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A p-th degree immersed finite element for boundary value problems with discontinuous coefficients ☆

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Abstract

In this manuscript we present a p-th degree immersed finite element method for solving boundary value problems with discontinuous coefficients. In this method, interface jump conditions are employed in the finite element basis functions, and the mesh does not have to be aligned with coefficient discontinuity. We show that under h refinement the immersed finite element solution converges to the true solution at the optimal $O(h^{p+1})$ and $O(h^p)$ rates in the L^2 and H^1 norms, respectively. Furthermore, numerical results suggest that the immersed finite element solution converges exponentially fast under p refinement. Numerical examples are provided to illustrate features of this immersed finite element method.

Keywords: Immersed finite elements; Higher order methods; Discontinuous coefficients

1. Introduction

In this article, we develop and analyze a class of p-th degree immersed finite element (IFE) spaces for boundary value problems (BVPs) with discontinuous coefficients, which will be called interface problems from now on. Mathematical modeling of a physical phenomenon in a domain consisting of multiple materials often leads to interface problems. It has been pointed out in the literature (for example, see [2]), that finite element methods should be designed/employed according to the problem to be solved. For interface problems, it is known, see [4,6] for example, that the conventional (including discontinuous Galerkin) finite element methods have to be tailored such that their meshes are aligned with the material interface, each element essentially contains only one material, while their basis functions are independent of the jump conditions. The recently developed immersed finite element (IFE) methods [5,9,12–16] employ an alternative approach to adapt the finite element methods for interface problems in which the basis functions are made to satisfy the jump conditions across the material interface, but the mesh itself can be independent of the interface. Since these methods allow the mesh to have elements containing multiple materials, the material interface is immersed inside these elements, and this leads to the name of immersed finite element.

Higher degree p finite element methods (where the mesh is kept fixed while the polynomial degree p is increased) and hp finite element methods (where the mesh is refined and the degree is increased simultaneously) can yield

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exponential convergence rates for analytic solutions and in the presence of singularities [3,10,11,17]. Thus, these methods can be very efficient. However, many applications involving inhomogeneous materials yield non-smooth solutions. Therefore, all the advantages of higher degree standard finite methods are lost unless the mesh is aligned with material discontinuities. Aligning the mesh with material interfaces may not be an obvious task and may lead to unnecessarily fine meshes in the presence, for instance, of small scale features such as thin coatings and/or fibers. The new p-th degree immersed finite element spaces in this article do not require the mesh to be aligned with material interfaces. Numerical methods based on these finite element spaces can be very efficient for solving multi-scale problems while achieving optimal convergence rates with respect to the polynomials employed in the finite element spaces.

To be specific, we consider the following model interface problem:

$$\begin{cases} -(\beta u')' = f, & \\ u(a) = u_a, \quad u(b) = u_b. \end{cases} \quad a < x < b, \tag{1.1a}$$

Without loss of generality, we assume that the domain $\Omega = (a, b)$ is separated into two sub-domains by an interface point $\alpha \in (a, b)$, across which the coefficient β has a jump:

$$\beta(x) = \begin{cases} \beta^-, & x \in (a, \alpha), \\ \beta^+, & x \in (\alpha, b), \end{cases} \tag{1.1b}$$

where $\beta^{\pm}$ are two positive constants. At the interface α, we assume that the solution u satisfies the following usual jump conditions:

$$[u](\alpha) = 0, \qquad [\beta u'](\alpha) = 0, \tag{1.2}$$

where $[v](\alpha) = v(\alpha^+) - v(\alpha^-)$. The results obtained here can be easily extended to the case in which the coefficient function β has multiple discontinuity points.

It has been shown [12–15] that, for elliptic interface problems (including two or three-dimensional problems), linear IFE functions on an interface element are uniquely determined by the nodal values and the interface jump conditions. Nevertheless, as observed in [5], these conditions are not enough to uniquely determine higher degree IFE functions. Improper choice of extra conditions to define higher degree IFE basis functions might decrease their approximation capability from the point of view of polynomials employed. Our effort in this article is to present a class of extra conditions that can uniquely determine p-th degree IFE functions capable of optimally resolving the non-smooth behavior of the solution across the interfaces without requiring the mesh to be aligned with the discontinuity.

This paper is organized as follows. In Section 2, we derive a set of extended jump conditions across the interface and show that with nodal values constraints they uniquely determine p-th degree IFE functions. In Section 3, we use multi-point Taylor expansion technique to derive estimates for the error in the IFE interpolation of a Sobolev function. We will further derive estimates for IFE solution of the model interface problem. In Section 4, we present numerical results to demonstrate features of these IFE spaces and conclude with a few remarks in Section 5.

2. p-th degree IFE basis functions

The main motivation of using higher degree finite element functions is for better approximation which in turn requires stronger smoothness of the exact solution of the interface problem. Within each subdomain the smoothness of the exact solution of the model interface problem (1.1a) is guaranteed by a corresponding smoothness of the right hand function f. If, for instance, $f \in C^{p-1}(a, b)$, $p \geqslant 0$ and β is a discontinuous piecewise constant function, then the exact solution u is only $C^0(a, b)$ while the flux $\beta u' \in C^p(a, b)$. Thus, at the interface the exact solution satisfies the following extended jump conditions:

$$[u]_\alpha = 0, \qquad \left[\beta u^{(j)}\right]_\alpha = 0, \quad j = 1, 2, \ldots, p. \tag{2.1}$$

Moreover, the conditions (2.1) also ensure consistency of the method, i.e. the p-th degree immersed finite element space contains piecewise p-th degree polynomial exact solutions. We will show that (2.1) together with the nodal value constraints can uniquely determine p-th degree IFE functions.

Without loss of generality we develop the p-th degree IFE space on the reference element $\hat{e} = [-1, 1]$ containing one interface point $-1 < \hat{\alpha} < 1$ with $\hat{e}^- = [-1, \hat{\alpha}]$ and $\hat{e}^+ = [\hat{\alpha}, 1]$. Then, the local IFE space on an actual interface element can be defined by the affine mapping following the usual procedure.

Consider the following linear spaces formed by piecewise polynomials of degree p

$$\tilde{\Pi}_p(\hat{e}) = \{\hat{v}(t) \mid \hat{v}|_{\hat{e}^\pm} \in \Pi_p(\hat{e}^\pm),\ [\hat{v}]_{\hat{\alpha}} = 0,\ [\beta \hat{v}^{(j)}]_{\hat{\alpha}} = 0,\ j = 1, 2, \ldots, p\}, \tag{2.2}$$

and

$$\hat{\Pi}_p = \Pi_p \times \Pi_p, \tag{2.3}$$

where Π_p denotes the set of polynomials of degree not exceeding p and $\Pi_p(\Gamma)$ emphases the discussion of Π_p over a given set Γ. Obviously, we have

$$\tilde{\Pi}_p(\hat{e}) \subset \tilde{\Pi}_q(\hat{e}), \quad \text{for } p \leqslant q.$$

