i(X)}{\partial x} \right] \\
&\left. + \sum_{i=1}^{2} \left[-(y_i - y)\phi_i(X) + (x_i - x)(y_i - y) \frac{\phi_i(X)}{\partial x} \right] \right] + \sum_{i=1}^{4} I_i \frac{\phi_i(X)}{\partial x}
\end{aligned}
\tag{3.49}
$$

$$
\begin{aligned}
\frac{\partial (I_{h,T} u(X) - u(X))}{\partial y} = &\sum_{i=3}^{4} \left[\big(N^+(\widetilde{A}_i) - N^+_{\overline{DE}} \big) \nabla u(X) \cdot (A_i - \widetilde{A}_i) \frac{\partial \phi_i(X)}{\partial y} \right. \\
&\left. - \big(N^+_{\overline{DE}} - I \big) \nabla u(X) \cdot \big(\widetilde{A}_i - X^{(i)}_{\overline{DE}} \big) \frac{\phi_i(X)}{\partial y} \right] \\
&- \frac{\partial^2 u^+(X)}{\partial x \partial y} \left[\sum_{i=3}^{4} \left[N^+_{\overline{DE}} \big(X^{(i)s}_{\overline{DE}} - X^s \big) \cdot \big(A_i - X^{(i)}_{\overline{DE}} \big) \frac{\phi_i(X)}{\partial y} \right.\right. \\
&+ N^+_{\overline{DE}} (-1,0)^T \cdot \big(A_i - X^{(i)}_{\overline{DE}} \big) \phi_i(X) + (x_i - \bar{x}_i)(y_i - \bar{y}_i) \frac{\phi_i(X)}{\partial y} \\
&\left. - (\bar{x}_i - x)\phi_i(X) + (\bar{x}_i - x)(\bar{y}_i - y) \frac{\phi_i(X)}{\partial y} \right] \\
&\left. + \sum_{i=1}^{2} \left[-(x_i - x)\phi_i(X) + (x_i - x)(y_i - y) \frac{\phi_i(X)}{\partial y} \right] \right] + \sum_{i=1}^{4} I_i \frac{\phi_i(X)}{\partial y}
\end{aligned}
\tag{3.50}
$$

where $X^{(i)}_{\overline{DE}} = (\bar{x}_i, \bar{y}_i)^T$, $i = 3, 4$ *are arbitrary points on* $\overline{DE}$.

Theorem 3.13. *There exits a constant* C *such that*

$$
\|I_{h,T} u - u\|_{0,T^+} \le C h^2 \|u\|_{2,T} \tag{3.51}
$$

and

$$
\left\| \frac{\partial (I_{h,T} u - u)}{\partial v} \right\|_{0,T^+} \le C h \|u\|_{2,T}, \quad v = x, y, \tag{3.52}
$$

for any $u \in PH^2_{\text{int}}(T)$ *where* T *is an arbitrary interface triangle.*

C. Interpolation Error on Ω

We now ready to derive the error estimates for the interpolation $I_h u$ in $S_h(\Omega)$.

Theorem 3.14. *There exists a constant C such that*

$$\|I_h u - u\|_{0,\Omega} \le Ch^2 \|u\|_{2,\Omega}, \tag{3.53}$$

$$\left\| \frac{\partial (I_h u - u)}{\partial s} \right\|_{0,\Omega} \le Ch\|u\|_{2,\Omega}, \quad s = x, y, \tag{3.54}$$

for any $u \in PH^2_{\text{int}}(\Omega)$ *and* $h > 0$ *small enough.*

Proof. First we have $\|I_h u - u\|^2_{0,\Omega} = \sum_{T\in\mathcal{T}_h} \|I_{h,T} u - u\|^2_{0,T}$. If T is a Type I interface element, by Theorem 3.2 and Theorem 3.7, we can get

$$\|I_{h,T} u - u\|^2_{0,T} = \|I_{h,T} u - u\|^2_{0,T^-} + \|I_{h,T} u - u\|^2_{0,T^+} \le Ch^4\|u\|^2_{2,T^-} + Ch^4\|u\|^2_{2,T^+} \le Ch^4\|u\|^2_{2,T}$$

