Received: 31 December 2017 | Revised: 4 July 2018 | Accepted: 21 November 2018 | Published on: 15 January 2019

DOI: 10.1002/num.22348

RESEARCH ARTICLE

WILEY

Approximation capabilities of immersed finite element spaces for elasticity Interface problems

Ruchi Guo[1] | Tao Lin[1] | Yanping Lin[2]

[1]Department of Mathematics, Virginia Tech, Blacksburg, Virginia

[2]Department of Applied Mathematics, Hong Kong Polytechnic University, Hong Kong

Correspondence

Tao Lin, Department of Mathematics, 460 McBryde Hall, Virginia Tech, 225 Stanger Street, Blacksburg, VA 24061-1026.

Email: tlin@vt.edu

Funding information

Hong Kong Special Administrative Region, GRF B-Q56D and B-Q40W.

We construct and analyze a group of immersed finite element (IFE) spaces formed by linear, bilinear, and rotated Q_1 polynomials for solving planar elasticity equation involving interface. The shape functions in these IFE spaces are constructed through a group of approximate jump conditions such that the unisolvence of the bilinear and rotated Q_1 IFE shape functions are always guaranteed regardless of the Lamé parameters and the interface location. The boundedness property and a group of identities of the proposed IFE shape functions are established. A multi-point Taylor expansion is utilized to show the optimal approximation capabilities for the proposed IFE spaces through the Lagrange type interpolation operators.

KEYWORDS

discontinuous coefficients, elasticity equations, immersed finite element method, interface problems

1 | INTRODUCTION

In many applications of sciences and engineering, we need to consider an elastic object formed with multiple materials which leads to a linear elasticity system with discontinuous coefficients whose values reflect the difference between materials. To be specific and without loss of generality, we consider an elastic object forming a domain $\Omega \subset \mathbb{R}^2$ separated by a smooth interface curve Γ into two subdomains Ω^- and Ω^+ each of which is occupied by a different material. Thus, the Lamé parameters of Ω are piecewise constant functions in the following forms:

$$\lambda = \begin{cases} \lambda^- & \text{if } X \in \Omega^-, \\ \lambda^+ & \text{if } X \in \Omega^+, \end{cases} \qquad \mu = \begin{cases} \mu^- & \text{if } X \in \Omega^-, \\ \mu^+ & \text{if } X \in \Omega^+. \end{cases} \tag{1.1}$$

As usual, we assume that the $\mathbf{u} = (u_1, u_2)^T$ is modeled by the planar linear elasticity equations:

$$-div\,\sigma(\mathbf{u}) = \mathbf{f} \text{ in } \Omega^- \cup \Omega^+, \tag{1.2}$$

$$\mathbf{u} = \mathbf{g} \text{ on } \partial\Omega, \tag{1.3}$$

where $\mathbf{f} = (f_1, f_2)^T$ and $\mathbf{g} = (g_1, g_2)^T$ represent the given body force and the displacement on the boundary, $\sigma(\mathbf{u}) = (\sigma_{ij}(\mathbf{u}))_{1 \le i,j \le 2}$ is the stress tensor given by

$$\sigma_{ij}(\mathbf{u}) = \lambda(\nabla \cdot \mathbf{u})\delta_{i,j} + 2\mu\varepsilon_{ij}(\mathbf{u}), \quad \text{with } \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{1.4}$$

being the strain tensor. Furthermore, the discontinuity in the Lamé parameters requires the displacement $\mathbf{u} = (u_1, u_2)^T$ to satisfy the jump conditions across the material interface Γ:

$$[\mathbf{u}]_\Gamma := (\mathbf{u}^+ - \mathbf{u}^-)|_\Gamma = \mathbf{0}, \tag{1.5}$$

$$[\sigma(\mathbf{u})\mathbf{n}]_\Gamma := (\sigma^+(\mathbf{u}^+)\mathbf{n} - \sigma^-(\mathbf{u}^-)\mathbf{n})|_\Gamma = \mathbf{0}, \tag{1.6}$$

where $\mathbf{u}^s = \mathbf{u}|_{\Omega^s}, \sigma^s(\mathbf{u}^s) = \sigma(\mathbf{u})|_{\Omega^s}, s = -, +$, and $\mathbf{n}$ is the normal vector to Γ.

We call (1.2)–(1.6) an elasticity interface problem for determining the displacement $\mathbf{u} = (u_1, u_2)^T$. Elasticity interface problems have a wide range of applications in engineering and science, such as the inverse problems [1–3] in which one needs to recover the location or geometry of buried cracks, cavities or inclusions, and the structure optimization problems [4, 5] in which one aims at optimizing the distribution of different elastic materials such that the overall structure compliance can be minimized, and additional elasticity problems can be found in [6–9], to name just a few.

Finite element methods [10–12] and discontinuous Galerkin methods [13–15] have been developed to solve elasticity interface problems, and these methods perform optimally provided that their mesh is interface-fitted [16, 17]. However, in some applications such as those inverse/design problems mentioned above solved by shape optimization methods, the shape or location of the interface usually involves large changes in the computation. In general, it is nontrivial and time consuming to generate an interface-fitting mesh again and again; therefore, solving (1.2)–(1.6) on interface-independent (noninterface-fitted) meshes has attracted research attentions. Both the finite element approach and the finite difference approach have been attempted. For example, an unfitted finite element method using the Nitsche's penalty along the interface to enforce the jump conditions is presented in [18, 19], and some immersed interface methods (IIM) based on finite difference formulation are presented in [20–22] which handle the jump conditions through a local coordinate transformation between subelements partitioned by the interface.

Immersed finite element (IFE) methods are developed for solving interface problems with interface-independent meshes. The key idea of an IFE space is to use standard polynomials on noninterface elements, but Hsieh–Clough–Tocher type [23, 24] macro polynomials constructed according to interface jump conditions on interface elements. There have been quite a few publications on IFE methods, for example, IFE methods for elliptic interface problems are discussed in [25–32], IFE methods for interface problems of other types partial differential equations are presented in [33–41]. In particular, for planar-elasticity interface problems described by (1.2)–(1.6), a nonconforming linear IFE space on a uniform triangular mesh is discussed in [21, 42, 43]. A conforming IFE space is developed in [21, 44] by extending the IFE shape functions in [21, 42, 43] to the neighborhood interface elements. A bilinear IFE space on a rectangular Cartesian mesh is discussed in [45]. A nonconforming IFE space using the rotated Q_1 polynomials is presented in [46] which leads to a locking-free IFE method. Most of the IFE methods for the planar-elasticity interface problems are Galerkin type, that is, the test and trial functions used in each of these methods are from the same IFE space, but the Petrov–Galerkin formulation can also be used, for example, a Petrov–Galerkin IFE scheme is developed in [47] that uses standard Lagrange polynomials as the test functions.

This article focuses on two issues in the research of IFE methods for interface problems of the planar elasticity. First, we develop a unified construction procedure for the shape functions of the IFE spaces defined on a triangular or rectangular mesh with the linear, bilinear, or rotated Q_1 polynomials. By this procedure, the coefficients in an IFE shape function satisfy a *Sherman–Morrison* linear system from which the unisolvence of the IFE shape functions can be readily deducted. Second, we derive a group of multi-point Taylor expansions for vector functions satisfying the jump conditions specified in (1.5) and (1.6) for the planar-elasticity interface problems, and we then employ them in the framework recently developed in [48] to show the optimal approximation capabilities for the IFE spaces considered in this article. However, the major challenge in applying it to the analysis of the approximation capability of the IFE spaces developed for the elasticity interface problems is the coupling of the two components in the displacement through the jump conditions. Because of this coupling, a multi-point Taylor expansion for each component (which is a scalar function) of the displacement individually is difficulty, if not impossible, to be used in the analysis of the approximation capability of the proposed vector IFE spaces. To deal with this challenge, more sophisticated identities have to be developed for transferring the quantities of a vector function from one side of the interface to the other according to the jump conditions, and these identities are nontrivial extensions of their scalar counterparts. To the best of our knowledge, this is the first time the approximation capability of IFE spaces formed with vector functions is analyzed, and this is an important step toward to the establishment of the theoretical foundation for IFE methods that can solve interface problems of the linear elasticity system with interface-independent (such as Cartesian) meshes.

We note that the unisolvence results established in this article can be used to show the existence of the IFE functions satisfying nonhomogeneous jump conditions such as those in [47] because coefficients in such an IFE function are determined by a linear system with the same matrix as the one for determining the coefficients in the IFE function satisfying the related homogeneous jump conditions. These IFE functions are not suitable for constructing IFE spaces because of their nonhomogeneity; therefore, their research is beyond the scope of this article. However, they can be used together with the homogeneous IFE spaces developed in this article to deal with the planar-elasticity interface problems with nonhomogeneous jump conditions in a homogenization technique such as one of those proposed in [28, 49].

This article consists of five additional sections. The next section is for some basic notations and assumptions. In Section 3, we establish a few fundamental geometric identities and estimates related to the interface. In Section 4, we derive the multi-point Taylor expansions for a vector function $\mathbf{u}$ satisfying the jump conditions (1.5) and (1.6) along the interface. In Section 5, we derive a *Sherman–Morrison* linear system for determining the coefficients in IFE shape functions on an interface element, study properties of these shape functions, and prove the optimal approximation capabilities for the IFE spaces considered in this article. In the last section, we present a group of numerical examples to illustrate the approximation features of these IFE spaces.

2 | PRELIMINARIES

We now describe terms and facts to be used in the discussion. Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain formed as union of finitely many rectangles/triangles, and without loss of generality, we assume Ω is separated by Γ into two subdomains Ω^+ and Ω^- such that $\overline{\Omega} = \overline{\Omega^+} \cup \overline{\Omega^-}$. For a measurable subset $\widetilde{\Omega} \subseteq \Omega$, we define the vector Sobolev space $\mathbf{W}^{k,p}(\widetilde{\Omega}) = [W^{k,p}(\widetilde{\Omega})]^2$ where $W^{k,p}(\widetilde{\Omega})$ is the standard Sobolev space, and the associated norm and semi-norm of $\mathbf{W}^{k,p}(\widetilde{\Omega})$ are such that for every $\mathbf{u} = (u_1, u_2)^T \in \mathbf{W}^{k,p}(\widetilde{\Omega})$,

$$\|\mathbf{u}\|_{k,p,\widetilde{\Omega}} = \|u_1\|_{k,p,\widetilde{\Omega}} + \|u_2\|_{k,p,\widetilde{\Omega}} \quad \text{and} \quad |\mathbf{u}|_{k,p,\widetilde{\Omega}} = \|D^\alpha u_1\|_{0,p,\widetilde{\Omega}} + \|D^\alpha u_2\|_{0,p,\widetilde{\Omega}}, \quad |\alpha| = k. \tag{2.1}$$

The related vector Hilbert space is denoted by $\mathbf{H}^k(\widetilde{\Omega}) = \mathbf{W}^{k,2}(\widetilde{\Omega})$. Let $\mathbf{C}^k(\widetilde{\Omega})$ be the collection of kth differentiable smooth vector functions. When $\widetilde{\Omega}^s = \widetilde{\Omega} \cap \Omega^s \neq \emptyset$, $s = \pm$, and $k \geq 1$, we define

$$\mathbf{PW}^{k,p}_{int}(\widetilde{\Omega}) = \{\mathbf{u} : \mathbf{u} \in \mathbf{W}^{k,p}(\widetilde{\Omega}^s), s = \pm; [\mathbf{u}]_\Gamma = \mathbf{0}, \text{ and } [\sigma(\mathbf{u})\mathbf{n}]_\Gamma = \mathbf{0}\}, \tag{2.2}$$

$$\mathbf{PC}^{k}_{int}(\widetilde{\Omega}) = \{\mathbf{u} : \mathbf{u} \in \mathbf{C}^{k}(\widetilde{\Omega}^s), s = \pm; [\mathbf{u}]_\Gamma = \mathbf{0}, \text{ and } [\sigma(\mathbf{u})\mathbf{n}]_\Gamma = \mathbf{0}\}. \tag{2.3}$$

Here, the definition implicitly implies that the zeroth- and first-order trace of $\mathbf{u}$ are well defined on Γ. We then define the following norms and semi-norms associated with these spaces:

$$\|\mathbf{u}\|_{k,p,\widetilde{\Omega}} = \sum_{i=1}^{2}(\|u_i\|_{k,p,\widetilde{\Omega}^-} + \|u_i\|_{k,p,\widetilde{\Omega}^+}), \quad \text{and} \quad |\mathbf{u}|_{k,p,\widetilde{\Omega}} = \sum_{i=1}^{2}(|u_i|_{k,p,\widetilde{\Omega}^-} + |u_i|_{k,p,\widetilde{\Omega}^+}),$$

$$\|\mathbf{u}\|_{k,\infty,\widetilde{\Omega}} = \max_{i=1,2}\{\max\{\|u_i\|_{k,\infty,\widetilde{\Omega}^-}, \|u_i\|_{k,\infty,\widetilde{\Omega}^+}\}\}, \quad \text{and} \quad |\mathbf{u}|_{k,\infty,\widetilde{\Omega}} = \max_{i=1,2}\{\max\{|u_i|_{k,\infty,\widetilde{\Omega}^-}, |u_i|_{k,\infty,\widetilde{\Omega}^+}\}\}.$$

Also we denote the corresponding Hilbert space $\mathbf{PH}^k(\widetilde{\Omega}) = \mathbf{PW}^{k,2}(\widetilde{\Omega})$ with the norm $\|\cdot\|_{k,\widetilde{\Omega}} = \|\cdot\|_{k,2,\widetilde{\Omega}}$ and the semi-norm $|\cdot|_{k,\widetilde{\Omega}} = |\cdot|_{k,2,\widetilde{\Omega}}$. Furthermore, for any vector function $\mathbf{v} = (v_1, v_2)^T \in \mathbf{H}^1(\widetilde{\Omega})$, let $\nabla\mathbf{v}$ be its 2-by-2 Jacobian matrix where the ith row is the row vector ∇v_i, $i = 1, 2$.

Let $\mathcal{T}_h$ be a Cartesian rectangular or triangular mesh of the domain Ω with a mesh size $h > 0$. An element $T \in \mathcal{T}_h$ is called an interface element if the intersection of the interior of T with the interface Γ is nonempty; otherwise, it is called a noninterface element. Let $\mathcal{T}_h^i$ and $\mathcal{T}_h^n$ be the sets of interface elements and noninterface elements, respectively. Similarly, let $\mathcal{E}_h^i$ and $\mathcal{E}_h^n$ be the sets of interface edges and noninterface edges, respectively. In addition, as in [50, 51], we assume that $\mathcal{T}_h$ satisfies the following hypotheses when the mesh size h is small enough:

Hypothesis 1 The interface Γ cannot intersect an edge of any element at more than two points unless the edge is part of Γ.

Hypothesis 2 If Γ intersects the boundary of an element at two points, these intersection points must be on different edges of this element.

Hypothesis 3 The interface Γ is a piecewise C^2 function, and the mesh $\mathcal{T}_h$ is formed such that the subset of Γ in every interface element $T \in \mathcal{T}_h^i$ is C^2.

Hypothesis 4 The interface Γ is smooth enough so that $\mathbf{PC}^2_{int}(T)$ is dense in $\mathbf{PH}^2_{int}(T)$ for every interface element $T \in \mathcal{T}_h^i$.