On the reference element $\hat{e} = [-1, 1]$, we introduce $p + 1$ equally spaced nodes

$$t_1 = -1 < t_2 < t_3 < \cdots < t_p < t_{p+1} = 1,$$

and let $i_{\hat{\alpha}}$ be the integer such that

$$t_{i_{\hat{\alpha}}} < \hat{\alpha} < t_{i_{\hat{\alpha}}+1}. \tag{2.4}$$

Applying the extended interface conditions (2.1) and the nodal value constraints $\hat{\phi}_i(t_j) = 0,\ j > i_{\hat{\alpha}}$, leads to the following formulas for the IFE basis functions $\hat{\phi}_i$ for $i = 1, 2, \ldots, i_{\hat{\alpha}}$:

$$\hat{\phi}_i(\hat{x}) = \begin{cases} \hat{\phi}_i^-(\hat{x}) = \hat{\phi}_i^+(\hat{\alpha}) + \dfrac{\beta^+}{\beta^-}\big(\hat{\phi}_i^+(\hat{x}) - \hat{\phi}_i^+(\hat{\alpha})\big), & \hat{x} \in [-1, \hat{\alpha}], \\ \hat{\phi}_i^+(\hat{x}) = \displaystyle\sum_{j=1}^{i_{\hat{\alpha}}} c_j L_j(\hat{x}), & \hat{x} \in [\hat{\alpha}, 1], \end{cases} \tag{2.5}$$

where $L_j(\hat{x})$ is the Lagrange cardinal polynomial defined by t_k, $k = 1, \ldots, p + 1$, such that $L_j(t_k) = \delta_{jk}$. Similarly, applying the extended interface conditions (2.1) and the nodal value constraints $\hat{\phi}_i(t_j) = 0,\ j \leqslant i_{\hat{\alpha}}$, leads to the following formulas for IFE basis functions $\hat{\phi}_i$ for $i = i_{\hat{\alpha}} + 1, i_{\hat{\alpha}} + 2, \ldots, p + 1$:

$$\hat{\phi}_i(\hat{x}) = \begin{cases} \hat{\phi}_i^-(\hat{x}) = \displaystyle\sum_{j=1+i_{\hat{\alpha}}}^{p+1} c_j L_j(\hat{x}), & \hat{x} \in [-1, \hat{\alpha}], \\ \hat{\phi}_i^+(\hat{x}) = \hat{\phi}_i^-(\hat{\alpha}) + \dfrac{\beta^-}{\beta^+}\big(\hat{\phi}_i^-(\hat{x}) - \hat{\phi}_i^-(\hat{\alpha})\big), & \hat{x} \in [\hat{\alpha}, 1]. \end{cases} \tag{2.6}$$

The coefficients c_j in these basis functions are to be determined by applying those nodal value constraints not used yet, details will be shown in the next theorem.

Now, we are ready to state and prove the essential properties of the IFE basis functions (2.5) and (2.6).

Theorem 2.1. *For the IFE basis functions introduced in* (2.5) *and* (2.6), *we have the following results*:

1. *The nodal constraints*

   $$\hat{\phi}_i(t_k) = \delta_{ik} \quad \text{for } 1 \leqslant k \leqslant i_{\hat{\alpha}}$$

   uniquely determine the coefficients c_j, $j = 1, 2, \ldots, i_{\hat{\alpha}}$, *in* $\hat{\phi}_i(\hat{x})$, $i = 1, 2, \ldots, i_{\hat{\alpha}}$, *given in* (2.5). *Similarly, the nodal constraints*

   $$\hat{\phi}_i(t_k) = \delta_{ik} \quad \text{for } i_{\hat{\alpha}} + 1 \leqslant k \leqslant p + 1$$

   uniquely determine the coefficients c_j, $j = i_{\hat{\alpha}} + 1, \ldots, p + 1$, *in* $\hat{\phi}_i(\hat{x})$, $i = i_{\hat{\alpha}} + 1, \ldots, p + 1$, *given in* (2.6).
2. *Moreover, there exists a constant* $C(\beta^-, \beta^+, p) > 0$ *independent of* $\hat{\alpha}$ *such that*

   $$|c_j| \leqslant C(\beta^-, \beta^+, p), \quad j = 1, 2, \ldots, p + 1. \tag{2.7}$$

Proof. We first consider those results related with IFE basis functions $\hat{\phi}_i, i > i_{\hat{\alpha}}$. Applying the following Lagrange interpolation conditions to $\hat{\phi}_i^+$ we have

$$\hat{\phi}_i^+(t_k) = \left(1 - \frac{\beta^-}{\beta^+}\right)\hat{\phi}_i^-(\hat{\alpha}) + \frac{\beta^-}{\beta^+}\hat{\phi}_i^-(t_k) = \delta_{ik}, \quad k = i_{\hat{\alpha}} + 1, \dots, p+1, \tag{2.8}$$

leads to a linear system for $\mathbf{c}^+ = (c_{i_{\hat{\alpha}}+1}, \dots, c_{p+1})^t$:

$$\mathbf{M}_{i_{\hat{\alpha}}}\mathbf{c}^+ = \mathbf{e}_i, \quad \mathbf{e}_i = (\delta_{i i_{\hat{\alpha}}+1}, \delta_{i i_{\hat{\alpha}}+2}, \dots, \delta_{i p+1})^t, \tag{2.9}$$

in which the coefficient matrix is

$$\begin{aligned}
\mathbf{M}_{i_{\hat{\alpha}}}^+ &= \begin{bmatrix} (1-r)L_{i_{\hat{\alpha}}+1}(\hat{\alpha}) + r & (1-r)L_{i_{\hat{\alpha}}+2}(\hat{\alpha}) & \cdots & (1-r)L_{p+1}(\hat{\alpha}) \\ (1-r)L_{i_{\hat{\alpha}}+1}(\hat{\alpha}) & (1-r)L_{i_{\hat{\alpha}}+2}(\hat{\alpha}) + r & \cdots & (1-r)L_{p+1}(\hat{\alpha}) \\ \vdots & \vdots & \ddots & \vdots \\ (1-r)L_{i_{\hat{\alpha}}+1}(\hat{\alpha}) & (1-r)L_{i_{\hat{\alpha}}+2}(\hat{\alpha}) & \cdots & (1-r)L_{p+1}(\hat{\alpha}) + r \end{bmatrix} \\
&= \begin{bmatrix} (1-r)L_{i_{\hat{\alpha}}+1}(\hat{\alpha}) & (1-r)L_{i_{\hat{\alpha}}+2}(\hat{\alpha}) & \cdots & (1-r)L_{p+1}(\hat{\alpha}) \\ (1-r)L_{i_{\hat{\alpha}}+1}(\hat{\alpha}) & (1-r)L_{i_{\hat{\alpha}}+2}(\hat{\alpha}) & \cdots & (1-r)L_{p+1}(\hat{\alpha}) \\ \vdots & \vdots & \ddots & \vdots \\ (1-r)L_{i_{\hat{\alpha}}+1}(\hat{\alpha}) & (1-r)L_{i_{\hat{\alpha}}+2}(\hat{\alpha}) & \cdots & (1-r)L_{p+1}(\hat{\alpha}) \end{bmatrix} + r\mathbf{I} \\
&= (1-r)\tilde{\mathbf{M}}_{i_{\hat{\alpha}}}^+ + r\mathbf{I},
\end{aligned}$$

with $r = \beta^-/\beta^+$ and

$$\begin{aligned}
\tilde{\mathbf{M}}_{i_{\hat{\alpha}}}^+ &= \begin{bmatrix} L_{i_{\hat{\alpha}}+1}(\hat{\alpha}) & L_{i_{\hat{\alpha}}+2}(\hat{\alpha}) & \cdots & L_{p+1}(\hat{\alpha}) \\ L_{i_{\hat{\alpha}}+1}(\hat{\alpha}) & L_{i_{\hat{\alpha}}+2}(\hat{\alpha}) & \cdots & L_{p+1}(\hat{\alpha}) \\ \vdots & \vdots & \ddots & \vdots \\ L_{i_{\hat{\alpha}}+1}(\hat{\alpha}) & L_{i_{\hat{\alpha}}+2}(\hat{\alpha}) & \cdots & L_{p+1}(\hat{\alpha}) \end{bmatrix} \\
&= (1, 1, \dots, 1)^t \big(L_{i_{\hat{\alpha}}+1}(\hat{\alpha}), L_{i_{\hat{\alpha}}+2}(\hat{\alpha}), \dots, L_{p+1}(\hat{\alpha})\big).
\end{aligned}$$