Similarly, if T is a Type II interface element, by Theorem 3.10 and Theorem 3.13, we can get

$$\|I_{h,T} u - u\|^2_{0,T} \le Ch^4\|u\|^2_{2,T}$$

If T is a noninterface element, by the standard finite element interpolation error theory, we can get

$$\|I_{h,T} u - u\|^2_{0,T} \le Ch^4\|u\|^2_{2,T}$$

Therefore, $\|I_h u - u\|^2_{0,\Omega} \le \sum_{T\in\mathcal{T}_h} Ch^4\|u\|^2_{2,T} = Ch^4\|u\|^2_{2,\Omega}$ which leads to (3.53). Similar derivation can be carried out to obtain (3.54). ∎

IV. NUMERICAL EXAMPLES

We now present a group of numerical results to illustrate features of the bilinear IFE space. Errors in both the bilinear IFE interpolant and the bilinear IFE solution to an interface problem will be given. The error estimation for the related finite element method will be provided in a forthcoming article.

For simplicity, we only present results obtained by using the bilinear IFE space based on uniformly rectangular Cartesian partitions in the rectangular domain $\Omega = (-1, 1) \times (-1, 1)$. The interface curve Γ is a circle with radius $r_0 = \pi/6.28$ which separates Ω into two subdomains Ω^- and Ω^+ with $\Omega^- = \{(x, y)|x^2 + y^2 \le r_0^2\}$.

First, we show numerical results for the bilinear IFE interpolant $I_h u$ of the following function

$$u(x, y) = \begin{cases} \dfrac{r^\alpha}{\beta^-}, & \text{if } r \le r_0, \\ \dfrac{r^\alpha}{\beta^+} + \left(\dfrac{1}{\beta^-} - \dfrac{1}{\beta^+}\right) r_0^\alpha, & \text{otherwise}, \end{cases} \tag{4.55}$$

with $\alpha = 5, r = \sqrt{x^2 + y^2}$.

Table I contains actual errors of the IFE interpolant $I_h u$ with various partition sizes h for $\beta^- = 1, \beta^+ = 10$ which represents a moderate discontinuity in the coefficient. By simple calculations, we can easily see that the data in this table satisfy

$$\|I_h u - u\|_0 \approx \frac{1}{4}\|I_{\hat{h}} u - u\|_0, \quad |I_h u - u|_1 \approx \frac{1}{2}|I_{\hat{h}} u - u|_1,$$

TABLE I. Errors in the interpolant $I_h u$ when $\beta^- = 1$, $\beta^+ = 10$.

h	$\|I_h u - u\|_0$	$\|I_h u - u\|_1$
1/16	0.001479	0.05848
1/32	3.715×10^{-4}	0.02918
1/64	9.321×10^{-5}	0.01453
1/128	2.334×10^{-5}	0.007264
1/256	5.840×10^{-6}	0.003635

for $h = \hat{h}/2$. Using linear regression, we can also see that the data in this table obey

$$\|I_h u - u\|_0 \approx 0.3750 h^{1.996}, \quad |I_h u - u|_1 \approx 0.9405 h^{1.002}$$

which clearly indicates that the interpolant converges to u with convergence rates $O(h^2)$ and $O(h)$ in the L^2 norm and H^1 norm, respectively, as predicted by Theorem 3.14.

Table II contains actual errors of the IFE interpolant $I_h u$ with various partition size h for $\beta^- = 1, \beta^+ = 10{,}000$ which represents a large discontinuity in the coefficient. Using linear regression again, we can see that

$$\|I_h u - u\|_0 \approx 0.09557 h^{1.954}, \quad |I_h u - u|_1 \approx 0.3582 h^{1.030},$$

which are also in agreement with the error estimates given in Theorem 3.14.

Since this bilinear IFE space has an $O(h^2)$ (in L^2-norm) and an $O(h)$ (in H^1-norm) approximation capability, we naturally expect the FE method based on this IFE space to perform accordingly. To confirm this numerically, we consider the interface problem defined by (1.1)–(1.4) in which the boundary condition function $g(x, y)$ and the source term $f(x, y)$ are chosen such that the function u given in (4.55) is the exact solution in the domain Ω with the interface curve Γ described above.