We will discuss IFE spaces formed by linear polynomials on triangular meshes and bilinear or rotated Q_1 polynomials on rectangular meshes. For each element T in a mesh $\mathcal{T}_h$, we introduce an index set $\mathcal{I} = \{1, 2, 3\}$ when T is triangular or $\mathcal{I} = \{1, 2, 3, 4\}$ when T is rectangular. Then, the local finite element space is denoted by $(T, \mathbf{\Pi}_T, \mathbf{\Sigma}_T)$, with $\mathbf{\Pi}_T = [\text{Span}\{1, x, y\}]^2$, $[\text{Span}\{1, x, y, xy\}]^2$, or $[\text{Span}\{1, x, y, x^2 - y^2\}]^2$ for the linear, bilinear, or rotated Q_1 polynomial space, respectively, and the local degrees of freedom $\mathbf{\Sigma}_T = \{\boldsymbol{\psi}_T(A_i) : i \in \mathcal{I}, \boldsymbol{\psi}_T \in \mathbf{\Pi}_T\}$, where A_is are the vertices of T for the conforming linear and bilinear cases, or midpoints of edges of T for the nonconforming linear (Crouzeix–Raviart elements) and rotated Q_1 case. For these finite element spaces, according to [10, 52–54], there exist vector shape functions $\boldsymbol{\psi}_{i,T} \in \mathbf{\Pi}_T$, $i = 1, 2, \ldots, 2|\mathcal{I}|$ such that

$\mathbf{\Pi}_T = \text{Span}\{\boldsymbol{\psi}_{i,T}, 1 \le i \le 2|\mathcal{I}|\}$ with

$$\boldsymbol{\psi}_{i,T}(A_j) = \begin{cases} \delta_{i,j}, \\ 0, \end{cases} \quad i = 1, \dots, |\mathcal{I}|, \quad \text{and} \quad \boldsymbol{\psi}_{i,T}(A_j) = \begin{cases} 0, \\ \delta_{i-|\mathcal{I}|,j}, \end{cases} \quad i = |\mathcal{I}|+1, \dots, 2|\mathcal{I}|, \tag{2.4}$$

$$|\boldsymbol{\psi}_{i,T}|_{k,\infty,T} \le Ch^{-k}, \quad k = 0, 1, 2. \tag{2.5}$$

In addition, we will use a vectorization mapping Vec: $\mathbb{R}^{m\times n} \to \mathbb{R}^{mn\times 1}$ such that for any $A = (a_{ij})_{i=1,j=1}^{m,n}$,

$$\text{Vec}(A) := (a_{11}, \dots, a_{m1}, a_{12}, \dots, a_{m2}, \dots, a_{1n}, \dots, a_{mn})^T,$$

and a Kronecker product $\otimes$: $\mathbb{R}^{m\times n} \times \mathbb{R}^{p\times q} \to \mathbb{R}^{mp\times nq}$ such that for any $A = (a_{ij})_{i=1,j=1}^{m,n} \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$, there holds

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}. \tag{2.6}$$

A well-known formula [55] about the Kronecker product and the vectorization operation is the following:

$$\text{Vec}(CDE) = (E^T \otimes C)\text{Vec}(D). \tag{2.7}$$

Throughout this article, we use the notation I_n to denote the n-by-n identity matrix and $0_{m\times n}$ to denote the m-by-n zero matrix for positive integers m and n, and to simplify the presentation, we adopt the notation $\partial_{x_k} = \frac{\partial}{\partial_{x_k}}$, $k = 1, 2$, for partial derivatives with $x_1 = x$, $x_2 = y$. Also, as usual, we will use C to denote generic constants independent with the mesh size h in all the discussion from now on.

3 | GEOMETRIC PROPERTIES OF THE INTERFACE

In this section, we derive a group of geometric properties on the interface elements for estimating interpolation errors of vector-valued functions. These properties are extensions of those developed in [48] for scalar functions. Let T be an interface element and l be a line connecting the intersection points of the interface Γ with ∂T. Let $\mathbf{n}(\widetilde{X}) = (\widetilde{n}_1(\widetilde{X}), \widetilde{n}_2(\widetilde{X}))^T$ and $\mathbf{t}(\widetilde{X}) = (\widetilde{n}_2(\widetilde{X}), -\widetilde{n}_1(\widetilde{X}))^T$ be the normal and tangential vectors of Γ at a point $\widetilde{X} \in \Gamma \cap T$, respectively, and let the normal and tangential vectors of l be $\bar{\mathbf{n}} = (\bar{n}_1, \bar{n}_2)^T$ and $\bar{\mathbf{t}} = (\bar{n}_2, -\bar{n}_1)^T$, respectively. Consider the following matrices:

$$N^s(\widetilde{X}) = \begin{bmatrix} (\lambda^s + 2\mu^s)\widetilde{n}_1(\widetilde{X}) & \mu^s\widetilde{n}_2(\widetilde{X}) & \mu^s\widetilde{n}_2(\widetilde{X}) & \lambda^s\widetilde{n}_1(\widetilde{X}) \\ \lambda^s\widetilde{n}_2(\widetilde{X}) & \mu^s\widetilde{n}_1(\widetilde{X}) & \mu^s\widetilde{n}_1(\widetilde{X}) & (\lambda^s + 2\mu^s)\widetilde{n}_2(\widetilde{X}) \\ -\widetilde{n}_2(\widetilde{X}) & 0 & \widetilde{n}_1(\widetilde{X}) & 0 \\ 0 & -\widetilde{n}_2(\widetilde{X}) & 0 & \widetilde{n}_1(\widetilde{X}) \end{bmatrix}, \quad s = \pm, \tag{3.1}$$

$$\overline{N}^s = \begin{bmatrix} (\lambda^s + 2\mu^s)\bar{n}_1 & \mu^s\bar{n}_2 & \mu^s\bar{n}_2 & \lambda^s\bar{n}_1 \\ \lambda^s\bar{n}_2 & \mu^s\bar{n}_1 & \mu^s\bar{n}_1 & (\lambda^s + 2\mu^s)\bar{n}_2 \\ -\bar{n}_2 & 0 & \bar{n}_1 & 0 \\ 0 & -\bar{n}_2 & 0 & \bar{n}_1 \end{bmatrix}, \quad s = \pm. \tag{3.2}$$

By straightforward calculation, we have

$$\text{Det}(N^s(\widetilde{X})) = \text{Det}(\overline{N}^s) = \mu^s(\lambda^s + 2\mu^s), \quad s = \pm. \tag{3.3}$$

Hence both the matrices $N^s(\widetilde{X})$ and $\overline{N}^s$ are nonsingular, and we can use them to define

$$M^-(\widetilde{X}) = (N^+(\widetilde{X}))^{-1}N^-(\widetilde{X}), \qquad M^+(\widetilde{X}) = (N^-(\widetilde{X}))^{-1}N^+(\widetilde{X}), \tag{3.4}$$

$$\overline{M}^- = (\overline{N}^+)^{-1}\overline{N}^-, \qquad \overline{M}^+ = (\overline{N}^-)^{-1}\overline{N}^+. \tag{3.5}$$

These matrices have the following properties.

Lemma 3.1 *For every* $\mathbf{u} \in \mathbf{PC}^2_{int}(T)$ *and* $\widetilde{X} \in \Gamma \cap T$, *there holds*

$$\mathrm{Vec}(\nabla \mathbf{u}^+(\widetilde{X})) = M^-(\widetilde{X})\mathrm{Vec}(\nabla \mathbf{u}^-(\widetilde{X})), \qquad \mathrm{Vec}(\nabla \mathbf{u}^-(\widetilde{X})) = M^+(\widetilde{X})\mathrm{Vec}(\nabla \mathbf{u}^+(\widetilde{X})). \tag{3.6}$$

Proof. To simplify the notations, we denote $\mathbf{n}(\widetilde{X}) = (\widetilde{n}_1, \widetilde{n}_2)^T$ in this proof. By direction calculations, we have

$$\sigma^s(\mathbf{u}(\widetilde{X}))\ \mathbf{n}(\widetilde{X}) = \begin{bmatrix} (\lambda^s + 2\mu^s)\widetilde{n}_1\partial_{x_1}u_1 + \mu^s\widetilde{n}_2\partial_{x_2}u_1 + \mu^s\widetilde{n}_2\partial_{x_1}u_2 + \lambda^s\widetilde{n}_1\partial_{x_2}u_2 \\ \lambda^s\widetilde{n}_2\partial_{x_1}u_1 + \mu^s\widetilde{n}_1\partial_{x_2}u_1 + \mu^s\widetilde{n}_1\partial_{x_1}u_2 + (\lambda^s + 2\mu^s)\widetilde{n}_2\partial_{x_1}u_2 \end{bmatrix}. \tag{3.7}$$

From the continuity jump (1.5), we have $\nabla u_i^+\ \mathbf{t}(\widetilde{X}) = \nabla u_i^-\ \mathbf{t}(\widetilde{X})$, $i = 1, 2$. Combining this with the stress jump (1.6) leads to $N^-(\widetilde{X})\mathrm{Vec}(\nabla \mathbf{u}^-(\widetilde{X})) = N^+(\widetilde{X})\mathrm{Vec}(\nabla \mathbf{u}^+(\widetilde{X}))$ from which we have (3.6) because of (3.4). ∎

Lemma 3.2 *The vectors* $\alpha_1 = [-\overline{n}_2, 0, \overline{n}_1, 0]^T$ *and* $\alpha_2 = [0, -\overline{n}_2, 0, \overline{n}_1]^T$ *are eigenvectors of* $(\overline{M}^s)^T$, $s = +$, *or* $-$, *such that*

$$(\overline{M}^s)^T\alpha_i = \alpha_i, \quad i = 1, 2. \tag{3.8}$$

Proof. The identities in (3.8) follow from direct calculations. ∎

As proved in the following lemma, the matrices $\overline{M}^s$ constructed on l can be used to approximate the matrices M^s constructed on the interface $\Gamma \cap T$, $s = +$ or $-$.

Lemma 3.3 *There exists a constant* C *independent of the interface location such that for every interface element* $T \in \mathcal{T}_h^i$ *and every point* $\widetilde{X} \in \Gamma \cap T$, $s = \pm$, *we have*

$$\|M^s(\widetilde{X})\| \leq C, \quad \|\overline{M}^s\| \leq C, \tag{3.9}$$

and

$$\|M^s(\widetilde{X}) - \overline{M}^s\| \leq Ch. \tag{3.10}$$

Proof. We only prove the case for $s = -$, and the argument for $s = +$ is similar. Since $\|\overline{\mathbf{n}}\| = 1$ and $\|\mathbf{n}(\widetilde{X})\| = 1$, we have $\|\overline{N}^-\| \leq C$ and $\|N^-(\widetilde{X})\| \leq C$. Besides, we note that

$$\|(N^+(\widetilde{X}))^{-1}\| = \frac{1}{\mathrm{Det}(N^+(\widetilde{X}))}\|\mathrm{adj}(N^+(\widetilde{X}))\| \leq C, \quad \text{and} \quad \|(\overline{N}^+)^{-1}\| = \frac{1}{\mathrm{Det}(\overline{N}^+)}\|\mathrm{adj}(\overline{N}^+)\| \leq C,$$

because $\mathrm{Det}(N^-(\widetilde{X})) = \mathrm{Det}(\overline{N}^-) = \mu^-(\lambda^- + 2\mu^-)$ and each term of the adjugate matrices is bounded by some constants C. Then, (3.9) follows from applying these estimates in the inequalities below:

$$\|M^-(\widetilde{X})\| \leq \|(N^+(\widetilde{X}))^{-1}\|\ \|N^-(\widetilde{X})\| \quad \text{and} \quad \|\overline{M}^-\| \leq \|(\overline{N}^+)^{-1}\|\ \|\overline{N}^-\|.$$

For (3.10), we note that

$$\begin{aligned} \|M^-(\widetilde{X}) - \overline{M}^-\| &= \|(N^+(\widetilde{X}))^{-1}N^-(\widetilde{X}) - (\overline{N}^+)^{-1}\overline{N}^-\| \\ &= \|(N^+(\widetilde{X}))^{-1}(N^-(\widetilde{X}) - \overline{N}^-) + (N^+(\widetilde{X}))^{-1}(\overline{N}^+ - N^+(\widetilde{X}))(\overline{N}^+)^{-1}\overline{N}^-\| \\ &\leq C\|N^-(\widetilde{X}) - \overline{N}^-\| + C\|\overline{N}^+ - N^+(\widetilde{X})\| \\ &\leq Ch \end{aligned}$$

in which we have used the estimate $\|\mathbf{n}(\widetilde{X}) - \overline{\mathbf{n}}\| \leq Ch$ given in Lemma 3.2 of [48]. ∎

In addition, we consider the following set

$$T_{int} = \cup\{l_t \cap T \; : \; l_t \text{ is a tangent line to } \Gamma \cap T\}, \tag{3.11}$$

which is the subelement swept by the tangent lines to $\Gamma \cap T$. We note that this set is equivalent to the one considered in [50]. Following an idea similar to that used in [48], we can show that this is actually a small set in the following lemma.

Lemma 3.4 *Assume h is small enough, then there exists a constant C independent of the interface location, such that $|T_{int}| \leq Ch^3$.*

Proof. Let κ be the maximal curvature of $\Gamma \cap T$. By the assumption, we can follow the idea in [48] to assume h is sufficiently small such $\kappa h \leq \epsilon$ for some $\epsilon \in (0, 1/2)$. Let D and E be the intersection points of Γ and ∂T, and recall l is the line connecting D and E. Let X be one point in T_{int}. According to the definition (3.11), there exists a point $Y \in \Gamma$ such that $\overline{XY}$ is tangent to Γ. Denote $X_\perp$ and $Y_\perp$ as the projection of X and Y onto l, respectively. Let $\theta \in [0, \pi/2]$ be the angle between $\overline{XY}$ and l. According to Lemma 3.2 in [48], we have $|\overline{YY_\perp}| \leq 2(1-2\epsilon^2)^{-3/2}\kappa h^2$. Using the fact $|\overline{XY}| \leq Ch$ and simply geometry, we obtain $|\overline{XX_\perp}| = |\overline{YY_\perp}| + |\overline{XY}| \sin(\theta) \leq Ch^2 + Ch\sin(\theta)$. In addition, using (3.5) in [48], there holds

$$\begin{aligned} \sin(\theta) &= (1 - (\bar{\mathbf{n}} \cdot \mathbf{n}(Y))^2)^{1/2} \\ &\leq (1 + (1-2\epsilon^2)^{-3/2})\kappa h(4 - (1 + (1-2\epsilon^2)^{-3/2})^2\kappa^2 h^2)^{1/2} \\ &\leq (1 + (1-2\epsilon^2)^{-3/2})(4 - (1 + (1-2\epsilon^2)^{-3/2})^2\epsilon^2)^{1/2}\kappa h \end{aligned} \tag{3.12}$$

where we have used $h\kappa \leq \epsilon$. It shows that $|\overline{XX_\perp}| \leq C(1+\kappa)h^2$ with C depending on ϵ, that is, the distance between X and $\overline{DE}$ is bounded by $C(1+\kappa)h^2$. Since $|\overline{DE}| \leq Ch$, we have $|T_{int}| \leq C(1+\kappa)h^3$. ∎

4 | MULTI-POINT TAYLOR EXPANSION

In this section, we present a multi-point Taylor expansion for the piecewise smooth vector functions satisfying (1.5) and (1.6) on interface elements and derive the estimates of the remainders in the expansion. The multi-point Taylor expansion idea was first employed in [29] for showing the approximation capabilities of the linear IFE spaces for the elliptic interface problems through the Lagrange type interpolation operator. Similar ideas were then used to study approximation capabilities for the bilinear IFE spaces [27, 56] and nonconforming IFE spaces [32, 50]. Recently, by generalizing this technique, the authors in [48] developed a unified framework to show the approximation capabilities of various IFE spaces, and we now extend this technique to IFE spaces of vector functions for solving the elasticity interface problems.