Note that $\tilde{M}_{i_{\hat{\alpha}}}^+$ is a rank-one matrix having a simple eigenvalue $\lambda_0 = \sum_{j=i_{\hat{\alpha}}+1}^{p} L_j(\hat{\alpha})$, and another eigenvalue $\lambda_1 = 0$ with multiplicity $p - i_{\hat{\alpha}}$. To find the bound for λ_0, we note that $P_{i_{\hat{\alpha}}}^+(\hat{x}) = \sum_{j=i_{\hat{\alpha}}+1}^{p} L_j(\hat{x})$ is a polynomial of degree p such that

$$P_{i_{\hat{\alpha}}}^+(t_i) = \begin{cases} 0, & i = 1, \dots, i_{\hat{\alpha}}, \\ 1, & i = i_{\hat{\alpha}} + 1, \dots, p+1. \end{cases}$$

Applying Rolle's theorem we show that the $(p-1)$-degree polynomial $P_{i_{\hat{\alpha}}}^{+\prime}(\hat{x})$ has $p-1$ zeros $\xi_i \in (t_i, t_{i+1})$, $i = 1, \dots, p$, $i \neq i_{\hat{\alpha}}$. Thus, $P_{i_{\hat{\alpha}}}'(\hat{x})$ does not change sign in $[t_{i_{\hat{\alpha}}}, t_{i_{\hat{\alpha}}+1}]$ which, combined with $P_{i_{\hat{\alpha}}}^+(t_{i_{\hat{\alpha}}}) = 0$ and $P_{i_{\hat{\alpha}}}^+(t_{i_{\hat{\alpha}}+1}) = 1$, establishes

$$0 < \lambda_0 = \sum_{j=i_{\hat{\alpha}}+1}^{p+1} L_j(\hat{\alpha}) < 1, \quad \forall \hat{\alpha} \in (t_{i_{\hat{\alpha}}}, t_{i_{\hat{\alpha}}+1}). \tag{2.10}$$

Hence, the matrix $\mathbf{M}_{i_{\hat{\alpha}}}^+$ has the following eigenvalues

$$r = \frac{\beta^-}{\beta^+}, \quad \text{of multiplicity } p - i_{\hat{\alpha}}, \tag{2.11}$$

$$(1-r)\sum_{j=i_{\hat{\alpha}}+1}^{p} L_j(\hat{\alpha}) + r = r + (1-r)P_{i_{\hat{\alpha}}}^+(\hat{\alpha}), \quad \text{of multiplicity } 1, \tag{2.12}$$

which implies that the spectrum of $\mathbf{M}_{i_{\hat{\alpha}}}^+$ satisfies

$$\sigma\big(\mathbf{M}_{i_{\hat{\alpha}}}^+\big) \subset \big[\min(1, r), \max(1, r)\big], \quad \forall \hat{\alpha} \in [t_{i_{\hat{\alpha}}}, t_{i_{\hat{\alpha}}+1}], \tag{2.13}$$

hence, $\mathbf{M}^+_{i_{\hat\alpha}}$ is invertible and the conditions (2.8) uniquely determine c_j, $j = i_{\hat\alpha} + 1, \dots, p+1$, in $\hat\phi_i(\hat x)$ for $i = i_{\hat\alpha} + 1, \dots, p+1$.

As for the boundedness of c_j's, applying Cramer's rule to (2.9) we can express c_j as a ratio of two determinants where the numerator is a polynomial of $\hat\alpha$ which can be bounded from above independently of $\hat\alpha \in [t_{i_{\hat\alpha}}, t_{i_{\hat\alpha}+1}]$, while the denominator is $\det(M^+_{i_{\hat\alpha}}) = r^{p-i_{\hat\alpha}}((r-1)P^+_{i_{\hat\alpha}}(\hat\alpha) + r) > r^{p-i_{\hat\alpha}} \min(1, r)$. This establishes (2.7) for $\hat\phi_i$, $i = i_{\hat\alpha} + 1, \dots, p+1$.

For the results related with IFE basis functions $\hat\phi_i$, $i = 1, 2, \dots, i_{\hat\alpha}$, we start by applying the nodal value constraints

$$\left(1 - \frac{1}{r}\right)\hat\phi_i^+(\hat\alpha) + \frac{1}{r}\hat\phi_i^+(t_k) = \hat\phi_i^-(t_k) = \delta_{ik}, \quad 1 \leqslant k \leqslant i_{\hat\alpha}.$$

Then, following the same procedure we can show that all statements of this theorem are also true for $\hat\phi_i, i = 1, 2, \dots, i_{\hat\alpha}$, and this concludes the proof. □

Now, let us define the linear functionals $\mathcal{L}_i : \hat\Pi_p \to \mathbb{R}$, such that for $\mathbf{f} = (f^-, f^+) \in \hat\Pi_p$

$$\mathcal{L}_i(\mathbf{f}) = \begin{cases} f^-(t_i), & i \leqslant i_{\hat\alpha}, \\ f^+(t_i), & i \geqslant i_{\hat\alpha} + 1, \end{cases} \qquad i = 1, \dots, p+1, \tag{2.14a}$$

and define the interface functionals $\mathcal{I}_i : \hat\Pi_p \to \mathbb{R}$ such that

$$\mathcal{I}_i(\mathbf{f}) = \begin{cases} f^+(\hat\alpha) - f^-(\hat\alpha), & i = 0, \\ \beta^+ f^{+(i)}(\hat\alpha) - \beta^- f^{-(i)}(\hat\alpha), & i = 1, \dots, p. \end{cases} \tag{2.14b}$$

We can then show that these linear functionals are linearly independent.

Corollary 2.1. *The functionals $\mathcal{L}_i$, $i = 1, 2, \dots, p+1$, and $\mathcal{I}_i$, $i = 0, 1, \dots, p$, defined in* (2.14) *are linearly independent in the space $\hat\Pi_p^*$.*

Proof. We start by assuming

$$\mathcal{F} = \sum_{j=1}^{p+1} a_j \mathcal{L}_j + \sum_{j=0}^{p} b_j \mathcal{I}_j = 0. \tag{2.15}$$

Using $\hat\phi_k$, we form $\mathbf{f}_k = (\hat\phi_k^-, \hat\phi_k^+)$, and from the properties of $\hat\phi_k$, we have

$$a_k = \mathcal{F}(\mathbf{f}_k) = 0, \quad k = 1, 2, \dots, p+1.$$

In addition, we use $\mathbf{g}_k = (0, \frac{(x-\hat\alpha)^k}{\beta^+ k!}) \in \hat\Pi_p$, $k = 0, 1, \dots, p$, consecutively to obtain

$$b_k = \sum_{j=0}^{p} b_j \mathcal{I}_j(\mathbf{g}_k) = \mathcal{F}(\mathbf{g}_k) = 0, \quad k = 0, 1, \dots, p. \tag{2.16}$$

All of the above lead to the linear independence of $\mathcal{L}_i$, $i = 1, 2, \dots, p+1$, and $\mathcal{I}_i$, $i = 0, 1, \dots, p$. □

Now we are ready to prove the existence and uniqueness of interpolation in the space $\tilde\Pi_p$ as stated in the following theorem.