Table III contains actual errors of the bilinear IFE solutions u_h with various partition size h for the interface problem with the coefficient function $\beta(x, y)$ with $\beta^- = 1, \beta^+ = 10$. We can easily see that the data in the second and third columns of this table satisfy

$$\|u_h - u\|_0 \approx \frac{1}{4}\|u_{\hat{h}} - u\|_0, \quad |u_h - u|_1 \approx \frac{1}{2}|u_{\hat{h}} - u|_1,$$

for $h = \hat{h}/2$. Using linear regression, we can also see that the data in this table obey

$$\|u_h - u\|_0 \approx 0.2622 h^{1.997}, \quad |u_h - u|_1 \approx 0.8957\, h^{0.9844},$$

which indicates that the bilinear IFE solution u_h converges to the exact solution with convergence rates $O(h^2)$ and $O(h)$ in the L^2 norm and H^1 norm, respectively. This is in agreement with those error estimates for the bilinear IFE interpolant obtained in the previous section.

TABLE II. Errors in the interpolant $I_h u$ when $\beta^- = 1$, $\beta^+ = 10{,}000$.

h	$\|I_h u - u\|_0$	$\|I_h u - u\|_1$
1/16	4.159×10^{-4}	0.02089
1/32	1.104×10^{-4}	0.01011
1/64	2.878×10^{-5}	0.004832
1/128	7.323×10^{-6}	0.002401
1/256	1.848×10^{-6}	0.001209

TABLE III. Errors of the IFE solutions for the case when $\beta^- = 1, \beta^+ = 10$.

h	$\|u_h - u\|_0$	$\|u_h - u\|_1$	$\|u_h - u\|_\infty$
1/16	0.001653	0.05882	9.500×10^{-4}
1/32	4.100×10^{-4}	0.02948	4.853×10^{-4}
1/64	1.017×10^{-4}	0.01482	3.256×10^{-4}
1/128	2.542×10^{-5}	0.007520	1.594×10^{-4}
1/256	6.559×10^{-6}	0.003841	7.410×10^{-5}

However, numerical experiments indicate that the bilinear IFE solution does not always have the second order convergence in the L^∞ norm because the data in the fourth column of Table III obey

$$|u_h - u|_\infty \approx 0.01173h^{0.8967}$$

which clearly shows that the rate at which u_h converges to u is not $O(h^2)$. The question under what conditions the bilinear IFE solution can have a second order convergence in the L^∞ norm is still open.

The bilinear IFE method also works well for the case in which the coefficient function has a large jump, see Table IV. The errors in this group of computations obey

$$\|u_h - u\|_0 \approx 0.1111h^{1.993}, \quad |u_h - u|_1 \approx 0.3807h^{1.040},$$

which again are in agreement with those error estimates for the bilinear IFE interpolant.

V. CONCLUSIONS

In this article, we have discussed a bilinear IFE space that can be used to solve interface problems of second-order elliptic partial differential equations. The partition of this bilinear IFE space can be formed without consideration of the interface location. With this bilinear IFE space, a Cartesian partition can be used to solve an interface problem with a rather arbitrary interface. The bilinear IFE space is closely related to the standard finite element space formed by bilinear polynomials except for functions over interface rectangles. Over an interface rectangle, piecewise bilinear IFE functions are formed according to the jump conditions of the interface problem to be solved. We have employed a multipoint Taylor expansion technique to analyze the interpolation errors of this bilinear IFE space for functions in the Sobolev space related to the interface problems. It has been shown that this bilinear IFE space has an approximation capability similar to that of the standard bilinear FE space. The estimates for the interpolation error obtained here are critical for deriving error estimates for the FE (or volume-element) solution to an interface problem based on this bilinear IFE space.

TABLE IV. Errors of the IFE solutions for the case when $\beta^- = 1, \beta^+ = 10{,}000$.

h	$\|u_h - u\|_0$	$\|u_h - u\|_1$	$\|u_h - u\|_\infty$
1/16	4.299×10^{-4}	0.02142	6.881×10^{-4}
1/32	1.150×10^{-4}	0.01049	2.400×10^{-4}
1/64	2.828×10^{-5}	0.004888	8.531×10^{-5}
1/128	7.026×10^{-6}	0.002419	3.286×10^{-5}
1/256	1.743×10^{-6}	0.001212	8.741×10^{-6}

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