In the following discussion, for every $T \in \mathcal{T}_h^i$, let Γ partition T into $T^\pm$ and let l partition T into $\overline{T}^\pm$. Define $\widetilde{T} = (T^+ \cap \overline{T}^-) \cup (T^- \cap \overline{T}^+)$ which is the subelement sandwiched by Γ and l. From [29], we know $|\widetilde{T}| \leq Ch^3$. In addition, as in [50], for every $X \in T \backslash T_{int}$, the segment $\overline{A_iX}$ intersects with $\Gamma \cap T$ either at only one point when A_i and X are on different sides of $\Gamma \cap T$ or no point when A_i and X are on the same side. Then, we define $T_*^s = (\overline{T}^s \cap T^s) \setminus T_{int})$, $s = \pm$, and let $T_* = T \setminus (T_*^- \cup T_*^+)$.

We further partition $\mathcal{I}$ into two sub index sets $\mathcal{I}^+ = \{i \; : \; A_i \in T^+\}$ and $\mathcal{I}^- = \{i \; : \; A_i \in T^-\}$. For every $X \in T$, we let $Y_i(t, X) = tA_i + (1-t)X$, $t \in [0, 1]$, $i \in \mathcal{I}$. Let $\widetilde{t}_i = \widetilde{t}_i(X) \in [0, 1]$ be such that $\widetilde{Y}_i = Y_i(\widetilde{t}_i, X)$ is on the curve $\Gamma \cap T$ if X and A_i are on different sides of T. We start from the following theorem that gives the expansion of $\mathbf{u}(A_i)$ about X if A_i and X are the same side of Γ, that is, $A_i \in T^s$ and $X \in T_*^s$, $s = \pm$.

Theorem 4.1 *For every interface element $T \in \mathcal{T}_h^i$ and $\mathbf{u} \in \mathbf{PC}_{int}^2(T)$, assume* $\mathrm{A_i} \in \mathrm{T^s}$, $\mathrm{s} = \pm$, *then*

$$\mathbf{u}^s(A_i) = \mathbf{u}^s(X) + ((A_i - X)^T \otimes I_2)Vec(\nabla \mathbf{u}^s(X)) + \mathbf{R}_i^s(X), \quad i \in \mathcal{I}^s, \ \forall X \in T_*^s, \tag{4.1}$$

$$\textit{where} \quad \mathbf{R}_i^s(X) = \int_0^1 (1-t)\frac{d^2}{dt^2}\mathbf{u}^s(Y_i(t,X))dt, \quad i \in \mathcal{I}^s, \ \forall X \in T_*^s. \tag{4.2}$$

Proof. Since $A_i \in T^s$ and $X \in T_*^s$, we know that $Y_i(t,X) \in T^s, \forall t \in [0,1]$. Applying the standard Taylor expansion with integral remainder to the components of $\mathbf{u}(X) = (u_1(X), u_2(X))^T$, we have

$$\mathbf{u}^s(A_i) = \mathbf{u}^s(X) + \nabla \mathbf{u}^s(X)(A_i - X) + \mathbf{R}_i^s(X), \quad i \in \mathcal{I}^s, \ \forall X \in T_*^s, \ s = \pm. \tag{4.3}$$

Then, we obtain (4.1) by applying the vectorization on each side of (4.3) and using the (2.7) with $C = I_2$, $E = A_i - X$, and $D = \nabla \mathbf{u}^s(X)$. ■

In the discussion from now on, we denote $s = \pm$ and $s' = \mp$, which means s and s' always take opposite signs when they appear in the same formula. And in the following theorem, we describe how to expand $\mathbf{u}(A_i)$ about X if they are the different sides of Γ, that is, $A_i \in T^s$ but $X \in T_*^{s'}$.

Theorem 4.2 *On every interface element $T \in \mathcal{T}_h^i$ and $\mathbf{u} \in \mathbf{PC}_{int}^2(T)$, assume* $\mathrm{A_i} \in \mathrm{T^{s'}}$, *then*

$$\begin{aligned}\mathbf{u}^{s'}(A_i) =& \mathbf{u}^s(X) + ((A_i - X)^T \otimes I_2)\mathrm{Vec}(\nabla \mathbf{u}^s(X)) \\ &+ ((A_i - \widetilde{Y}_i)^T \otimes I_2)(M^s - I_4)\mathrm{Vec}(\nabla \mathbf{u}^s(X)) + \mathbf{R}_i^s(X), i \in \mathcal{I}^{s'}, \forall X \in T_*^s, s = \pm,\end{aligned} \tag{4.4}$$

where $\mathbf{R}_i^s = \mathbf{R}_{i1}^s + \mathbf{R}_{i2}^s + \mathbf{R}_{i3}^s$, *with*

$$\begin{cases} \mathbf{R}_{i1}^s(X) = \int_0^{\widetilde{t}_i} (1-t)\frac{d^2}{dt^2}\mathbf{u}^s(Y_i(t,X))dt, \\ \mathbf{R}_{i2}^s(X) = \int_{\widetilde{t}_i}^1 (1-t)\frac{d^2}{dt^2}\mathbf{u}^{s'}(Y_i(t,X))dt, \\ \mathbf{R}_{i3}^s(X) = (1-\widetilde{t}_i)((A_i - X)^T \otimes I_2)(M^s(\widetilde{Y}_i) - I_4)\int_0^{\widetilde{t}_i} \frac{d}{dt}Vec(\nabla \mathbf{u}^s(Y_i(t,X)))dt. \end{cases} \tag{4.5}$$

Proof. Without loss of generality, we only discuss the case $A_i \in T^+$ and $X \in T_*^-$. Following a procedure similar to that used in [29], we have

$$\begin{aligned}\mathbf{u}^+(A_i) &= \mathbf{u}^-(X) + \int_0^{\widetilde{t}_i} \frac{d}{dt}\mathbf{u}^-(Y_i(t,X))dt + \int_{\widetilde{t}_i}^1 \frac{d}{dt}\mathbf{u}^+(Y_i(t,X))dt \\ &= \mathbf{u}^-(X) + \nabla \mathbf{u}^-(X)(A_i - X) - \nabla \mathbf{u}^-(\widetilde{Y}_i)(A_i - \widetilde{Y}_i) + \nabla \mathbf{u}^+(\widetilde{Y}_i)(A_i - \widetilde{Y}_i) \\ &\quad + \int_0^{\widetilde{t}_i} (1-t)\frac{d^2}{dt^2}\mathbf{u}^-(Y_i(t,X))dt + \int_{\widetilde{t}_i}^1 (1-t)\frac{d^2}{dt^2}\mathbf{u}^+(Y_i(t,X))dt,\end{aligned} \tag{4.6}$$

where the last two terms are actually $\mathbf{R}_{i1}^-$ and $\mathbf{R}_{i2}^-$. For the second and the third term on the right hand side of (4.6), by applying (2.7), we have

$$\begin{aligned}\nabla \mathbf{u}^-(X)(A_i - X) &= ((A_i - X)^T \otimes I_2)\mathrm{Vec}(\nabla \mathbf{u}^-(X)), \\ \nabla \mathbf{u}^-(\widetilde{Y}_i)(A_i - \widetilde{Y}_i) &= ((A_i - \widetilde{Y}_i)^T \otimes I_2)\mathrm{Vec}(\nabla \mathbf{u}^-(\widetilde{Y}_i)).\end{aligned} \tag{4.7}$$

For the fourth term on the right hand side of (4.6), by applying (2.7) and Lemma 3.1, we have

$$\nabla\mathbf{u}^+(\widetilde{Y}_i)(A_i-\widetilde{Y}_i)=((A_i-\widetilde{Y}_i)^T\otimes I_2)\mathrm{Vec}(\nabla\mathbf{u}^+(\widetilde{Y}_i))$$
$$=(1-\widetilde{t}_i)((A_i-X)^T\otimes I_2)M^-(\widetilde{Y}_i)\mathrm{Vec}(\nabla\mathbf{u}^-(\widetilde{Y}_i)). \quad (4.8)$$

Moreover we note that

$$\nabla\mathbf{u}^-(\widetilde{Y}_i)=\int_0^{\widetilde{t}_i}\frac{d}{dt}\nabla\mathbf{u}^-(Y_i(t,x))dt+\nabla\mathbf{u}^-(X). \quad (4.9)$$

Finally, expansion (4.4) follows from substituting (4.9), (4.8), (4.7) into (4.6). ■

For $X\in T_*$, we consider another group of expansion which only involves the first derivative of $\mathbf{u}$.

Theorem 4.3 *On every interface element $T\in\mathcal{T}_h^i$, $\mathbf{u}\in\mathbf{PC}^2_{int}(T)$, for each $X\in T_*$, we have*

$$\mathbf{u}(A_i)=\mathbf{u}(X)+\widetilde{\mathbf{R}}_i(X),\quad \textit{with}\ \ \widetilde{\mathbf{R}}_i(X)=\int_0^1\frac{d}{dt}\mathbf{u}(Y_i(t,X))dt. \quad (4.10)$$

Proof. The proof follows from a straightforward application of the same arguments used in [50] that only relies on the continuity of $\mathbf{u}$. ■

We proceed to estimate the remainders in (4.2) and (4.5) in terms of the Hilbert norms associated with $\mathbf{PH}^2_{int}(T)$. For every scalar function u, let $\nabla^2 u$ be its Hessian matrix. Then we note that

$$\frac{d^2}{dt^2}\mathbf{u}(Y_i(t,X))=\begin{bmatrix}(A_i-X)^T\nabla^2u_1\ (A_i-X)\\(A_i-X)^T\nabla^2u_2\ (A_i-X)\end{bmatrix}, \quad (4.11)$$

$$\frac{d}{dt}(\nabla\mathbf{u}(Y_i(t,X)))=\begin{bmatrix}(A_i-X)^T\nabla^2u_1\\(A_i-X)^T\nabla^2u_2\end{bmatrix}. \quad (4.12)$$

Therefore, we have

Lemma 4.1 *Let $\mathbf{u}\in\mathbf{PC}^2_{int}(T)$, there exist constants $C>0$ independent of interface location such that*

$$\|\mathbf{R}_i^s\|_{0,T_*^s}\le Ch^2|\mathbf{u}|_{2,T},\ \ i\in\mathcal{I}^s,\ s=\pm,$$
$$\|\mathbf{R}_{i1}^s\|_{0,T_*^s}\le Ch^2|\mathbf{u}|_{2,T},\ \ \|\mathbf{R}_{i2}^s\|_{0,T_*^s}\le Ch^2|\mathbf{u}|_{2,T},\ \ i\in\mathcal{I}^{s'},\ s=\pm. \quad (4.13)$$

Proof. Let $\mathbf{R}_i^s=(R_i^{1s},R_i^{2s})^T$, then according to (4.11), using Minkowski inequality and the fact $\|A_i-X\|\le h$, we have

$$R_i^{js}(X)=\left(\int_{T_*^s}\left(\int_0^1(1-t)(A_i-X)^T\nabla^2u_j^s(Y_i(t,X))(A_i-X)dt\right)^2dX\right)^{\frac12}$$
$$\le Ch^2\int_0^1\left(\int_{T_*^s}(1-t)^2\sum_{k,l=1}^2|\partial_{x_kx_l}u_j^s|^2\right)^{\frac12}dt\le Ch^2|u_j|_{2,T},\quad j=1,2,$$

where we have used arguments similar to those used for the Lemma 4.1 in [48], and these estimates lead to the first estimate in (4.13). The derivations for the estimates of $\mathbf{R}_{i1}^s$ and $\mathbf{R}_{i2}^s$ are similar. ■

Lemma 4.2 *Let* $\mathbf{u} \in \mathbf{PC}^2_{int}(T)$*, there exist constants* $C>0$ *independent of interface location such that*

$$\|\mathbf{R}^s_{i3}\|_{0,T^s_*} \le Ch^2|\mathbf{u}|_{2,T}, \quad i \in \mathcal{I}^{s'},\ s=\pm. \tag{4.14}$$

Proof. Let $\mathbf{R}^s_{i3} = (R^{1s}_{i3}, R^{2s}_{i3})^T$. Using (4.12), (3.9), the fact $\|A_i - X\| \le h$ and $0 \le 1 - \widetilde{t}_i(X) \le 1-t$, we have

$$\|R^{js}_{i3}\|_{0,T^s_*} \le Ch^2\left(\int_{T^s_*}\left(\int_0^{\widetilde{t}_i}(1-t)\sum_{k,l=1}^{2}\sum_{j=1}^{2}|\partial_{x_kx_l}u_j|dt\right)^2 dX\right)^{\frac{1}{2}}.$$

Then, applying the Minkowski inequality and Lemma 4.1 in [48] to the inequality above yields

$$\|R^{js}_{i3}\|_{0,T^s_*} \le Ch^2\int_0^{\widetilde{t}_i}\left(\sum_{k,l=1}^{2}\sum_{j=1}^{2}\int_{T^s_*}(1-t)^2|\partial_{x_kx_l}u_j|^2dX\right)^{\frac{1}{2}}dt \le Ch^2(|u_1|_{2,T}+|u_2|_{2,T}),$$

from which (4.14) readily follows. ∎

In addition, since $\mathbf{u} \in [H^2(T^s)]^2$, the Sobolev embedding theorem indicates $\mathbf{u} \in [W^{1,6}(T^s)]^2$, $s=\pm$. Therefore we can bound the remainder $\widetilde{\mathbf{R}}_i$ in (4.10) in terms of $W^{1,6}$-norm.