Theorem 2.2. *For any y_i, $i = 1, 2, \dots, p+1$, there exists one and only one $\hat v \in \tilde\Pi_p(\hat e)$ such that*

$$\hat v(t_i) = y_i, \quad i = 1, 2, \dots, p+1. \tag{2.17}$$

Proof. The existence follows directly from Theorem 2.1. First, we can form

$$\hat v(\hat x) = \sum_{i=1}^{p+1} y_i \hat\phi_i(\hat x).$$

Then, it is obvious that $\hat{v} \in \tilde{\Pi}_p(\hat{e})$ such that (2.17) is satisfied.

To show the uniqueness, we only need to show that if $\hat{v} \in \tilde{\Pi}_p(\hat{e})$ is such that $\hat{v}(t_i) = 0$, $i = 1, 2, \dots, p+1$, then $\hat{v}(\hat{x}) = 0$, $\forall \hat{x}$. In fact, applying the functionals defined by (2.14) we have

$$\mathcal{L}_j\big((\hat{v}^-, \hat{v}^+)\big) = 0, \quad j = 1, \dots, p+1,$$
$$\mathcal{I}_j\big((\hat{v}^-, \hat{v}^+)\big) = 0, \quad j = 0, 1, \dots, p,$$

where $\hat{v}^\pm = \hat{v}|_{\hat{e}^\pm}$. This leads to $(\hat{v}^-, \hat{v}^+) = (0, 0)$ because, by Corollary 2.1, $\mathcal{L}_i$, $i = 1, 2, \dots, p+1$, and $\mathcal{I}_i$, $i = 0, 1, \dots, p$, form a basis of dual space of $\hat{\Pi}_p$. □

As an important consequence of Theorem 2.2, we can show that the IFE basis functions defined by (2.5) and (2.6) have the property of partition of unity.

Corollary 2.2. *For the IFE basis functions defined by* (2.5) *and* (2.6)*, we have*

$$\sum_{i=1}^{p+1} \hat{\phi}_i(\hat{x}) = 1, \quad \forall \hat{x} \in [-1, 1]. \tag{2.18}$$

Proof. Let $\hat{v}(\hat{x}) = \sum_{i=1}^{p+1} \hat{\phi}_i(\hat{x})$. Then, we can easily verify that $\hat{v} \in \tilde{\Pi}_p(\hat{e})$ and

$$\hat{v}(t_i) = 1, \quad i = 1, 2, \dots, p+1.$$

On the other hand, $\hat{g}(\hat{x}) = 1 \in \tilde{\Pi}_p(\hat{e})$ also satisfies

$$\hat{g}(t_i) = 1, \quad i = 1, 2, \dots, p+1.$$

Then, we must have $\hat{v}(\hat{x}) = \hat{g}(\hat{x}) = 1$, $\forall \hat{x} \in [-1, 1]$ by the uniqueness stated in Theorem 2.2. □

Remarks.

- From the discussion above, we can see that
$$\tilde{\Pi}_p(\hat{e}) = \text{span}\{\hat{\phi}_1, \hat{\phi}_2, \dots, \hat{\phi}_{p+1}\}.$$
- Also, we can see that the p-th degree IFE space discussed above can be described by $(\hat{e}, \tilde{\Pi}_p(\hat{e}), \Sigma)$ where $\Sigma = \{\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_{p+1}\}$.

The procedure to construct $\hat{\phi}_i(\hat{x})$, $i = 1, 2, \dots, p+1$, can be extended to handle the case in which $\hat{e} = [-1, 1]$ contains multiple interface points $\hat{\alpha}_k$, $k = 1, 2, \dots, M$, such that

$$-1 = \hat{\alpha}_0 < \hat{\alpha}_1 < \hat{\alpha}_2 < \dots < \hat{\alpha}_k < \dots < \hat{\alpha}_M < \hat{\alpha}_{M+1} = 1,$$

with

$$\beta(\hat{x}) = \beta_k, \quad \hat{x} \in [\hat{\alpha}_{k-1}, \hat{\alpha}_k],\ k = 1, 2, \dots, M+1.$$

We now let $\hat{e}_k = [\hat{\alpha}_{k-1}, \hat{\alpha}_k]$, $k = 1, 2, \dots, M+1$. For $i = 1, 2, \dots, p+1$, we let

$$\hat{\phi}_i(\hat{x}) = \hat{\phi}_{i,k}(\hat{x}) \in \Pi_p, \quad \text{for } \hat{x} \in \hat{e}_k,\ k = 1, 2, \dots, M+1.$$

We note that $\hat{\phi}_i(\hat{x})$ has $(p+1)(M+1)$ coefficients which can be determined by the nodal value configuration:

$$\hat{\phi}_i(t_j) = \delta_{ij}, \quad j = 1, 2, \dots, p+1,$$

and the jump conditions at the interfaces:

$$[\hat{\phi}_i]_{\hat{\alpha}_k} = 0, \quad \big[\beta\hat{\phi}_i^{(j)}\big]_{\hat{\alpha}_k} = 0, \quad j = 1, 2, \dots, p,\ k = 1, 2, \dots, M. \tag{2.19}$$

For example, let us consider the case of two interface points $-1 < \hat{\alpha}_1 < \hat{\alpha}_2 < 1$ with $t_{i_{\hat{\alpha}_l}} < \hat{\alpha}_l < t_{i_{\hat{\alpha}_l+1}}$, $l = 1, 2$. Thus, for $i > i_{\hat{\alpha}_1}$ we have

$$\hat{\phi}_i(\hat{x}) = \begin{cases} f_1(\hat{x}) = \displaystyle\sum_{j=i_{\hat{\alpha}_1}+1}^{p+1} c_j L_j(\hat{x}), & \hat{x} \in [-1, \hat{\alpha}_1], \\ f_2(\hat{x}) = f_1(\hat{\alpha}_1) + r_1\big(f_1(\hat{x}) - f_1(\hat{\alpha}_1)\big), & \hat{x} \in [\hat{\alpha}_1, \hat{\alpha}_2], \\ f_3(\hat{x}) = f_2(\hat{\alpha}_2) + r_2\big(f_2(\hat{x}) - f_2(\hat{\alpha}_2)\big), & \hat{x} \in [\hat{\alpha}_2, 1], \end{cases}$$

where $r_l = \beta^-/\beta^+$ at the interface $\hat{x} = \hat{\alpha}_l$, $l = 1, 2$.

We note that by construction $\hat{\phi}_i(\hat{x})$ satisfies $\hat{\phi}_i(t_j) = 0$, $j = 1, \ldots, i_{\hat{\alpha}_1}$ and the interface conditions (2.19). We determine $\mathbf{c} = (c_{i_{\hat{\alpha}_1}+1}, \ldots, c_{p+1})^t$ from the Lagrange constraints

$$\hat{\phi}_i(t_j) = \delta_{ij}, \quad j = i_{\hat{\alpha}_1} + 1, \ldots, p + 1.$$

For $i \leqslant i_{\hat{\alpha}_1}$, we use the nodal conditions $\hat{\phi}_i(t_j) = 0$, $j > i_{\hat{\alpha}_2}$, and the interface conditions (2.19) to write

$$\hat{\phi}_i(\hat{x}) = \begin{cases} f_1(\hat{x}) = f_2(\hat{\alpha}_1) + r_1^{-1}\big(f_2(\hat{x}) - f_2(\hat{\alpha}_1)\big), & \hat{x} \in [-1, \hat{\alpha}_1], \\ f_2(\hat{x}) = f_3(\hat{\alpha}_2) + r_2^{-1}\big(f_3(\hat{x}) - f_3(\hat{\alpha}_2)\big), & \hat{x} \in [\hat{\alpha}_1, \hat{\alpha}_2], \\ f_3(\hat{x}) = \displaystyle\sum_{j=1}^{i_{\hat{\alpha}_2}} c_j L_j(\hat{x}), & \hat{x} \in [\hat{\alpha}_2, 1]. \end{cases}$$

We determine the coefficients $\mathbf{c} = (c_1, \ldots, c_{i_{\hat{\alpha}_2}})$ from the Lagrange constraints

$$\hat{\phi}_i(t_j) = \delta_{ij}, \quad j = 1, \ldots, i_{\hat{\alpha}_2}.$$

In order to construct the p-th degree IFE space on the whole domain $\Omega = (a, b)$, we partition $[a, b]$ into $N - 1$ subintervals as

$$a = x_1 < x_2 < \cdots < x_N = b, \tag{2.20}$$

with

$$h = \max_{1 \leqslant i \leqslant N-1} (x_{i+1} - x_i). \tag{2.21}$$

On each element $e_i = [x_i, x_{i+1}]$, $i = 1, 2, \ldots, N - 1$, we introduce $p + 1$ equally spaced local nodes $x_{i,j} \in e_i$ including the end points of e_i, and let $L_j(x)$, $j = 1, 2, \ldots, p + 1$, be the standard Lagrange cardinal polynomials defined by $x_{i,j} \in e_i$, $j = 1, 2, \ldots, p + 1$. Then, we define the local IFE basis functions on the element e_i to be

$$\phi_{i,j}(x) = \begin{cases} L_j(x), & \text{if } e_i \text{ is not an interface element}, \\ \phi_j(x) = \hat{\phi}_j\big(F^{-1}(x)\big), & \text{if } e_i \text{ is an interface element}, \end{cases} \quad j = 1, 2, \ldots, p + 1,$$

where F denotes the affine map between the reference element $\hat{e} = [-1, 1]$ and e_i

$$F : \hat{e} = [-1, 1] \to e_i,$$
$$x = F(\hat{x}) = \frac{x_{i+1} - x_i}{2}(\hat{x} + 1) + x_i. \tag{2.22}$$

One can easily verify that $\phi_j(x)$ defined on the physical interface element satisfies the homogeneous interface conditions (1.2).

Next, we define the p-th degree local IFE space on an element e_i by

$$S_{h,p}(e_i) = \operatorname{span}\{\phi_{i,j},\ j = 1, 2, \ldots, p + 1\}, \quad i = 1, 2, \ldots, N - 1. \tag{2.23}$$

As usual, we use the local IFE space to define the global p-th degree IFE space on the whole domain $\Omega = (a, b)$ as

$$S_{h,p}(\Omega) = \big\{v_h \in H^1(\Omega) \mid v_h|_{e_i} \in S_{h,p}(e_i),\ i = 1, 2, \ldots, N - 1\big\}. \tag{2.24}$$

As an application, we apply these p-th degree IFE spaces with the Galerkin method for the model interface problem (1.1a) which consists of finding $u_h \in S_{h,p,E}(\Omega)$ such that

$$\langle v_h', \beta u_h' \rangle = \langle v_h, f \rangle, \quad \forall v_h \in S_{h,p}(\Omega) \cap H_0^1(\Omega), \tag{2.25a}$$

where

$$S_{h,p,E} = \{ v_h \in S_{h,p}, \mid v_h(a) = u_a, \ v_h(b) = u_b \}. \tag{2.25b}$$

These IFE spaces can also be used with other finite element methods. For instance, Adjerid and Lin [1] have applied them to discontinuous Galerkin methods for solving the model interface problem. There, the authors have developed a more stable hierarchical basis functions spanning the same IFE space which lead to better conditioned linear systems. In addition to the interface conditions (2.1), extra orthogonality constraints are applied to construct this hierarchical basis. Please refer to [7,8] for general discussions of discontinuous Galerkin methods.

3. Approximation capability of the p-th degree IFE space

Here we investigate the approximation capability of the p-th degree IFE space introduced in the previous section and derive *a priori* IFE interpolation error bounds. Then, we apply these results to establish *a priori* finite element error estimates for the IFE solution given by (2.25) for the model interface problem.

In our analysis we will use the following function spaces on a set Γ containing an interface α, with $\Gamma^- = \Gamma \cap [a, \alpha]$ and $\Gamma^+ = \Gamma \cap [\alpha, b]$

$$\tilde{C}^{p,q}(\Gamma) = \{ v \mid v|_{\Gamma^\pm} \in C^q(\Gamma^\pm), [v]_\alpha = 0, [\beta v^{(j)}]_\alpha = 0, \ j = 1, \dots, p \}, \quad q \geqslant 0, \ p = 0, 1, \dots, q, \tag{3.1}$$

and

$$\tilde{H}^{p+1}(\Gamma) = \{ v \mid v|_{\Gamma^\pm} \in H^{p+1}(\Gamma^\pm), [v]_\alpha = 0, [\beta v^{(j)}]_\alpha = 0, \ j = 1, \dots, p \}, \quad p = 0, 1, \dots, \tag{3.2}$$

where $C^q(\Gamma)$ is the space of functions v such that $v^{(k)}$, $k = 0, 1, \dots, q$, are continuous on Γ and $H^p(\Gamma)$ is the standard Sobolev space of functions v such that $v^{(k)}$, $k = 0, 1, \dots, p$, are square integrable on Γ equipped with the norm

$$\|u\|_{p,\Gamma}^2 = \sum_{k=0}^{p} \|u^{(k)}\|_\Gamma^2,$$

and semi norm

$$|u|_{p,\Gamma}^2 = \|u^{(p)}\|_\Gamma^2,$$

where

$$\|u\|_\Gamma^2 = \int_\Gamma u(x)^2 \, dx.$$

The space $\tilde{H}^{p+1}(\Gamma)$ is equipped with the following norm and semi-norm

$$\|u\|_{p+1,\Gamma}^2 = \|u\|_{p+1,\Gamma^-}^2 + \|u\|_{p+1,\Gamma^+}^2,$$
$$|u|_{p+1,\Gamma}^2 = |u|_{p+1,\Gamma^-}^2 + |u|_{p+1,\Gamma^+}^2.$$

For simplicity, we may omit Γ in the notations $\|u\|_{p+1,\Gamma}$ and $|u|_{p+1,\Gamma}$ if $\Gamma = \Omega$.

For $u \in C^0(\bar{\Omega})$, we define its p-th degree IFE piecewise interpolant $I_{\alpha,p}u(x)$ such that

$$I_{\alpha,p}u|_{e_i} = \sum_{j=1}^{p+1} u(x_{i,j})\phi_{i,j}(x), \quad \forall x \in e_i, \ i = 1, 2, \dots, N-1. \tag{3.3}$$

In particular, using the notations on the reference element $\hat{e} = [-1, 1]$, we can see that if e_i is the interface element, then for $x \in e_i$, we have

$$I_{\alpha,p}u(x)=\sum_{j=1}^{p+1}u(x_{i,j})\phi_{i,j}(x)=\sum_{j=1}^{p+1}\hat{u}(t_j)\hat{\phi}_j(\hat{x})=\hat{I}_{\alpha,p}\hat{u}(\hat{x}),$$

where

$$\hat{u}(\hat{x})=u\big(F(\hat{x})\big),\quad t_j=F^{-1}(x_{i,j}),\quad \hat{x}=F^{-1}(x).$$

In order to derive *a priori* estimates for IFE interpolation error on interface elements, we start by developing basic estimates on the reference element $\hat{e}=[-1,1]$ and map them to the physical element. However, before doing that we will state and prove several preliminary results in a series of lemmas.