Lemma 4.3 *There exists a constant* C *independent of the interface location such that when* h *is small enough there holds*

$$\|\widetilde{\mathbf{R}}_i\|_{0,T_*} \le Ch^2\|\mathbf{u}\|_{1,6,T}. \tag{4.15}$$

Proof. We note that $T_* = \widetilde{T} \cup T_{int}$, and it is a small set such that $|T_*| \le Ch^3$ when the mesh is fine enough because $|\widetilde{T}| \le Ch^3$ [29] and $|T_*| \le Ch^3$ by Lemma 3.4. Note that $\widetilde{\mathbf{R}}_i = (\widetilde{R}^1_i, \widetilde{R}^2_i)^T$, and, by using (4.12), we have

$$\widetilde{R}^j_i(X) = \int_0^1 \nabla u_j(Y_i(t,X))\,(A_i - X)dt, \quad j=1,2.$$

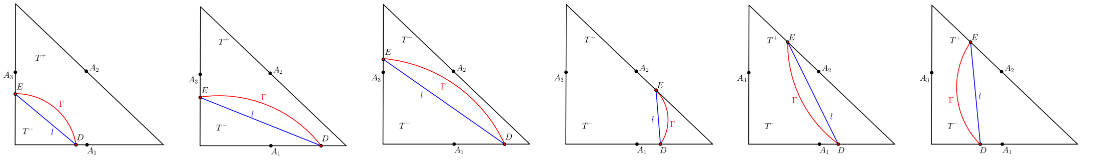
Then, applying arguments similar to those used for Lemma 3.2 in [50] and using the fact $|T_*| \le Ch^3$, we have $\|\widetilde{R}^j_i\|_{0,T_*} \le Ch^2\|u_j\|_{1,6,T}$ for $j=1, 2$ from which (4.15) follows. ∎

5 | CONSTRUCTION OF IFE SPACES

In this section, we construct local IFE spaces corresponding to their related finite element spaces $(T, \mathbf{\Pi}_T, \mathbf{\Sigma}_T)$ described in Section 2. As usual the local IFE space on every noninterface element T is the standard vector polynomial space, that is,

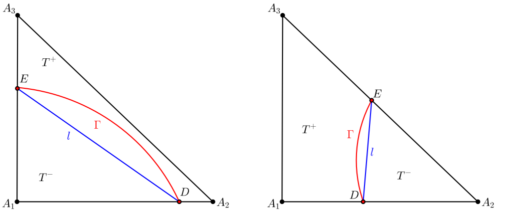
$$\mathbf{S}_h(T) = \text{Span}\{\boldsymbol{\psi}_{i,T},\ \boldsymbol{\psi}_{i+|\mathcal{I}|,T} : i \in \mathcal{I}\}, \tag{5.1}$$

where $\boldsymbol{\psi}_{i,T}$ are given by (2.4). We note that a procedure to construct the local IFE spaces formed by piecewise linear polynomials on interface elements is discussed in [43, 45], and a similar procedure is presented in [45] for the local IFE spaces formed by piecewise bilinear polynomials. However, according to the example presented in [45], the linear system for determining an IFE shape function in these procedures can be singular in some cases. We now propose a new procedure so that the bilinear



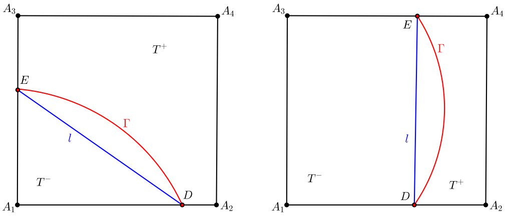
(a) Case 1 (b) Case 2 (c) Case 3 (d) Case 4 (e) Case 5 (f) Case 5

FIGURE 1 Typical nonconforming linear elements [Color figure can be viewed at wileyonlinelibrary.com]



(a) Case 1 (b) Case 2

FIGURE 2 Typical conforming linear elements [Color figure can be viewed at wileyonlinelibrary.com]



(a) Case 1 (b) Case 2

FIGURE 3 Typical bilinear elements [Color figure can be viewed at wileyonlinelibrary.com]

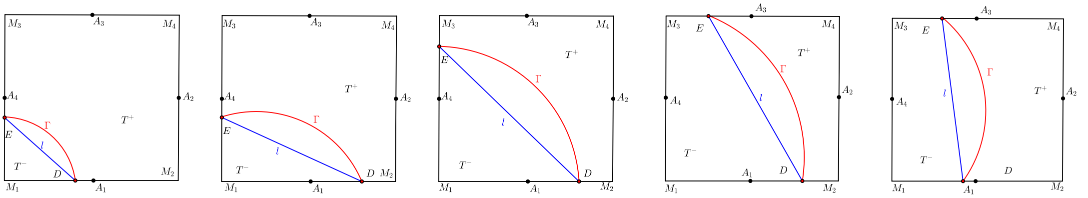
or the rotated Q_1 IFE shape functions on every interface element can always be uniquely determined by the local degrees of freedom $\mathbf{\Sigma}_T$.

5.1 | Local IFE spaces

Without loss of generality, we consider a typical interface element $T \in \mathcal{T}_h^i$ with $A_1 = (0,0)^T$, $A_2 = (h,0)^T$, $A_3 = (0,h)^T$ when the conforming linear polynomials are discussed on a triangular T, $A_1 = (h/2,0)^T$, $A_2 = (h/2,h/2)^T$, $A_3 = (0,h/2)^T$ when the nonconforming linear polynomials are discussed on a triangular T, $A_1 = (0,0)^T$, $A_2 = (h,0)^T$, $A_3 = (0,h)^T$, $A_4 = (h,h)^T$ for the bilinear case on a rectangular T, and $A_1 = (h/2,0)^T$, $A_2 = (h,h/2)^T$, $A_3 = (h/2,h)^T$, $A_4 = (0,h/2)^T$ for the rotated Q_1 case on a rectangular T. We further denote the vertices of the rectangular element by $M_1 = (0,0)^T$, $M_2 = (h,0)^T$, $M_3 = (0,h)^T$, and $M_4 = (h,h)^T$, when the rotated Q_1 elements are considered. According to [48], by considering rotation, there are six possible cases of the interface element configuration for the nonconforming linear case, two possible cases for the conforming linear and bilinear cases, and five possible cases for the rotated Q_1 case, as illustrated in Figures 1–4.

On an interface element T, we consider the elasticity IFE functions as piecewise vector polynomials in the following format:

$$\phi_T(X) = \begin{cases} \phi_T^-(X) \in \mathbf{\Pi}_T, & \text{if} \quad X \in \overline{T}^-, \\ \phi_T^+(X) \in \mathbf{\Pi}_T, & \text{if} \quad X \in \overline{T}^+, \end{cases} \tag{5.2}$$



(a) Case 1 (b) Case 2 (c) Case 3 (d) Case 4 (e) Case 5

FIGURE 4 Typical rotated Q_1 elements [Color figure can be viewed at wileyonlinelibrary.com]

with $\phi_T^+(X)$ and $\phi_T^-(X)$ satisfying that

$$\begin{cases} \phi_T^-|_l = \phi_T^+|_l, & \text{(for the linear case)}, \\ \phi_T^-|_l = \phi_T^+|_l, \quad d(\phi_T^-) = d(\phi_T^+), & \text{(for the bilinear/rotated } Q_1 \text{ case)}, \end{cases} \tag{5.3}$$

$$\sigma^+(\phi_T^+)(F)\ \overline{\mathbf{n}} = \sigma^-(\phi_T^-)(F)\ \overline{\mathbf{n}}, \tag{5.4}$$

where F is a point on l which will be specified later and $d(\boldsymbol{\psi})$ is a vector formed by the coefficients of the second degree term of $\boldsymbol{\psi} \in \boldsymbol{\Pi}_T$, that is, the coefficient of xy for a bilinear polynomial or the coefficient of $x^2 - y^2$ for a rotated Q_1 polynomial. Given a set of nodal-value vectors $\mathbf{v}_i$, $i \in \mathcal{I}$, we further impose the nodal value condition:

$$\phi_T(A_i) = \mathbf{v}_i. \tag{5.5}$$

Let $\Psi_{i,T} = [\boldsymbol{\psi}_{i,T}, \boldsymbol{\psi}_{i+|\mathcal{I}|,T}]$, $i \in \mathcal{I}$ which is a 2-by-2 matrix basis function and let $L(X) = 0$ be the equation of the line l with $L(X) = \overline{\mathbf{n}} \cdot (X - D)$. It is easy to see that $\Psi_{i,T}(A_j) = \delta_{i,j} I_2$, $i, j \in \mathcal{I}$. Without loss of generality, we assume that $|\mathcal{I}^+| \geq |\mathcal{I}^-|$. Then by (5.5), (5.3), we can express (5.2) as

$$\phi_T(X) = \begin{cases} \phi_T^-(X) = \phi_T^+(X) + L(X)\mathbf{c}_0 & \text{if } X \in \overline{T}^-, \\ \phi_T^+(X) = \sum\limits_{i \in \mathcal{I}^+} \Psi_{i,T}(X)\mathbf{v}_i + \sum\limits_{i \in \mathcal{I}^-} \Psi_{i,T}(X)\mathbf{c}_i & \text{if } X \in \overline{T}^+, \end{cases} \tag{5.6}$$

where $\mathbf{c}_0 = (c_0^1, c_0^2)^T$ and $\mathbf{c}_i = (c_i^1, c_i^2)^T$, $i \in \mathcal{I}^-$ are to be determined. Applying the jump condition for the stress tensor (5.4)–(5.6), we obtain

$$\sigma^-(L\mathbf{c}_0)(F)\overline{\mathbf{n}} = \hat{\sigma}(\phi_T^+)(F)\overline{\mathbf{n}}, \tag{5.7}$$

where $\hat{\sigma}(\mathbf{v})(X)$ for a vector function $\mathbf{v}$ is defined as follows:

$$\hat{\sigma}(\mathbf{v}) = (\hat{\sigma}_{ij}(\mathbf{v}))_{1 \leq i,j \leq 2}, \quad \hat{\sigma}_{ij}(\mathbf{v}) = \hat{\lambda}(\nabla \cdot \mathbf{v})\delta_{i,j} + 2\hat{\mu}\varepsilon_{ij}(\mathbf{v}), \quad \text{with } \hat{\lambda} = \lambda^+ - \lambda^-, \hat{\mu} = \mu^+ - \mu^-. \tag{5.8}$$

Also, in $\hat{\sigma}(\phi_T^+)(X)$, the function ϕ_T^+ is a polynomial so that it can be evaluated for any X, and this meaning applies to similar situations from now on. By direct calculations, we have

$$\sigma^-(L\mathbf{c}_0)(F) = \begin{bmatrix} \overline{n}_1^2(\lambda^- + \mu^-) + \mu^- & \overline{n}_1\overline{n}_2(\lambda^- + \mu^-) \\ \overline{n}_1\overline{n}_2(\lambda^- + \mu^-) & \overline{n}_2^2(\lambda^- + \mu^-) + \mu^- \end{bmatrix} \begin{bmatrix} c_0^1 \\ c_0^2 \end{bmatrix} := K\mathbf{c}_0. \tag{5.9}$$

Then we note that

$$K = Q\mathcal{P}^- Q^T, \quad \text{with } \mathcal{P}^- = \begin{bmatrix} (\lambda^- + 2\mu^-) & 0 \\ 0 & \mu^- \end{bmatrix}, \quad Q = [\overline{\mathbf{n}}, \overline{\mathbf{t}}], \tag{5.10}$$

which is obviously nonsingular. Hence $\mathbf{c}_0$ is determined by

$$\mathbf{c}_0 = K^{-1}\hat{\sigma}(\phi_T^+)(F)\overline{\mathbf{n}}. \tag{5.11}$$

Next we apply the nodal value (5.5) for $j \in \mathcal{I}^-$ and (5.11) to (5.6) to obtain

$$K\mathbf{c}_j + L(A_j)\sum_{i\in\mathcal{I}^-}\hat{\sigma}(\Psi_{i,T}\mathbf{c}_i)(F)\overline{\mathbf{n}} = K\mathbf{v}_j - L(A_j)\sum_{i\in\mathcal{I}^+}\hat{\sigma}(\Psi_{i,T}\mathbf{v}_i)(F)\overline{\mathbf{n}},\ \ j \in \mathcal{I}^-. \tag{5.12}$$

We now put equations in (5.12) into a matrix form. To this end, we first let $(j_1, j_2, \dots, j_{|\mathcal{I}|})$ be a permutation of $(1, 2, \dots, |\mathcal{I}|)$ such that $j_k \in \mathcal{I}^-$ for $1 \le k \le |\mathcal{I}^-|$ but $j_k \in \mathcal{I}^+$ for $|\mathcal{I}^-| + 1 \le k \le |\mathcal{I}|$. Consider three vectors $\mathbf{c}$, $\mathbf{v}^-$, and $\mathbf{v}^+$ such that

$$\mathbf{c} = [\mathbf{c}_{j_k}]_{k=1}^{|\mathcal{I}^-|} \in \mathbb{R}^{2|\mathcal{I}^-|},\ \ \mathbf{v}^- = [\mathbf{v}_{j_k}]_{k=1}^{|\mathcal{I}^-|} \in \mathbb{R}^{2|\mathcal{I}^-|},\ \ \mathbf{v}^+ = [\mathbf{v}_{j_k}]_{k=|\mathcal{I}^-|+1}^{|\mathcal{I}|} \in \mathbb{R}^{2|\mathcal{I}^+|}.$$

We adopt the following notations:

$$\overline{K} = I_{|\mathcal{I}^-|} \otimes K \in \mathbb{R}^{2|\mathcal{I}^-|\times 2|\mathcal{I}^-|},\ \ \overline{L} = [L(A_{j_k})I_2]_{k=1}^{|\mathcal{I}^-|} \in \mathbb{R}^{2|\mathcal{I}^-|\times 2}, \tag{5.13a}$$

$$\overline{\Psi}^- = [\overline{\Psi}_{j_k}]_{k=1}^{|\mathcal{I}^-|} \in \mathbb{R}^{2|\mathcal{I}^-|\times 2},\ \ \overline{\Psi}^+ = [\overline{\Psi}_{j_k}]_{k=|\mathcal{I}^-|+1}^{|\mathcal{I}|} \in \mathbb{R}^{2|\mathcal{I}^+|\times 2}, \tag{5.13b}$$

$$\text{with}\ \ \overline{\Psi}_j = \left[\hat{\sigma}(\boldsymbol{\psi}_{j,T})(F)\overline{\mathbf{n}}\ \ \hat{\sigma}(\boldsymbol{\psi}_{j+|\mathcal{I}|,T})(F)\overline{\mathbf{n}}\right]^T \in \mathbb{R}^{2\times 2},\ \ 1 \le j \le |\mathcal{I}|. \tag{5.13c}$$

For any vector $\mathbf{r} \in \mathbb{R}^{2\times 1}$, we note the identity $\hat{\sigma}(\Psi_{i,T}\mathbf{r})(F)\overline{\mathbf{n}} = \overline{\Psi}_i^T\mathbf{r}$. Hence by using the matrices defined in (5.13a)–(5.13c), we can represent equations in (5.12) as follows:

$$(\overline{K} + \overline{L}\,\overline{\Psi}^{-T})\mathbf{c} = \mathbf{b}, \tag{5.14}$$

$$\text{with}\quad \mathbf{b} = \overline{K}\mathbf{v}^- - \overline{L}\,\overline{\Psi}^{+T}\mathbf{v}^+. \tag{5.15}$$

We note that the coefficient matrix in (5.14) is a generalized *Sherman–Morrison* matrix formed by matrices $\overline{K}, \overline{L}$ and $\overline{\Psi}$. Here, we proceed to discuss the unisolvence for the bilinear and the rotated Q_1 IFE functions, that is, the invertibility of the matrix in (5.14), which can be always guaranteed with a suitable choice for F through the proposed new construction procedure. The discussion for the linear case will be left to Remark 5.3.

First, for the rotated Q_1 IFE functions in Case 1 as illustrated in Figure 4a, we note that there is no $\mathbf{c}_i$, $i \in \mathcal{I}^-$ coefficients in the formulation (5.6) and $\mathbf{c}_0$ is uniquely determined by (5.11), and this means that the unisolvence for this case is always guaranteed. To discuss other cases, we define two parameters d and e for describing the interface-element intersection points D and E for those typical rectangular interface elements illustrated in Figures 3 and 4:

- We let $d = \|D - A_1\|/h$, $e = \|E - A_1\|/h$ for Case 1 in Figure 3 and $d = \|D - M_1\|/h$, $e = \|E - M_1\|/h$ for Case 2 and Case 3 in Figure 4.
- We let $d = \|D - A_1\|/h$, $e = \|E - A_3\|/h$ for Case 2 in Figure 3 and $d = \|D - M_1\|/h$, $e = \|E - M_3\|/h$ for Case 4 and Case 5 in Figure 4.