Lemma 3.1. *Let us assume that $\hat{u}\in\tilde{C}^{p,p+1}(\hat{e})$ and let $i_{\hat{\alpha}}$, $1<i_{\hat{\alpha}}<p$, be the integer such that*

$$t_{i_{\hat{\alpha}}}\leqslant\hat{\alpha}\leqslant t_{i_{\hat{\alpha}}+1}.$$

Then, for $\hat{x}\in\hat{e}^-$ we have

$$\begin{aligned}\hat{I}_{\alpha,p}\hat{u}(\hat{x})&=\sum_{i=1}^{p+1}\hat{u}(t_i)\hat{\phi}_i(\hat{x})\\&=\sum_{i=1}^{p+1}\left(\sum_{j=0}^{p}\frac{1}{j!}\hat{u}^{(j)}(\hat{x})(t_i-\hat{x})^j\right)\hat{\phi}_i(\hat{x})\\&\quad+\left(\frac{\beta^-}{\beta^+}-1\right)\sum_{i=i_{\hat{\alpha}}+1}^{p+1}\left(\sum_{j=1}^{p}\frac{1}{j!}\left(\sum_{k=0}^{p-j}\frac{1}{k!}\hat{u}^{(j+k)}(\hat{x})(\hat{\alpha}-\hat{x})^k\right)(t_i-\hat{\alpha})^j\right)\hat{\phi}_i(\hat{x})\\&\quad+\sum_{i=1}^{i_{\hat{\alpha}}}R^-_{p,i}\hat{\phi}_i(\hat{x})+\sum_{i=i_{\hat{\alpha}}+1}^{p+1}\left(\left(\frac{\beta^-}{\beta^+}-1\right)\left(\sum_{j=1}^{p}\frac{R^-_{p,i,j}}{j!}\right)+R^-_{p,i,0}+R^-_{p,i}\right)\hat{\phi}_i(\hat{x}).\end{aligned}\tag{3.4}$$

For $\hat{x}\in\hat{e}^+$ we have

$$\begin{aligned}\hat{I}_{\alpha,p}\hat{u}(\hat{x})&=\sum_{i=1}^{p+1}\hat{u}(t_i)\hat{\phi}_i(\hat{x})\\&=\sum_{i=1}^{p+1}\left(\sum_{j=0}^{p}\frac{1}{j!}\hat{u}^{(j)}(\hat{x})(t_i-\hat{x})^j\right)\hat{\phi}_i(\hat{x})\\&\quad+\left(\frac{\beta^+}{\beta^-}-1\right)\sum_{i=1}^{i_{\hat{\alpha}}}\left(\sum_{j=1}^{p}\frac{1}{j!}\left(\sum_{k=0}^{p-j}\frac{1}{k!}\hat{u}^{(j+k)}(\hat{x})(\hat{\alpha}-\hat{x})^k\right)(t_i-\hat{\alpha})^j\right)\hat{\phi}_i(\hat{x})\\&\quad+\sum_{i=1}^{i_{\hat{\alpha}}}\left(\left(\frac{\beta^+}{\beta^-}-1\right)\left(\sum_{j=1}^{p}\frac{R^+_{p,i,j}}{j!}\right)+R^+_{p,i,0}+R^+_{p,i}\right)\hat{\phi}_i(\hat{x})+\sum_{i=i_{\hat{\alpha}}+1}^{p+1}R^+_{p,i}\hat{\phi}_i(\hat{x}).\end{aligned}\tag{3.5}$$

Here $R^{\pm}_{p,i}$, $R^{\pm}_{p,i,j}$ are remainders to be given in the proof below.

Proof. For a function $U(t)$ with enough smoothness, using integration by parts, we can show that

$$U(\tilde{t})=U(0)-\sum_{j=1}^{k}\frac{1}{j!}(1-t)^jU^{(j)}(t)\big|_0^{\tilde{t}}+\frac{1}{k!}\int_0^{\tilde{t}}(1-t)^kU^{(k+1)}(t)\,dt.\tag{3.6}$$

$$U(1) = U(\tilde{t}) + \sum_{j=1}^{k} \frac{1}{j!}(1-\tilde{t})^j U^{(j)}(\tilde{t}) + \frac{1}{k!}\int_{\tilde{t}}^{1} (1-t)^k U^{(k+1)}(t)\,dt. \tag{3.7}$$

$$U(1) = \sum_{j=0}^{k} \frac{1}{j!}U^{(j)}(0) + \frac{1}{k!}\int_{0}^{1} (1-t)^k U^{(k+1)}(t)\,dt. \tag{3.8}$$

First, we derive the expansions for $\hat{x} \in \hat{e}^-$. Consider those nodes in $\hat{e}^-$:

$$t_i \in [t_1, \hat{\alpha}], \quad i = 1, 2, \ldots, i_{\hat{\alpha}}.$$

Let

$$U(t) = \hat{u}\big(\hat{x} + t(t_i - \hat{x})\big), \quad i = 1, 2, \ldots, i_{\hat{\alpha}}.$$

Then, applying (3.8) to $U(t)$, we have

$$\hat{u}(t_i) = \sum_{j=0}^{p} \frac{1}{j!}\hat{u}^{(j)}(\hat{x})(t_i - \hat{x})^j + R^-_{p,i}, \tag{3.9}$$

$$R^-_{p,i} = \frac{1}{p!}\int_0^1 (1-t)^p \hat{u}^{(p+1)}\big(\hat{x} + t(t_i - \hat{x})\big)(t_i - \hat{x})^{p+1}\,dt, \\ i = 1, 2, \ldots, i_{\hat{\alpha}}. \tag{3.10}$$

Now, we consider those nodes in $\hat{e}^+$:

$$t_i \in [\hat{\alpha}, t_{p+1}], \quad i = i_{\hat{\alpha}} + 1, i_{\hat{\alpha}} + 2, \ldots, p+1.$$

As before, we let

$$U(t) = \hat{u}\big(\hat{x} + t(t_i - \hat{x})\big), \quad i = i_{\hat{\alpha}} + 1, i_{\hat{\alpha}} + 2, \ldots, p+1,$$

and let $t_{\hat{\alpha}}$ be such that

$$\hat{\alpha} = \hat{x} + t_{\hat{\alpha}}(t_i - \hat{x}).$$

Applying (3.6) to $U(t)$ with $\tilde{t} = t_{\hat{\alpha}}$, we have

$$\hat{u}(\hat{\alpha}-) = \hat{u}(\hat{x}) + \sum_{j=1}^{p} \frac{1}{j!}\hat{u}^{(j)}(\hat{x})(t_i - \hat{x})^j - \sum_{j=1}^{p} \frac{1}{j!}\hat{u}^{(j)}(\hat{\alpha}-)(t_i - \hat{\alpha})^j + R^-_{p,i,0}, \tag{3.11}$$

$$R^-_{p,i,0} = \frac{1}{p!}\int_0^{t_{\hat{\alpha}}} (1-t)^p \hat{u}^{(p+1)}\big(\hat{x} + t(t_i - \hat{x})\big)(t_i - \hat{x})^{p+1}\,dt, \quad i = i_{\hat{\alpha}} + 1, i_{\hat{\alpha}} + 2, \ldots, p+1. \tag{3.12}$$

Applying (3.7) to $U(t)$ with $\tilde{t} = t_{\hat{\alpha}}$ and using the jump conditions at the interface and (3.11), we have

$$\begin{aligned} \hat{u}(t_i) &= \sum_{j=0}^{p} \frac{1}{j!}\hat{u}^{(j)}(\hat{\alpha}+)(t_i - \hat{\alpha})^j + R^-_{p,i} \\ &= \hat{u}(\hat{\alpha}-) + \frac{\beta^-}{\beta^+}\sum_{j=1}^{p} \frac{1}{j!}\hat{u}^{(j)}(\hat{\alpha}-)(t_i - \hat{\alpha})^j + R^-_{p,i} \\ &= \hat{u}(\hat{x}) + \sum_{j=1}^{p} \frac{1}{j!}\hat{u}^{(j)}(\hat{x})(t_i - \hat{x})^j + \left(\frac{\beta^-}{\beta^+} - 1\right)\sum_{j=1}^{p} \frac{1}{j!}\hat{u}^{(j)}(\hat{\alpha}-)(t_i - \hat{\alpha})^j + R^-_{p,i,0} + R^-_{p,i}, \end{aligned} \tag{3.13}$$

where

$$R^-_{p,i} = \frac{1}{p!}\int_{t_{\hat{\alpha}}}^{1} (1-t)^p \hat{u}^{(p+1)}\big(\hat{x}+t(t_i-\hat{x})\big)(t_i-\hat{x})^{p+1}\,dt, \quad i = i_{\hat{\alpha}}+1, i_{\hat{\alpha}}+2, \ldots, p+1. \tag{3.14}$$

Now, let $V(t) = \hat{u}(\hat{x}+t(\hat{\alpha}-\hat{x}))$. For $j = 1, 2, \ldots, p$, applying (3.8) to $V^{(j)}(t)$, we have

$$\hat{u}^{(j)}(\hat{\alpha}-)(\hat{\alpha}-\hat{x})^j = \sum_{k=0}^{p-j} \frac{1}{k!}\hat{u}^{(j+k)}(\hat{x})(\hat{\alpha}-\hat{x})^{j+k} + \frac{1}{(p-j)!}\int_0^1 (1-t)^{p-j}\hat{u}^{(p+1)}\big(\hat{x}+t(\hat{\alpha}-\hat{x})\big)(\hat{\alpha}-\hat{x})^{p+1}\,dt,$$

$$\hat{u}^{(j)}(\hat{\alpha}-) = \sum_{k=0}^{p-j} \frac{1}{k!}\hat{u}^{(j+k)}(\hat{x})(\hat{\alpha}-\hat{x})^{k} + R^-_{p,i,j}, \tag{3.15}$$

$$R^-_{p,i,j} = \frac{1}{(p-j)!}\int_0^1 (1-t)^{p-j}\hat{u}^{(p+1)}\big(\hat{x}+t(\hat{\alpha}-\hat{x})\big)(\hat{\alpha}-\hat{x})^{p-j+1}\,dt, \quad i = i_{\hat{\alpha}}+1, i_{\hat{\alpha}}+2, \ldots, p+1. \tag{3.16}$$

Substituting (3.15) in (3.13), we finally obtain:

$$\begin{aligned}\hat{u}(t_i) = {} & \sum_{j=0}^{p} \frac{1}{j!}\hat{u}^{(j)}(\hat{x})(t_i-\hat{x})^j + \left(\frac{\beta^-}{\beta^+}-1\right)\sum_{j=1}^{p}\frac{1}{j!}\left(\sum_{k=0}^{p-j}\frac{1}{k!}\hat{u}^{(j+k)}(\hat{x})(\hat{\alpha}-\hat{x})^k\right)(t_i-\hat{\alpha})^j \\ & + \left(\frac{\beta^-}{\beta^+}-1\right)\left(\sum_{j=1}^{p}\frac{R^-_{p,i,j}}{j!}\right) + R^-_{p,i,0} + R^-_{p,i}, \quad i = i_{\hat{\alpha}}+1, i_{\hat{\alpha}}+2, \ldots, p+1.\end{aligned} \tag{3.17}$$

Hence, using (3.9), (3.17), and the definition of IFE interpolation, we establish (3.4).

We now consider the expansion for $\hat{x} \in \hat{e}^+$. Following similar arguments, we can show that

$$\begin{aligned}\hat{u}(t_i) = U(1) &= \sum_{j=0}^{p}\frac{1}{j!}U^{(j)}(0) + \frac{1}{p!}\int_0^1 (1-t)^p U^{(p+1)}(t)\,dt \\ &= \sum_{j=0}^{p}\frac{1}{j!}\hat{u}^{(j)}(\hat{x})(t_i-\hat{x})^j + \frac{1}{p!}\int_0^1 (1-t)^p \hat{u}^{(p+1)}\big(\hat{x}+t(t_i-\hat{x})\big)(t_i-\hat{x})^{p+1}\,dt \\ &= \sum_{j=0}^{p}\frac{1}{j!}\hat{u}^{(j)}(\hat{x})(t_i-\hat{x})^j + R^+_{p,i}, \quad i = i_{\hat{\alpha}}+1, i_{\hat{\alpha}}+2, \ldots, p+1,\end{aligned} \tag{3.18}$$

where

$$R^+_{p,i} = \frac{1}{p!}\int_0^1 (1-t)^p \hat{u}^{(p+1)}\big(\hat{x}+t(t_i-\hat{x})\big)(t_i-\hat{x})^{p+1}\,dt, \quad i = i_{\hat{\alpha}}+1, i_{\hat{\alpha}}+2, \ldots, p+1, \tag{3.19}$$

and

$$\begin{aligned}\hat{u}(t_i) = {} & \sum_{j=0}^{p} \frac{1}{j!}\hat{u}^{(j)}(\hat{x})(t_i-\hat{x})^j + \left(\frac{\beta^+}{\beta^-}-1\right)\sum_{j=1}^{p}\frac{1}{j!}\left(\sum_{k=0}^{p-j}\frac{1}{k!}\hat{u}^{(j+k)}(\hat{x})(\hat{\alpha}-\hat{x})^k\right)(t_i-\hat{\alpha})^j \\ & + \left(\frac{\beta^+}{\beta^-}-1\right)\left(\sum_{j=1}^{p}\frac{R^+_{p,i,j}}{j!}\right) + R^+_{p,i,0} + R^+_{p,i}, \quad i = 1, 2, \ldots, i_{\hat{\alpha}},\end{aligned} \tag{3.20}$$

with

$$R^+_{p,i} = \frac{1}{p!}\int_{t_{\hat{\alpha}}}^{1} (1-t)^p \hat{u}^{(p+1)}\big(\hat{x} + t(t_i - \hat{x})\big)(t_i - \hat{x})^{p+1}\,dt, \tag{3.21}$$

$$R^+_{p,i,0} = \frac{1}{p!}\int_{0}^{t_{\hat{\alpha}}} (1-t)^p \hat{u}^{(p+1)}\big(\hat{x} + t(t_i - \hat{x})\big)(t_i - \hat{x})^{p+1}\,dt, \tag{3.22}$$

$$R^+_{p,i,j} = \frac{1}{(p-j)!}\int_{0}^{1} (1-t)^{p-j} \hat{u}^{(p+1)}\big(\hat{x} + t(\hat{\alpha} - \hat{x})\big)(\hat{\alpha} - \hat{x})^{p-j+1}\,dt,$$
$$i = 1, 2, \ldots, i_{\hat{\alpha}}. \tag{3.23}$$

Finally, combining (3.18) and (3.20) we establish (3.5). □

The results of Lemma 3.1 will be used to establish important identities given in the following lemmas.