We start from some estimates for the following two auxiliary functions:

$$g_n(X) = \sum_{i\in\mathcal{I}^-} L(A_i)\nabla\psi_{i,T}(X)\cdot\overline{\mathbf{n}},\quad g_t(X) = \sum_{i\in\mathcal{I}^-} L(A_i)\nabla\psi_{i,T}(X)\cdot\overline{\mathbf{t}}. \tag{5.16}$$

Lemma 5.1 *On each rectangular interface element $T \in \mathcal{T}_h^i$, let $F_0 = t_0 D + (1 - t_0)E$ such that*

- *when it is a bilinear element in Case 1 illustrated in Figure 3, assume $t_0 = e/(d + e)$,*
- *when it is a bilinear element in Case 2 illustrated in Figure 3, assume $t_0 = 1 - e$ if $d \ge e$, $t_0 = 1 - d$ if $e > d$,*

- *when it is a rotated* Q_1 *element in Case 2 or Case 3 illustrated in Figure 4, assume* $t_0 = 1$ *when* $d \geq e$ *or* $t_0 = 0$ *when* $e > d$,
- *when it is a rotated* Q_1 *element in Case 4 or Case 5 illustrated in Figure 4, assume* $t_0 = 1/2$.

Then

$$(1 - g_n(F_0))^2 - g_t^2(F_0) \geq 0, \quad g_n^2(F_0) - g_t^2(F_0) \geq 0, \tag{5.17a}$$

$$g_n(F_0) \in [0, 1], \quad g_t(F_0) \in [-1, 1]. \tag{5.17b}$$

Proof. Here, we only provide a proof for the Case 1 of the bilinear elements and similar arguments can be applied to other cases, see more details provided in the appendix. By direct calculation, we can verify that

$$g_n^2(F_0) - g_t^2(F_0) = \frac{4d^3e^3(d+e-de)^2}{(d^2+e^2)^2(d+e)^2} \geqslant 0;$$

$$(1 - g_n(F_0))^2 - g_t^2(F_0) = \frac{-d^2e^2(e^2(1-d) - d^2(1-e))^2 + (e^2(1-d) + d^2(1-e+e^2))^2(d+e)^2}{(d^2+e^2)^2(d+e)^2} \geq 0,$$

which lead to (5.17a). For (5.17b), the first inequality is just a special case of Lemma 5.1 in [48] and the second inequality is a consequence of (5.17a). ∎

Lemma 5.2 *The matrix in the linear system (5.14) is nonsingular if and only if the following matrix is nonsingular:*

$$\Xi(F) = \mathcal{P}^- + \begin{bmatrix} (\hat{\lambda} + 2\hat{\mu}) g_n(F) & \hat{\lambda} g_t(F) \\ \hat{\mu} g_t(F) & \hat{\mu} g_n(F) \end{bmatrix}. \tag{5.18}$$

Proof. Note that the matrix in (5.14) is in a generalized *Sherman–Morrison* format. Since $\overline{K}$ is invertible, the linear system (5.14) is nonsingular if and only if the matrix

$$I_2 + \overline{\Psi}^{-T}\overline{K}^{-1}\overline{L} = I_2 + \sum_{j \in \mathcal{I}^-} L(A_j)\overline{\Psi}_j^T K^{-1} = \left(K + \sum_{j \in \mathcal{I}^-} L(A_j)\overline{\Psi}_j \right) K^{-1} \tag{5.19}$$

is invertible. Then by using (5.10), we can directly verify that

$$Q \left(K + \sum_{j \in \mathcal{I}^-} L(A_j)\overline{\Psi}_j \right) Q^T = P^- + \begin{bmatrix} (\hat{\lambda} + 2\hat{\mu}) g_n(F) & \hat{\lambda} g_t(F) \\ \hat{\mu} g_t(F) & \hat{\mu} g_n(F) \end{bmatrix} \tag{5.20}$$

which leads to the conclusion of this lemma because Q is invertible. ∎

Lemma 5.3 *With the* F_0 *specified in Lemma 5.1, we have*

$$\mathrm{Det}(\Xi(F_0)) > 2(\min\{\mu^+, \mu^-\})^2. \tag{5.21}$$

Proof. By (5.18) and direct calculations, we have

$$\begin{aligned} \mathrm{Det}(\Xi(F_0)) =& \lambda^+\mu^+(g_n^2 - g_t^2) + \lambda^-\mu^-((1-g_n)^2 - g_t^2) \\ &+ \lambda^-\mu^+((1-g_n)g_n + g_t^2) + \lambda^+\mu^-(g_n(1-g_n) + g_t^2) \\ &+ 2(\mu^+)^2 g_n^2 + 2(\mu^-)^2(1-g_n)^2 + 4\mu^+\mu^-(1-g_n)g_n, \end{aligned}$$

in which $g_n = g_n(F_0)$ and $g_t = g_t(F_0)$. Then, applying estimates in Lemma 5.1 to the above, we have $\mathrm{Det}(\Xi(F_0)) \geq 2(\mu^+)^2 g_n^2 + 2(\mu^-)^2(1-g_n)^2 > 2(\min\{\mu^+, \mu^-\})^2$. ∎

Finally we can prove the following main theorem in this section.

Theorem 5.1 **(Unisolvence).** *Let $T \in \mathcal{T}_h^i$ be a rectangular interface element with* $\mathrm{F} = \mathrm{F}_0$ *specified in Lemma 5.1. Then given any vector* $\mathbf{v} \in \mathbb{R}^{2|\mathcal{I}^-|\times 1}$, *for the bilinear and rotated* Q_1 *elements, there exists one and only one IFE shape function satisfying (5.2)–(5.5).*

Proof. The proof is directly based on Lemmas 5.2 and 5.3. ■

Remark 5.1 According to the generalized Sherman–Morrison formula and (5.19), (5.20), we can give an analytical formula for the coefficients **c** in (5.14) as

$$\begin{aligned}\mathbf{c} &= \overline{K}^{-1}\mathbf{b} - \overline{K}^{-1}\overline{L}(I_2 + \overline{\Psi}^{-T}\overline{K}^{-1}\overline{L})^{-1}\overline{\Psi}^{-T}\overline{K}^{-1}\mathbf{b} \\ &= \overline{K}^{-1}\mathbf{b} - \overline{K}^{-1}\overline{L}KQ^T\Xi^{-1}Q\overline{\Psi}^{-T}\overline{K}^{-1}\mathbf{b}.\end{aligned} \tag{5.22}$$

Here it is important to note that $\overline{K}$ is a diagonal block matrix formed only by the 2-by-2 matrix K and Ξ is also a 2-by-2 matrix so that their inverses are easy to calculate analytically. Hence, if preferred, there is no need to solve for **c** numerically because of (5.22).

Remark 5.2 When $F = (D + E)/2$, the linear and bilinear IFE shape functions given by (5.6) with the coefficients determined by (5.11) and (5.14) coincide with those in [45].

Remark 5.3 For the linear IFE functions, because each side in (5.4) is a constant vector which is therefore independent of the location of F, the new construction procedure proposed above is the same as the one considered in [43, 45], that is, the one in Remark 5.2, regardless of the choice of F_0. In this case, the authors in [45] identified a specific interface element configuration such that the conforming linear IFE shape functions cannot be uniquely determined by the Lagrange type degrees of freedom, that is, the nodal values, and they also showed that the unisolvence can be conditionally guaranteed by some assumptions on the Lamé parameters (Theorem 4.7 in [45]). The immersed nonconforming linear elements, that is, the linear Crouzeix–Raviart IFE elements, also have a conditional unisolvence, which can be discussed similarly as Theorem 4.7 in [45].

Remark 5.4 In the bilinear and rotated Q_1 cases, the unisolvence of the IFE shape functions depends on suitable choices of the point F_0 stated in Lemma 5.1. It is easy to see that a small perturbation of a suitable choice of $F = F_0$ given in Lemma 5.1 can also yield $\det(\Xi(F)) > 0$ since the $\Xi(F)$ in (5.18) is a continuous function of F for a fixed interface location in an element. This means that each choice of F_0 is not unique and Lemma 5.1 only provides sufficient conditions for the unisolvence. We also note that, because of the continuous dependence of **c** given in (5.22) and $\mathbf{c}_0$ given in (5.11) on F, a small perturbation of $F = F_0$ should lead to a small change of the coefficients, and thus the corresponding IFE shape functions will not change much.

By taking the nodal value vector **v** to be unit vectors, we construct the IFE shape functions satisfying the weak jump (5.3) and (5.4) and

$$\phi_{i,T}(A_j) = \begin{cases} \delta_{i,j}, \\ 0, \end{cases} \quad i = 1, \dots, |\mathcal{I}|, \quad \text{and} \quad \phi_{i,T}(A_j) = \begin{cases} 0, \\ \delta_{i-|\mathcal{I}|,j}, \end{cases} \quad i = |\mathcal{I}| + 1, \dots, 2|\mathcal{I}|. \tag{5.23}$$

The local IFE spaces on interface elements $T \in \mathcal{T}_h^i$ are then defined as

$$\mathbf{S}_h(T) = \mathrm{Span}\{\phi_{i,T}, \ \phi_{i+|\mathcal{I}|,T} : \ i \in \mathcal{I}\}. \tag{5.24}$$

The local IFE spaces defined by (5.1) and (5.24) can be used to construct an IFE space over the whole domain Ω according to the need of a finite element scheme. For example, by enforcing the continuity at the mesh nodes, we can consider the following global IFE space:

$$\mathbf{S}_h(\Omega) = \{\mathbf{v} \in [L^2(\Omega)]^2 : \mathbf{v}|_T \in \mathbf{S}_h(T); \\ \mathbf{v}|_{T_1}(N) = \mathbf{v}|_{T_2}(N) \forall N \in \mathcal{N}_h, \forall T_1, T_2 \in \mathcal{T}_h \text{ such that } N \in T_1 \cap T_2\}. \tag{5.25}$$

5.2 | Properties of IFE shape functions

In this subsection, we discuss some fundamental properties of the proposed IFE shape functions. We tacitly assume that, on each interface element $T \in \mathcal{T}_h^i$, $\phi_{i,T}, 1 \leq i \leq 2 \mid \mathcal{I} \mid$ are the bilinear or the rotated Q_1 IFE shape functions constructed according to Theorem 5.1 or they are the conforming/nonconforming linear IFE shape functions which uniquely exist according to their degrees of freedom under some conditions on the Lamé parameters, see Remark 5.3.

Theorem 5.2 **(Boundedness).** *There exists a constant* C *such that the following estimates are valid for IFE shape functions on each interface element:*

$$|\phi_{i,T}|_{k,\infty,T} \leq Ch^{-k}, \quad k = 0, 1, 2, \quad 1 \leq i \leq 2 \mid \mathcal{I} \mid, \quad \forall T \in \mathcal{T}_h^i. \tag{5.26}$$

Proof. For the bilinear or the rotated Q_1 IFE shape functions, we note that (5.10) yields $\|\overline{K}\| \leq C$. And (5.13a) shows $\|\overline{L}\| \leq Ch$ because $|L|_{0,\infty,T} \leq Ch$ and $\|\overline{\Psi}^s\| \leq Ch^{-1}$. So we have $\|\mathbf{b}\| \leq C$, of which the constants C only depends on Lamé parameters. Next (5.17b), (5.21) suggest $\|\Xi^{-1}\| \leq C$. So by the (5.22), we have $\|\mathbf{c}\| \leq C$ and then use (5.11) to show $\|\mathbf{c}_0\| \leq Ch^{-1}$. Therefore, $\|\mathbf{c}_0 L\|_{0,\infty,T} \leq C$ and $|\mathbf{c}_0 L|_{1,\infty,T} \leq Ch^{-1}$ because $|L|_{0,\infty,T} \leq Ch$ and $|L|_{1,\infty,T} \leq C$. In addition, it is easy to see $|\mathbf{c}_0 L|_{2,\infty,T} = 0$, since L is a linear function. Finally, applying these estimates and (2.5) to (5.6) leads to (5.26). Similar arguments can be used for the linear IFE shape functions. ■

For simplicity of presentation, we denote the following matrix shape functions:

$$\Phi_{i,T}(X) = [\phi_{i,T}(X), \phi_{i+|\mathcal{I}|,T}(X)], \quad i \in \mathcal{I}. \tag{5.27}$$

Theorem 5.3 **(Partition of unity).** *On each interface element $T \in \mathcal{T}_h^i$, we have*

$$\sum_{i \in \mathcal{I}} \Phi_{i,T}(X) = I_2, \quad \sum_{i \in \mathcal{I}} \partial_{x_j} \Phi_{i,T}(X) = \mathbf{0}_{2\times 2}, \quad \sum_{i \in \mathcal{I}} \partial_{x_j x_k} \Phi_{i,T}(X) = \mathbf{0}_{2\times 2}, \quad j, k = 1, 2. \tag{5.28}$$

Proof. By direct verifications, we can see that vector functions $\boldsymbol{\phi}^1 = (1, 0)^T$ and $\boldsymbol{\phi}^2 = (0, 1)^T$ satisfy the weak jump (5.3) and (5.4) exactly; hence, they are in the IFE space $\mathbf{S}_h(T)$. Then the unisolvence of the IFE function leads to the first identity in (5.28). And the second and third identity in (5.28) are just the derivatives of the first one. ■

Remark 5.5 The first identity in (5.28) is proved in [45] for the linear and bilinear IFE shape functions by direct verifications.

For the proposed IFE shape functions, we consider the following 2-by-4 matrix functions:

$$\Lambda_-(X) = \sum_{i \in \mathcal{I}} ((A_i - X)^T \otimes (\Phi_{i,T}^-(X))) + \sum_{i \in \mathcal{I}^+} ((A_i - \overline{X}_i)^T \otimes (\Phi_{i,T}^-(X)))(\overline{M}^- - I_4), \tag{5.29a}$$

$$\Lambda_+(X) = \sum_{i\in\mathcal{I}}((A_i - X)^T \otimes (\Phi_{i,T}^+(X))) + \sum_{i\in\mathcal{I}^-}((A_i - \overline{X}_i)^T \otimes (\Phi_{i,T}^+(X)))(\overline{M}^+ - I_4), \tag{5.29b}$$

where $\overline{X}_i,\ \ i \in \mathcal{I}$ are arbitrary points on l, and we further use them to define

$$\Lambda^+(X) = \Lambda_+(X), \qquad \Lambda^-(X) = \Lambda_-(X)\overline{M}^+. \tag{5.30}$$

First we show that both $\Lambda^-(X)$ and $\Lambda^+(X)$ are well defined that is, they are independent of $\overline{X}_i \in l$, $i \in \mathcal{I}$.

Lemma 5.4 *The matrix functions $\Lambda^-(\mathrm{X})$ and $\Lambda^+(\mathrm{X})$ are independent with the points $\overline{X}_i \in l,\ \ i \in \mathcal{I}$.*

Proof. Let $\overline{X}_i$ and $\overline{X}_i'$ be two points on l. Then, by Lemma 3.2, we have

$$\begin{aligned}&((A_i - \overline{X}_i)^T \otimes (\Phi_i^s(X)))(\overline{M}^s - I_4) - ((A_i - \overline{X}_i')^T \otimes (\Phi_i^s(X)))(\overline{M}^s - I_4)\\ &\quad = \|\overline{X}_i - \overline{X}_i'\|[\Phi_i^s, \Phi_i^s]\,[\alpha_1, \alpha_2](\overline{M}^s - I_4) = \mathbf{0}, \quad s = \pm,\end{aligned} \tag{5.31}$$

where, as in Lemma 3.2, $\alpha_1 = (-\overline{n}_2, 0, \overline{n}_1, 0)^T$ and $\alpha_2 = (0, -\overline{n}_2, 0, \overline{n}_1)^T$. Hence, by (5.29) and (5.30), functions $\Lambda^-(X)$ and $\Lambda^+(X)$ are independent of $\overline{X}_i,\ \ i \in \mathcal{I}$. ■

Lemma 5.4 allows us to consolidate $\overline{X}_i, i \in \mathcal{I}$ in $\Lambda^-(X)$ and $\Lambda^+(X)$ into a single point $\overline{X} \in l$. Then, by using (5.28), we rewrite these two functions as follows:

$$\Lambda^-(X) = \sum_{i\in\mathcal{I}^-}(A_i - \overline{X})^T \otimes \Phi_i^-(X)\overline{M}^+ + \sum_{i\in\mathcal{I}^+}(A_i - \overline{X})^T \otimes \Phi_i^-(X) - ((X - \overline{X})^T \otimes I_2)\overline{M}^+. \tag{5.32a}$$

$$\Lambda^+(X) = \sum_{i\in\mathcal{I}^-}(A_i - \overline{X})^T \otimes \Phi_i^+(X)\overline{M}^+ + \sum_{i\in\mathcal{I}^+}(A_i - \overline{X})^T \otimes \Phi_i^+(X) - (X - \overline{X})^T \otimes I_2, \tag{5.32b}$$

For every fixed $\overline{X}$, we consider the following piecewise 2-by-4 matrix function:

$$V(X) = \begin{cases}(X - \overline{X})^T \otimes I_2 & \text{if } X \in \overline{T}^+,\\ ((X - \overline{X})^T \otimes I_2)\overline{M}^+ & \text{if } X \in \overline{T}^-.\end{cases} \tag{5.33}$$

Lemma 5.5 *On every interface element $T \in \mathcal{T}_h^i$, each column of $\mathrm{V}(\mathrm{X})$ is in the local IFE space $\mathbf{S}_\mathrm{h}(\mathrm{T})$.*

Proof. Clearly, each column of $V(X)$ restricted on either $\overline{T}^+$ or $\overline{T}^-$ is in the corresponding polynomial space $\mathbf{\Pi}_T$. Furthermore, we note that $V^-(\overline{X}) = V^+(\overline{X})$ and

$$\begin{bmatrix}\partial_{x_1}V^+\\ \partial_{x_2}V^+\end{bmatrix} = I_4 = \overline{M}^+\overline{M}^- = \begin{bmatrix}\partial_{x_1}V^-\\ \partial_{x_2}V^-\end{bmatrix}\overline{M}^-,$$

which, together with the fact $d(V) = 0_{2\times 4}$, shows each column of V satisfies (5.3) and (5.4) simultaneously; thus, it is in the corresponding IFE space. ■

Theorem 5.4 *For every interface element $T \in \mathcal{T}_h$, we have the identities $\Lambda_+ = 0_{2\times 4}$ and $\Lambda_- = 0_{2\times 4}$. And let $\mathrm{I}^1 = [\mathrm{I}_2, 0_{2\times 2}]$, $\mathrm{I}^2 = [0_{2\times 2}, \mathrm{I}_2]$, for $\mathrm{j, k} = 1, 2$, $\mathrm{s} = \pm$, we also have*

$$\sum_{i\in\mathcal{I}}((A_i - X)^T \otimes (\partial_{x_j}\Phi_i^s(X))) + \sum_{i\in\mathcal{I}^{s'}}((A_i - \overline{X}_i)^T \otimes (\partial_{x_j}\Phi_i^s(X)))(\overline{M}^s - I_4) = I^j, \tag{5.34a}$$

$$\sum_{i\in\mathcal{I}}((A_i - X)^T \otimes (\partial_{x_jx_k}\Phi_i^s(X))) + \sum_{i\in\mathcal{I}^{s'}}((A_i - \overline{X}_i)^T \otimes (\partial_{x_jx_k}\Phi_i^s(X)))(\overline{M}^s - I_4) = 0_{2\times 4}. \tag{5.34b}$$

Proof. We construct a piecewise defined 2-by-4 matrix function:

$$\Lambda(X) = \begin{cases} \Lambda^-(X) & \text{if } X \in \overline{T}^-, \\ \Lambda^+(X) & \text{if } X \in \overline{T}^+. \end{cases} \tag{5.35}$$

According to (5.32b) and (5.32a), the first two terms in $\Lambda^-(X)$ and $\Lambda^+(X)$ are the linear combination of the IFE shape functions with the same coefficients. Lemma 5.5 shows that their last terms together form a function in the local IFE space. So each column of the piecewise defined matrix function (5.35) belongs to $\mathbf{S}_h(T)$ for every interface element T. In addition, by (5.29b) and (5.29a), we can see that $\Lambda(A_i) = 0_{2\times 4},\ i \in \mathcal{I}$. Hence, by the unisolvence of IFE functions, we know that each column of $\Lambda(X)$ must be $0_{2\times 1}$ which leads to $\Lambda_\pm(X) = 0_{2\times 4}$ because $\overline{M}^+$ is nonsingular. Furthermore, (5.34a) and (5.34b) can be obtained by differentiating (5.29b) and (5.29a). ∎

5.3 | Interpolation error analysis

In this subsection, we use the results above to show the optimal approximation capabilities of the proposed IFE spaces by estimating the errors of the Lagrange type interpolation operators. Again, we assume the conditions for the unisolvence of IFE shape functions are satisfied, that is, $F = F_0$ given in Lemma 5.1 in the construction of the bilinear and rotated Q_1 IFE shape functions and the conditions on Lamé parameters are satisfied, see Remark 5.3, in the construction of the conforming/nonconforming linear IFE shape functions. We also assume the mesh size h is small enough such that the geometric estimates in Section 3 hold. To study the approximation capabilities, we consider the following local Lagrange type interpolation operator: $I_{h,T} : \mathbf{C}^0(T) \to \mathbf{S}_h(T)$ with

$$I_{h,T}\mathbf{u}(X) = \begin{cases} \sum_{i\in\mathcal{I}} \Psi_{i,T}(X)\mathbf{u}(A_i), & \text{if } T \in \mathcal{T}_h^n, \\ \sum_{i\in\mathcal{I}} \Phi_{i,T}(X)\mathbf{u}(A_i), & \text{if } T \in \mathcal{T}_h^i; \end{cases} \quad \forall \mathbf{u} \in \mathbf{C}^0(T). \tag{5.36}$$

The global interpolation operator I_h on $\mathbf{C}^0(\Omega)$ can be defined in a usual piecewise manner such that

$$(I_h\mathbf{u})|_T = I_{h,T}\mathbf{u}, \qquad \forall T \in \mathcal{T}_h,\ \forall \mathbf{u} \in \mathbf{C}^0(\Omega). \tag{5.37}$$

Applying the standard scaling argument [10, 52, 54] onto each component of the vector function $\mathbf{u} = (u_1, u_2)^T \in \mathbf{H}^2(T)$, we can show that for all the noninterface elements, there holds

$$\|I_{h,T}u_i - u_i\|_{0,T} + h|I_{h,T}u_i - u_i|_{1,T} + h^2|I_{h,T}u_i - u_i|_{2,T} \le Ch^2|u_i|_{2,T},\ \ i = 1, 2,\ \ \forall T \in \mathcal{T}_h^n. \tag{5.38}$$

To estimate the interpolation errors on an interface element $T \in \mathcal{T}_h^i$, the two components of $\mathbf{u}$ have to be treated together with the jump (1.5) and (1.6). In the following two theorems, we derive some expansions of the interpolation on $T \in \mathcal{T}_h^i$, and one key idea is to use the second-order expansion on T_*^s (a major subelement) and the first order expansion on T_* (a small subelement).

Theorem 5.5 *On each interface element $T \in \mathcal{T}_h^i$, assume $\mathbf{u} \in \mathbf{PC}^2_{int}(T)$, then, for any $\overline{X}_i \in l$, the following expansions hold for every $X \in T_*^s$:*

$$I_{h,T}\mathbf{u}(X) - \mathbf{u}(X) = \sum_{i\in\mathcal{I}^{s'}} \Phi_{i,T}(X)(\mathbf{E}_i^s(X) + \mathbf{F}_i^s(X)) + \sum_{i\in\mathcal{I}} \Phi_{i,T}(X)\mathbf{R}_i^s(X), s = \pm, \tag{5.39a}$$

$$\partial_{x_j}(I_{h,T}\mathbf{u}(X) - \mathbf{u}(X)) = \sum_{i\in\mathcal{I}^{s'}} \partial_{x_j}\Phi_{i,T}(X)(\mathbf{E}_i^s(X) + \mathbf{F}_i^s(X)) + \sum_{i\in\mathcal{I}} \partial_{x_j}\Phi_{i,T}(X)\mathbf{R}_i^s(X), s = \pm, \tag{5.39b}$$

$$\partial_{x_jx_k} I_{h,T}\mathbf{u}(X) = \sum_{i\in\mathcal{I}^{s'}} \partial_{x_jx_k}\Phi_{i,T}(X)(\mathbf{E}_i^s(X)+\mathbf{F}_i^s(X)) + \sum_{i\in\mathcal{I}} \partial_{x_jx_k}\Phi_{i,T}(X)\mathbf{R}_i^s(X), s=\pm, \tag{5.39c}$$

where $j,k=1,2$, $\mathbf{R}_i^s(X)$ *are given in (4.2), (4.5) and*

$$\mathbf{E}_i^s(X) = ((A_i-\widetilde{Y}_i)^T\otimes I_2)(M^s(\widetilde{Y}_i)-\overline{M}^s)Vec(\nabla\mathbf{u}^s(X)),$$
$$\mathbf{F}_i^s(X) = -((\widetilde{Y}_i-\overline{X}_i)^T\otimes I_2)(\overline{M}^s-I_4)Vec(\nabla\mathbf{u}^s(X)). \tag{5.40}$$

Proof. The argument is similar to [48] by applying the fundamental identity Theorem 5.4 onto the interpolation operator (5.36). Expanding the nodal values $\mathbf{u}(A_i)$, $i\in\mathcal{I}$, about $X\in T_*^s$ in the interpolation operator (5.36) by (4.1) and (4.4), we obtain

$$I_{h,T}\mathbf{u}(X) = \sum_{i\in\mathcal{I}}\Phi_{i,T}(X)\mathbf{u}(X) + \left(\sum_{i\in\mathcal{I}}\Phi_{i,T}(X)((A_i-X)^T\otimes I_2)\right)\mathrm{Vec}(\nabla\mathbf{u}^s(X))$$
$$+\left(\sum_{i\in\mathcal{I}^{s'}}\Phi_{i,T}(X)((A_i-\widetilde{Y}_i)^T\otimes I_2)(M^s-I_4)\right)\mathrm{Vec}(\nabla\mathbf{u}^s(X)) + \sum_{i\in\mathcal{I}}\Phi_{i,T}(X)\mathbf{R}_i^s. \tag{5.41}$$

Note that for any vector $\mathbf{r}\in\mathbb{R}^{2\times 1}$, there holds

$$\Phi_{i,T}(X)(\mathbf{r}^T\otimes I_2) = \mathbf{r}^T\otimes\Phi_{i,T}(X). \tag{5.42}$$

Then we apply Theorem 5.4 onto the second term in (5.41) to have

$$I_{h,T}\mathbf{u}(X) = \sum_{i\in\mathcal{I}}\Phi_{i,T}(X)\mathbf{u}(X) - \left(\sum_{i\in\mathcal{I}^{s'}}((A_i-\overline{X}_i)^T\otimes\Phi_{i,T}(X))(\overline{M}^s-I_4)\right)\mathrm{Vec}(\nabla\mathbf{u}^s(X))$$
$$+\left(\sum_{i\in\mathcal{I}^{s'}}((A_i-\widetilde{Y}_i)^T\otimes\Phi_{i,T}(X))(M^s-I_4)\right)\mathrm{Vec}(\nabla\mathbf{u}^s(X)) + \sum_{i\in\mathcal{I}}\Phi_{i,T}(X)\mathbf{R}_i^s. \tag{5.43}$$

Then, (5.39a) follows by applying partition of unity, the fact $A_i-\overline{X}_i = (A_i-\widetilde{Y}_i)+(\widetilde{Y}_i-\overline{X}_i)$, and the identity (5.42) and (5.43). For (5.39b), we apply (4.1) and (4.4) to $\partial_{x_j}I_{h,T}\mathbf{u}(X) = \sum_{i\in\mathcal{I}}\partial_{x_j}\Phi_{i,T}(X)\mathbf{u}(A_i)$ to obtain

$$\partial_{x_j}I_{h,T}\mathbf{u}(X) = \sum_{i\in\mathcal{I}}\partial_{x_j}\Phi_{i,T}(X)\mathbf{u}(X) + \left(\sum_{i\in\mathcal{I}}\partial_{x_j}\Phi_{i,T}(X)((A_i-X)^T\otimes I_2)\right)\mathrm{Vec}(\nabla\mathbf{u}^s(X))$$
$$+\left(\sum_{i\in\mathcal{I}^{s'}}\partial_{x_j}\Phi_{i,T}(X)((A_i-\widetilde{Y}_i)^T\otimes I_2)(M^s-I_4)\right)\mathrm{Vec}(\nabla\mathbf{u}^s(X)) + \sum_{i\in\mathcal{I}}\partial_{x_j}\Phi_{i,T}(X)\mathbf{R}_i^s.$$

By using (5.28), (5.34a) and (5.42) in the above, we have

$$\partial_{x_j}I_{h,T}\mathbf{u}(X) = I^j\mathrm{Vec}(\nabla\mathbf{u}^s(X)) - \left(\sum_{i\in\mathcal{I}^{s'}}((A_i-\overline{X}_i)^T\otimes\partial_{x_j}\Phi_{i,T}(X))(\overline{M}^s-I_4)\right)\mathrm{Vec}(\nabla\mathbf{u}^s(X))$$
$$+\left(\sum_{i\in\mathcal{I}^{s'}}((A_i-\widetilde{Y}_i)^T\otimes\partial_{x_j}\Phi_{i,T}(X))(M^s-I_4)\right)\mathrm{Vec}(\nabla\mathbf{u}^s(X)) + \sum_{i\in\mathcal{I}}\partial_{x_j}\Phi_{i,T}(X)\mathbf{R}_i^s,$$

which is in the same format as (5.43) because $I^j\mathrm{Vec}(\nabla\mathbf{u}^s(X) = \partial_{x_j}\mathbf{u}(X)$. Therefore, (5.39b) follows from arguments used to derive (5.39a) from (5.43). Finally, (5.39c) can be derived very similarly by applying (4.1) and (4.4) in $\partial_{x_jx_k}I_{h,T}\mathbf{u}(X) = \sum_{i\in\mathcal{I}}\partial_{x_jx_k}\Phi_{i,T}(X)\mathbf{u}(A_i)$ and then using (5.28), (5.34b) and (5.42). ■

Remark 5.6 We note that (5.39c) is trivial for the linear IFE shape functions $\Phi_{i,T}$ since each side is simply a zero vector. And the nontrivial one is the bilinear case with $j=1$, $k=2$ and the rotated-Q_1 case with $j=k=1$ or $j=k=2$.

In addition, for $X \in T_*$, we consider a simpler expansion as the following.

Theorem 5.6 *On each interface element $T \in \mathcal{T}_h^i$, assume $\mathbf{u} \in \mathbf{PC}^2_{int}(T)$, the following expansions hold for every* $X \in T_*$*:*

$$I_{h,T}\mathbf{u}(X) - \mathbf{u}(X) = \sum_{i \in \mathcal{I}} \Phi_{i,T}(X)\widetilde{\mathbf{R}}_i(X), \tag{5.44a}$$

$$\partial_{x_j}(I_{h,T}\mathbf{u}(X) - \mathbf{u}(X)) = -\partial_{x_j}\mathbf{u}(X) + \sum_{i \in \mathcal{I}} \partial_{x_j}\Phi_{i,T}(X)\widetilde{\mathbf{R}}_i(X), \tag{5.44b}$$

$$\partial_{x_j x_k}(I_{h,T}\mathbf{u}(X) - \mathbf{u}(X)) = -\partial_{x_j x_k}\mathbf{u}(X) + \sum_{i \in \mathcal{I}} \partial_{x_j x_k}\Phi_{i,T}(X)\widetilde{\mathbf{R}}_i(X), \tag{5.44c}$$

where $j, k = 1, 2$ *and* $\widetilde{\mathbf{R}}_i$ *is given in (4.10).*

Proof. They can be directly verified by applying (4.10) to the interpolation operator (5.36). ■

Now we are ready to derive estimates for the interpolation error.

Theorem 5.7 *There exists a constant* C *independent of the interface location such that the following estimate holds for every* $\mathbf{u} \in \mathbf{PH}^2_{int}(T)$*:*

$$\|I_{h,T}\mathbf{u} - \mathbf{u}\|_{0,T_*^s} + h|I_{h,T}\mathbf{u} - \mathbf{u}|_{1,T_*^s} + h^2|I_{h,T}\mathbf{u} - \mathbf{u}|_{2,T_*^s} \le Ch^2(|\mathbf{u}|_{1,T} + |\mathbf{u}|_{2,T}), \quad s = \pm, \ \forall T \in \mathcal{T}_h^i. \tag{5.45}$$

Proof. First by (3.10) and $\|A_i - \widetilde{Y}_i\| \le Ch$, we have $\|\mathbf{E}_i^s\|_{0,T_*^s} \le Ch^2|\mathbf{u}|_{1,T}, \ i \in \mathcal{I}, \ s = \pm$. Noticing $\|\widetilde{Y}_i - \overline{X}_i\| \le Ch^2$ from Lemma 3.2 in [48], we have $\|\mathbf{F}_i^s\|_{0,T_*^s} \le Ch^2|\mathbf{u}|_{1,T}, \ i \in \mathcal{I}, \ s = \pm$. Now putting these estimates, Lemmas 4.1, 4.2 and Theorem 5.2 into (5.39a) and (5.39b), for $s = \pm, j = 1, 2$, we have

$$\|I_{h,T}\mathbf{u} - \mathbf{u}\|_{0,T_*^s} \le \sum_{i \in \mathcal{I}^{s'}} C(\|\mathbf{E}_i^s\|_{0,T_*^s} + \|\mathbf{F}_i^s\|_{0,T_*^s}) + \sum_{i \in \mathcal{I}} C\|\mathbf{R}_i^s\|_{0,T_*^s} \le Ch^2(|\mathbf{u}|_{1,T} + |\mathbf{u}|_{2,T}),$$

$$\|\partial_{x_j} I_{h,T}\mathbf{u} - \partial_{x_j}\mathbf{u}\|_{0,T_*^s} \le \sum_{i \in \mathcal{I}^{s'}} Ch^{-1}(\|\mathbf{E}_i^s\|_{0,T_*^s} + \|\mathbf{F}_i^s\|_{0,T_*^s}) + \sum_{i \in \mathcal{I}} Ch^{-1}\|\mathbf{R}_i^s\|_{0,T_*^s} \le Ch(|\mathbf{u}|_{1,T} + |\mathbf{u}|_{2,T}).$$

In addition, by (5.39c), for $j, k = 1, 2$, we have

$$\begin{aligned}\|\partial_{x_j x_k} I_{h,T}\mathbf{u} - \partial_{x_j x_k}\mathbf{u}\|_{0,T_*^s} &\le \|\partial_{x_j x_k}\mathbf{u}\|_{0,T_*^s} + \sum_{i \in \mathcal{I}^{s'}} Ch^{-2}(\|\mathbf{E}_i^s\|_{0,T_*^s} + \|\mathbf{F}_i^s\|_{0,T_*^s}) + \sum_{i \in \mathcal{I}} Ch^{-2}\|\mathbf{R}_i^s\|_{0,T_*^s} \\ &\le C(|\mathbf{u}|_{1,T} + |\mathbf{u}|_{2,T}).\end{aligned}$$

These estimates lead to the desired result for $\mathbf{u} \in \mathbf{PC}^2_{int}(T)$. Then the estimation for $\mathbf{u} \in \mathbf{PH}^2_{int}(T)$ can be obtained from the density Hypothesis 4. ■

Theorem 5.8 *There exists a constant* C *independent of the interface location such that the following estimate holds for every* $\mathbf{u} \in \mathbf{PH}^2_{int}(T)$*:*

$$\|I_{h,T}\mathbf{u} - \mathbf{u}\|_{0,T_*} + h|I_{h,T}\mathbf{u} - \mathbf{u}|_{1,T_*} + h^2|I_{h,T}\mathbf{u} - \mathbf{u}|_{2,T_*} \le Ch^2(\|\mathbf{u}\|_{1,6,T} + \|\mathbf{u}\|_{2,T}), \ \ \forall T \in \mathcal{T}_h^i. \tag{5.46}$$

Proof. By the same arguments used for the proof of Lemma 4.3, we know that $|T_*| \leq Ch^3$ for a mesh fine enough. Then, according to (2.5), Lemma 4.3 and Theorem 5.6, for $j, k = 1, 2$, we have

$$\|\Phi_{i,T}\widetilde{\mathbf{R}}_i\|_{0,T_*} \leq Ch^2\|\mathbf{u}\|_{1,6,T}, \quad \|\partial_{x_j}\Phi_{i,T}\widetilde{\mathbf{R}}_i\|_{0,T_*} \leq Ch\|\mathbf{u}\|_{1,6,T} \quad \text{and} \quad \|\partial_{x_jx_k}\Phi_{i,T}\widetilde{\mathbf{R}}_i\|_{0,T_*} \leq C\|\mathbf{u}\|_{1,6,T}.$$

Besides, the Hölder's inequality implies

$$\left(\int_{T_*}(\partial_{x_j}u_m)^2 dX\right)^{\frac{1}{2}} \leq \left(\int_{T_*}1^{\frac{3}{2}} dX\right)^{\frac{1}{3}}\left(\int_{T_*}(\partial_{x_j}u_m)^6 dX\right)^{\frac{1}{6}} \leq Ch\|u_m\|_{1,6,T}, m = 1, 2,$$

where we have used the fact $|T_*| \leq Ch^3$. For the second term, it is easy to see that

$$\left(\int_{T_*}(\partial_{x_jx_k}u_j)^2 dX\right)^{\frac{1}{2}} \leq C\|u_j\|_{2,T}, \quad j, k = 1, 2.$$

By applying the estimates above to (5.44a)–(5.44c), we have (5.46) for all $\mathbf{u} \in \mathbf{PC}^2_{int}(T)$. Again the result for $\mathbf{u} \in \mathbf{PH}^2_{int}(T)$ follows from the density Hypothesis 4. ■

Finally, by combing the results above, we can prove the optimal approximation capabilities for the proposed IFE space through the following error estimation for the global interpolation operator.

Theorem 5.9 *There exists a constant* C *independent of the interface location such that the following estimate holds for every* $\mathbf{u} \in \mathbf{PH}^2_{int}(T)$*:*

$$\|I_h\mathbf{u} - \mathbf{u}\|_{0,\Omega} + h|I_h\mathbf{u} - \mathbf{u}|_{1,h,\Omega} + h^2|I_h\mathbf{u} - \mathbf{u}|_{2,h,\Omega} \leq Ch^2\|\mathbf{u}\|_{2,\Omega}, \tag{5.47}$$

where $|\cdot|_{1,h,\Omega}$ *and* $|\cdot|_{2,h,\Omega}$ *are the usual discrete semi-norms defined according to the mesh* $\mathcal{T}_h$.

Proof. By putting (5.45) and (5.46) together over all the elements T, we have

$$\|I_h\mathbf{u} - \mathbf{u}\|_{0,\Omega} + h|I_h\mathbf{u} - \mathbf{u}|_{1,h,\Omega} + h^2|I_h\mathbf{u} - \mathbf{u}|_{2,h,\Omega} \leq Ch^2(\|\mathbf{u}\|_{2,\Omega} + \|\mathbf{u}\|_{1,6,\Omega}).$$

Then using the inequality $\|w\|^2_{1,p,\Omega} \leq C\|w\|^2_{2,\Omega}$ for any $w \in W^{1,p}(\Omega)$ from [57], we have (5.47). ■

6 | NUMERICAL EXAMPLES

In this section we demonstrate the optimal approximation capabilities of the IFE spaces by numerical examples. We use an example similar to that given in [45] in which the solution domain is $\Omega = [-1, 1] \times [-1, 1]$ and the exact solution $\mathbf{u}$ to the elasticity interface problem described by (1.2)–(1.6) is

$$\mathbf{u}(x_1, x_2) = \begin{bmatrix} u_1(x_1, x_2) \\ u_2(x_1, x_2) \end{bmatrix} = \begin{cases} \begin{bmatrix} u_1^-(x_1, x_2) \\ u_2^-(x_1, x_2) \end{bmatrix} = \begin{bmatrix} \frac{a^2b^2}{\lambda^-} r^{\alpha_1} \\ \frac{a^2b^2}{\lambda^-} r^{\alpha_2} \end{bmatrix} & \text{if } X \in \Omega^-, \\ \begin{bmatrix} u_1^+(x_1, x_2) \\ u_2^+(x_1, x_2) \end{bmatrix} = \begin{bmatrix} \frac{a^2b^2}{\lambda^+} r^{\alpha_1} + \left(\frac{1}{\lambda^-} - \frac{1}{\lambda^+}\right) a^2b^2 \\ \frac{a^2b^2}{\lambda^+} r^{\alpha_2} + \left(\frac{1}{\lambda^-} - \frac{1}{\lambda^+}\right) a^2b^2 \end{bmatrix} & \text{if } X \in \Omega^+, \end{cases} \tag{6.1}$$

TABLE 1 IFE interpolation errors and rates for the bilinear IFE functions

h	$\|\mathbf{u}-I_h\mathbf{u}\|_{0,\Omega}$	Rate	$\|\mathbf{u}-I_h\mathbf{u}\|_{1,\Omega}$	Rate
1/10	5.6990E-1		6.8680E+0	
1/20	1.4528E-1	1.9719	3.4933E+0	0.9753
1/40	3.6502E-2	1.9928	1.7544E+0	0.9936
1/80	9.1372E-3	1.9981	8.7822E-1	0.9984
1/160	2.2851E-3	1.9995	4.3924E-1	0.9996
1/320	5.7132E-4	1.9999	2.1964E-1	0.9999
1/640	1.4283E-4	2.0000	1.0982E-1	1.0000
1/1280	3.5709E-5	2.0000	5.4911E-2	1.0000

TABLE 2 IFE solution errors and rates for the bilinear IFE functions

h	$\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}$	Rate	$\|\mathbf{u}-\mathbf{u}_h\|_{1,\Omega}$	Rate
1/10	6.6120E-1		6.8668E+0	
1/20	1.6880E-1	1.9698	3.4932E+0	0.9751
1/40	4.2380E-2	1.9938	1.7545E+0	0.9935
1/80	1.0599E-2	1.9995	8.7833E-1	0.9982
1/160	2.6485E-3	2.0007	4.3933E-1	0.9995
1/320	6.6160E-4	2.0011	2.1972E-1	0.9997
1/640	1.6493E-4	2.0041	1.0991E-1	0.9994
1/1280	4.1100E-5	2.0047	5.4990E-2	0.9990

where $\lambda^-=1$, $\lambda^+=5$, $\mu^-=2$ and $\mu^+=10$, $a=b=\pi/6.28$, $\alpha_1=5$, $\alpha_2=7$ and $r(x_1,x_2) = (x_1^2/a^2+x_2^2/b^2)^{1/2}$, the interface Γ is a circle defined by the zero level set $r(x_1,x_2)-1=0$ and $\Omega^-=\{(x_1,x_2)^T : r(x_1,x_2)<1\}$, $\Omega^+=\{(x_1,x_2)^T : r(x_1,x_2)>1\}$. We note that the exact solution (6.1) is simply a constant multiplier of the one used in [45], and here we used a circle with different radius as the interface. All the numerical results presented below are generated by the proposed bilinear IFE space on Cartesian meshes. The errors are measured in both the L^2 and semi-H^1 norms over a sequence a meshes with the size specified by h.

We first present the numerical results for the interpolation operator $I_h\mathbf{u}$ defined by (5.36) and (5.37) in Table 1. The convergence rate r is estimated from the errors computed on two consecutive meshes. As expected, the numerical results clearly show that the interpolation errors converge optimally.

Next, for the IFE solution to the elasticity interface problem, we consider an IFE Galerkin scheme discussed in [43, 45]: find $\mathbf{u}_h\in\mathbf{S}_h(\Omega)$ such that $\mathbf{u}_h=I_h\mathbf{g}$ on $\partial\Omega$,

$$a(\mathbf{u}_h,\mathbf{v}_h)=L(v_h),\quad \forall \mathbf{v}_h\in\mathbf{S}_{h,0}(\Omega), \tag{6.2}$$

where $\mathbf{S}_{h,0}=\{\mathbf{v}_h\in\mathbf{S}_h : \mathbf{v}|_{\partial\Omega}=\mathbf{0}\}$ and

$$a(\mathbf{u}_h,\mathbf{v}_h)=\sum_{T\in\mathcal{T}_h}\int_T 2\mu\epsilon(\mathbf{u}_h) : \epsilon(\mathbf{v}_h)+\lambda div(\mathbf{u}_h)div(\mathbf{v}_h)dX,\ \text{ and }\ L(\mathbf{v}_h)=\sum_{T\in\mathcal{T}_h}\int_T \mathbf{f}\cdot\mathbf{v}_h dX. \tag{6.3}$$

Errors of the IFE solution are listed in Table 2 in which, again, we use the errors generated from two consecutive meshes to estimate the convergence rate. The data in this table clearly demonstrate that the IFE solutions $\mathbf{u}_h$ also converge to the exact solution $\mathbf{u}$ optimally.

ACKNOWLEDGMENT

This research was partially supported by GRF B-Q56D and B-Q40W of HKSAR.

ORCID

Tao Lin https://orcid.org/0000-0002-7771-3936

REFERENCES

[1] A. B. Abda, H. B. Ameur, M. Jaoua, *Identification of 2D cracks by elastic boundary measurements*, Inverse Problems vol. 15 (1999) pp. 67–77.
[2] C. J. S. Alves, T. Ha-Duong, *Inverse scattering for elastic plane cracks*, Inverse Probl. vol. 15 (1999) pp. 91–97.
[3] H. B. Ameur, M. Burger, B. Hackl, *Level set methods for geometric inverse problems in linear elasticity*, Inverse Probl. vol. 20(3) (2004) pp. 673–696.
[4] M. P. Bendsøe, O. Sigmund, *Topology optimization: Theory, methods and applications*, Springer-Verlag, Berlin, 2003.
[5] A. Cherkaev, *Variational methods for structural optimization*, vol. 140. 1st ed., Springer New York, New York, NY, 2000.
[6] H. Gao, Y. Huang, F. F. Abraham, *Continuum and atomistic studies of intersonic crack propagation*, J. Mech. Phys. Solids vol. 49(9) (2001) pp. 2113–2132.
[7] H. J. Jou, P. H. Leo, J. S. Lowengrub, *Microstructural evolution in inhomogeneous elastic media*, J. Comput. Phys. vol. 131(1) (1997) pp. 109–148.
[8] P. H. Leo, J. S. Lowengrub, Q. Nie, *Microstructural evolution in orthotropic elastic media*, J. Comput. Phys. vol. 157(1) (2000) pp. 44–88.
[9] A. P. Sutton, R. W. Balluffi, *Interfaces in crystalline materials*, Oxford Science Publications, Oxford, 1995.
[10] S. C. Brenner, L. R. Scott, "*The mathematical theory of finite element methods*," in *Texts in applied mathematics*, vol. 15. 3rd ed., Springer, New York, 2008.
[11] P. G. Ciarlet, "*Mathematical elasticity volume I: Three-dimensional elasticity*," in *Studies in mathematics and its applications*, vol. 20, North-Holland Publishing Co., Amsterdam, 1988.
[12] O. C. Zienkiewicz, R. L. Taylor, "*The finite element method*," in *Solid mechanics*, vol. 2. 5th ed., Butterworth-Heinemann, Oxford, 2000.
[13] B. Cockburn, D. Schötzau, J. Wang, *Discontinuous Galerkin methods for incompressible elastic materials*, Comput. Methods Appl. Mech. Engrg. vol. 195(25–28) (2006) pp. 3184–3204.
[14] P. Hansbo, M. G. Larson, *Discontinuous Galerkin and the Crouzeix–Raviart element: Application to elasticity*, Math. Model. Numer. Anal. vol. 37(1) (2003) pp. 63–72.
[15] T. P. Wihler, *Locking-free adaptive discontinuous Galerkin FEM for linear elasticity problems*, Math. Comp. vol. 75(255) (2006) pp. 1087–1102.
[16] I. Babuška, J. E. Osborn, *Can a finite element method perform arbitrarily badly*, Math. Comp. vol. 69(230) (2000) pp. 443–462.
[17] Z. Chen, J. Zou, *Finite element methods and their convergence for elliptic and parabolic interface problems*, Numer. Math. vol. 79(2) (1998) pp. 175–202.
[18] R. Becker, E. Burman, P. Hansbo, *A Nitsche extended finite element method for incompressible elasticity with discontinuous modulus of elasticity*, Comput. Methods Appl. Mech. Engrg. vol. 198(41–44) (2009) pp. 3352–3360.
[19] A. Hansbo, P. Hansbo, *A finite element method for the simulation of strong and weak discontinuities in solid mechanics*, Comput. Methods Appl. Mech. Engrg. vol. 193(33–35) (2004) pp. 3523–3540.
[20] P. Angot, Z. Li, *An augmented IIM & preconditioning technique for jump embedded boundary conditions*, Int. J. Numer. Anal. Model. vol. 14(4–5) (2017) pp. 712–729.
[21] X. Yang, Immersed interface method for elasticity problems with interfaces, Ph.D. Thesis, North Carolina State University, 2004.
[22] X. Yang, L. Bo, Z. Li, *The immersed interface method for elasticity problems with interfaces*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. vol. 10(5) (2003) pp. 783–808.
[23] D. Braess, *Finite elements: Theory, fast solvers, and applications in solid mechanics*. 2nd ed., Cambridge University Press, Cambridge, 2001, Translated from the 1992 German edition by Larry L. Schumaker.
[24] R. W. Clough, J. L. Tocher, "*Finite element stiffness matrices for analysis of plates in bending*," in *Matrix Methods in Structural Mechanics, The Proceedings of the Conference (Wright-Parrterson Air Force Base, Ohio, 26–28, October 1965)*, J. S. Przemieniecki et al. (Editors), Air Force Flight Dynamics Laboratory, Washington, 1966, pp. 515–545.
[25] S. Adjerid, R. Guo, T. Lin, *High degree immersed finite element spaces by a least squares method*, Int. J. Numer. Anal. Model. vol. 14(4–5) (2017) pp. 604–626.
[26] R. Guo, T. Lin, Y. Lin, *A fixed mesh method with immersed finite elements for solving interface inverse problems*, J. Sci. Comput. (2018). https://doi.org/10.1007/s10915-018-0847-y.
[27] X. He, T. Lin, Y. Lin, *Approximation capability of a bilinear immersed finite element space*, Numer. Methods Partial Differential Equations vol. 24(5) (2008) pp. 1265–1300.

[28] X. He, T. Lin, Y. Lin, *Immersed finite element methods for elliptic interface problems with non-homogeneous jump conditions*, Int. J. Numer. Anal. Model. vol. 8(2) (2011) pp. 284–301.

[29] Z. Li et al., *An immersed finite element space and its approximation capability*, Numer. Methods Partial Differential Equations vol. 20(3) (2004) pp. 338–367.

[30] Z. Li, T. Lin, X. Wu, *New Cartesian grid methods for interface problems using the finite element formulation*, Numer. Math. vol. 96(1) (2003) pp. 61–98.

[31] T. Lin, Y. Lin, X. Zhang, *Partially penalized immersed finite element methods for elliptic interface problems*, SIAM J. Numer. Anal. vol. 53(2) (2015) pp. 1121–1144.

[32] X. Zhang, Nonconforming immersed finite element methods for interface problems, Ph.D. Thesis, Virginia Polytechnic Institute and State University, 2013.

[33] S. Adjerid, N. Chaabane, T. Lin, *An immersed discontinuous finite element method for stokes interface problems*, Comput. Methods Appl. Mech. Engrg. vol. 293(15) (2015) pp. 170–190.

[34] J. Bai et al., *Modeling and an immersed finite element method for an interface wave equation*, Comput. Math. Appl. vol. 76(7) (2018) pp. 1625–1638.

[35] Z. Chen, W. Zedong, Y. Xiao, *An adaptive immersed finite element method with arbitrary Lagrangian–Eulerian scheme for parabolic equations in time variable domains*, Int. J. Numer. Anal. Model. vol. 12 (2015) pp. 567–591.

[36] Y. C. Chu et al., *An immersed-finite-element particle-in-cell simulation tool for plasma surface interaction*, Int. J. Numer. Anal. Model. vol. 14 (2017) pp. 175–200.

[37] X. He et al., *Immersed finite element methods for parabolic equations with moving interface*, Numer. Methods Partial Differential Equations vol. 29(2) (2013) pp. 619–646.

[38] M. Lin, T. Lin, H. Zhang, *Error analysis of an immersed finite element method for Euler–Bernoulli beam interface problems*, Int. J. Numer. Anal. Model. vol. 14(6) (2017) pp. 822–841.

[39] T. Lin et al., *Immersed finite element methods for 4th order differential equations*, J. Comput. Appl. Math. vol. 235(13) (2011) pp. 3953–3964.

[40] K. Moon, Immersed discontinuous Galerkin methods for acoustic wave propagation in inhomogeneous media, Thesis, Oxford University Press, 2016.

[41] H. Zhang, T. Lin, Y. Lin, *Linear and quadratic immersed finite element methods for the multi-layer porous wall model for coronary drug-eluting stents*, Int. J. Numer. Anal. Model. vol. 15(1–2) (2018) pp. 48–73.

[42] Y. Gong, Immersed-interface finite-element methods for elliptic and elasticity interface problems, Ph.D. Thesis, North Carolina State University, 2007.

[43] Y. Gong, Z. Li, *Immersed interface finite element methods for elasticity interface problems with non-homogeneous jump conditions*, Numer. Math. Theory Methods Appl. vol. 3(1) (2010) pp. 23–39.

[44] Z. Li, X. Yang, "*An immersed finite element method for elasticity equations with interfaces*," in *Recent advances in adaptive computation*, Contemp. Math., vol. 383, Amer. Math. Soc., Providence, RI, 2005, pp. 285–298.

[45] T. Lin, X. Zhang, *Linear and bilinear immersed finite elements for planar elasticity interface problems*, J. Comput. Appl. Math. vol. 236(18) (2012) pp. 4681–4699.

[46] T. Lin, D. Sheen, X. Zhang, *A locking-free immersed finite element method for planar elasticity interface problems*, J. Comput. Phys. vol. 247 (2013) pp. 228–247.

[47] S. Hou et al., *A numerical method for solving elasticity equations with interfaces*, Commun. Comput. Phys. vol. 12(2) (2012) pp. 595–612.

[48] R. Guo, T. Lin, *A group of immersed finite element spaces for elliptic interface problems*, IMA J. Numer. Anal. (2017). https://doi.org/10.1093/imanum/drx074

[49] Y. Gong, L. Bo, Z. Li, *Immersed-interface finite-element methods for elliptic interface problems with nonhomogeneous jump conditions*, SIAM J. Numer. Anal. vol. 46(1) (2007/2008) pp. 472–495.

[50] R. Guo, T. Lin, X. Zhang, *Nonconforming immersed finite element spaces for elliptic interface problems*, Comput. Math. Appl. vol. 75(6) (2018) pp. 2002–2016.

[51] X. He, T. Lin, Y. Lin, *A bilinear immersed finite volume element method for the diffusion equation with discontinuous coefficient*, Commun. Comput. Phys. vol. 6(1) (2009) pp. 185–202.

[52] P. G. Ciarlet, "*The finite element method for elliptic problems*," in *Studies in mathematics and its applications*, vol. 4, North-Holland Publishing Co., Amsterdam/New York/Oxford, 1978.

[53] M. Crouzeix, P.-A. Raviart, *Conforming and nonconforming finite element methods for solving the stationary Stokes equations. I*, Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge vol. 7(R-3) (1973) pp. 33–75.

[54] R. Rannacher, S. Turek, *Simple nonconforming quadrilateral Stokes element*, Numer. Methods Partial Differential Equations vol. 8(2) (1992) pp. 97–111.

[55] K. M. Abadir, J. R. Magnus, *Matrix algebra*, vol. 1, Cambridge University Press, New York, 2005.

[56] X. He, Bilinear immersed finite elements for interface problems, Ph.D. Thesis, Virginia Polytechnic Institute and State University, 2009.

[57] X. Ren, J. Wei, *On a two-dimensional elliptic problem with large exponent in nonlinearity*, Trans. Amer. Math. Soc. vol. 343(2) (1994) pp. 749–763.

How to cite this article: Guo R, Lin T, Lin Y. Approximation capabilities of immersed finite element spaces for elasticity Interface problems. *Numer Methods Partial Differential Eq.* 2019;35:1243–1268. https://doi.org/10.1002/num.22348

APPENDIX A: SOME TECHNICAL DETAILS FOR THE PROOF OF LEMMA 5.1

The proof is based on direct calculation of which we provide the detailed results for all the cases in Figures 3 and 4. Note that the first inequality in (5.17b) is just a special case of Lemma 5.1 in [48] and the second inequality is a consequence of (5.17a). So we only need to discuss (5.17a).

Bilinear elements

Case 2. Because of the symmetry, we assume $d \geq e$ and take $t_0 = 1 - e$.

$$g_n^2(F_0) - g_t^2(F_0) = \frac{1}{(1+d^2-2de+e^2)^2}(-(2d^2(-1+e)+d(1+2e-4e^2)+e(-1+2e^2))^2 + (d^3(-1+e)+d^2(1+2e-3e^2)-e^2(-2+e^2)+d(1-3e-e^2+3e^3))^2) \geqslant 0;$$

$$(1-g_n(F_0))^2 - g_t^2(F_0) = \frac{1}{(1+d^2-2de+e^2)^2}(-(2d^2(-1+e)+d(1+2e-4e^2)+e(-1+2e^2))^2 + (-1+d^3(-1+e)+d^2(2-3e)e+e^2-e^4+d(1-e-e^2+3e^3))^2) \geq 0.$$

Rotated Q_1 elements

Case 1 in Figure 4 is trivial since there are no unknown coefficients involved.

Case 2 Note that $1 \geq d \geq 1/2 \geq e \geq 0$. Then

$$g_n^2(F_0) - g_t^2(F_0) = \frac{(e-2de)^2(4d^3-4d^4-3e^2-4de(2+e)+d^2(3+16e+4e^2))}{4(d^2+e^2)^2} \geq 0;$$

$$(1-g_n(F_0))^2 - g_t^2(F_0) = \frac{1}{4(d^2+e^2)^2}(32d^5e^2-16d^6e^2-3e^4+24de^4+d^2e^2(7+32e-8e^2) + 8d^3e(1+e-12e^2-4e^3)+4d^4(1-4e-6e^2+16e^3+4e^4)) \geq \frac{1}{4}.$$

Case 3 Note that $1 \geq d \geq e \geq 1/2$. Then

$$g_n^2(F_0) - g_t^2(F_0) = \frac{1}{4(d^2+e^2)^2}(8d^5-4d^6+8d(-1+e)e^3-3e^4-8d^3e(1+6e) + 2d^2e^2(13+4e-2e^2)+d^4(-3+8e+24e^2)) \geq 0;$$

$$(1-g_n(F_0))^2 - g_t^2(F_0) = \frac{1}{4(d^2+e^2)^2}(8d^5-4d^6+8d^3(1-6e)e-3e^4+8de^3(1+e) - 2d^2e^2(-13+4e+2e^2)+d^4(-3-8e+24e^2)) \geq \frac{1}{4}.$$

Case 4 Note that $1 \geq d \geq 1/2 \geq e \geq 0$. Then

$$g_n^2(F_0) - g_t^2(F_0) = \frac{1}{4(1+d^2-2de+e^2)^2}(-d^6+2d^5(1+e)+d^4(10-6e+e^2)-4d^3(4+2e-e^2+e^3) + d^2(11+24e-12e^2+4e^3+e^4)+e(6+3e-8e^2+2e^3+2e^4-e^5) + 2d(-1-9e+4e^3-3e^4+e^5)) \geq \frac{1}{16};$$

$$(1-g_n(F_0))^2 - g_t^2(F_0) = \frac{1}{4(1+d^2-2de+e^2)^2}(-d^6 + 2d^5(1+e) + d^4(2-6e+e^2) - 4d^3(2-2e-e^2+e^3) + d^2(3-12e^2+4e^3+e^4) + 2d(3-9e+12e^2-4e^3-3e^4+e^5) - e(2-11e+16e^2-10e^3-2e^4+e^5)) \geq \frac{1}{16}.$$

Case 5 Note that $1/2 \geq d \geq 0$ and $1/2 \geq e \geq 0$. Then

$$g_n^2(F_0) - g_t^2(F_0) = \frac{(d^2 + 2d(-1+e) + (-2+e)e)^2(1-(d-e)^2)}{4(1+d^2-2de+e^2)^2} \geq 0;$$

$$(1-g_n(F_0))^2 - g_t^2(F_0) = \frac{-(d^3 + d^2(-2+e) - de^2 - (-2+e)e^2)^2 + (2+3d^2-2e+3e^2-2d(1+e))^2}{4(1+d^2-2de+e^2)^2} \geq \frac{1}{4}.$$