Lemma 3.2. *For any $\hat{u} \in \tilde{C}^{p,p+1}(\hat{e})$, we have*

$$\hat{u}(\hat{x}) = \begin{cases} \displaystyle\sum_{i=1}^{p+1}\left(\sum_{j=0}^{p}\frac{1}{j!}\hat{u}^{(j)}(\hat{x})(t_i-\hat{x})^j\right)\hat{\phi}_i(\hat{x}) \\ \displaystyle\quad + \left(\frac{\beta^-}{\beta^+} - 1\right)\sum_{i=i_{\hat{\alpha}}+1}^{p+1}\left(\sum_{j=1}^{p}\frac{1}{j!}\left(\sum_{k=0}^{p-j}\frac{1}{k!}\hat{u}^{(j+k)}(\hat{x})(\hat{\alpha}-\hat{x})^k\right)(t_i-\hat{\alpha})^j\right)\hat{\phi}_i(\hat{x}), & \hat{x} \in [-1, \hat{\alpha}], \\ \displaystyle\sum_{i=1}^{p+1}\left(\sum_{j=0}^{p}\frac{1}{j!}\hat{u}^{(j)}(\hat{x})(t_i-\hat{x})^j\right)\hat{\phi}_i(\hat{x}) \\ \displaystyle\quad + \left(\frac{\beta^+}{\beta^-} - 1\right)\sum_{i=1}^{i_{\hat{\alpha}}}\left(\sum_{j=1}^{p}\frac{1}{j!}\left(\sum_{k=0}^{p-j}\frac{1}{k!}\hat{u}^{(j+k)}(\hat{x})(\hat{\alpha}-\hat{x})^k\right)(t_i-\hat{\alpha})^j\right)\hat{\phi}_i(\hat{x}), & \hat{x} \in [\hat{\alpha}, 1]. \end{cases} \tag{3.24}$$

Proof. For any $\hat{x} \in \hat{e}$, we let $L(\hat{y}) \in \tilde{\Pi}_p(\hat{e})$ be such that

$$L^{(j)}(\hat{x}) = \hat{u}^{(j)}(\hat{x}), \quad j = 0, 1, \ldots, p.$$

First, applying Theorem 2.2, we have

$$L(\hat{y}) = \hat{I}_{\alpha,p}L(\hat{y}) = \sum_{i=1}^{p+1} L(t_i)\hat{\phi}_i(\hat{y}), \quad \forall \hat{y} \in [-1, 1].$$

Then, the result of this lemma follows from applying (3.4) and (3.5) to $L(\hat{y}) = \hat{I}_{\alpha,p}L(\hat{y})$ and then letting $\hat{y} = \hat{x}$, because

$$L^{(j)}(\hat{y}) = 0, \quad \forall \hat{y} \neq \hat{\alpha}, j = p+1, p+2, \ldots. \quad \Box$$

Lemma 3.3. *The IFE basis functions satisfy the following identity*:

$$0=\begin{cases}\displaystyle\sum_{i=1}^{p+1}\left(\frac{1}{p!}(t_i-\hat{x})^p\right)\hat{\phi}_i(\hat{x}) \\ \displaystyle\quad+\left(\frac{\beta^-}{\beta^+}-1\right)\sum_{i=i_{\hat{\alpha}}+1}^{p+1}\left(\sum_{j=1}^{p}\frac{1}{j!}\frac{(\hat{\alpha}-\hat{x})^{p-j}(t_i-\hat{\alpha})^j}{(p-j)!}\right)\hat{\phi}_i(\hat{x}), \quad \hat{x}\in[-1,\hat{\alpha}], \\ \displaystyle\sum_{i=1}^{p+1}\left(\frac{1}{p!}(t_i-\hat{x})^p\right)\hat{\phi}_i(\hat{x}) \\ \displaystyle\quad+\left(\frac{\beta^+}{\beta^-}-1\right)\sum_{i=1}^{i_{\hat{\alpha}}}\left(\sum_{j=1}^{p}\frac{1}{j!}\frac{(\hat{\alpha}-\hat{x})^{p-j}(t_i-\hat{\alpha})^j}{(p-j)!}\right)\hat{\phi}_i(\hat{x}), \quad \hat{x}\in[\hat{\alpha},1].\end{cases} \tag{3.25}$$

Proof. For any $\hat{x}\in\hat{e}$, we let $L(\hat{y})\in\tilde{\Pi}_p([-1,1])$ be such that

$$L^{(j)}(\hat{x})=0,\quad j=0,1,\dots,p-1,\qquad L^{(p)}(\hat{x})=C\neq 0,\quad \hat{x}\neq\hat{\alpha}.$$

Then applying (3.4) or (3.5) to $L(\hat{y})$ and then letting $\hat{y}=\hat{x}$, we have

$$0=\begin{cases}\displaystyle\sum_{i=1}^{p+1}\left(\frac{1}{p!}L^{(p)}(\hat{x})(t_i-\hat{x})^p\right)\hat{\phi}_i(\hat{x}) \\ \displaystyle\quad+\left(\frac{\beta^-}{\beta^+}-1\right)\sum_{i=i_{\hat{\alpha}}+1}^{p+1}\left(\sum_{j=1}^{p}\frac{1}{j!}\frac{1}{(p-j)!}L^{(p)}(\hat{x})(\hat{\alpha}-\hat{x})^{p-j}(t_i-\hat{\alpha})^j\right)\hat{\phi}_i(\hat{x}), \quad \hat{x}\in[-1,\hat{\alpha}], \\ \displaystyle\sum_{i=1}^{p+1}\left(\frac{1}{p!}L^{(p)}(\hat{x})(t_i-\hat{x})^p\right)\hat{\phi}_i(\hat{x}) \\ \displaystyle\quad+\left(\frac{\beta^+}{\beta^-}-1\right)\sum_{i=1}^{i_{\hat{\alpha}}}\left(\sum_{j=1}^{p}\frac{1}{j!}\frac{1}{(p-j)!}L^{(p)}(\hat{x})(\hat{\alpha}-\hat{x})^{p-j}(t_i-\hat{\alpha})^j\right)\hat{\phi}_i(\hat{x}), \quad \hat{x}\in[\hat{\alpha},1],\end{cases}$$

which leads to (3.25) because $L^{(p)}(\hat{x})=C\neq 0$. □

Lemma 3.4. *For $\hat{u}\in\tilde{C}^{p,p+2}(\hat{e})$, the remainders $R^s_{p,i}$, $R^s_{p,i,j}$, $i=1,2,\dots,p+1$, $j=0,1,\dots,p+1$, $s=\pm$, satisfy the following identity*:

$$0=\begin{cases}\displaystyle\sum_{i=1}^{i_{\hat{\alpha}}}\frac{\partial R^-_{p,i}}{\partial\hat{x}}\hat{\phi}_i(\hat{x}) \\ \displaystyle\quad+\sum_{i=i_\alpha+1}^{p+1}\left(\left(\frac{\beta^-}{\beta^+}-1\right)\left(\sum_{j=1}^{p}\frac{1}{j!}\frac{\partial R^-_{p,i,j}}{\partial\hat{x}}\right)+\frac{\partial R^-_{p,i,0}}{\partial\hat{x}}+\frac{\partial R^-_{p,i}}{\partial\hat{x}}\right)\hat{\phi}_i(\hat{x}), \quad \hat{x}\in\hat{e}^-, \\ \displaystyle\sum_{i=1}^{i_{\hat{\alpha}}}\left(\left(\frac{\beta^+}{\beta^-}-1\right)\left(\sum_{j=1}^{p}\frac{1}{j!}\frac{\partial R^+_{p,i,j}}{\partial\hat{x}}\right)+\frac{\partial R^+_{p,i,0}}{\partial\hat{x}}+\frac{\partial R^+_{p,i}}{\partial\hat{x}}\right)\hat{\phi}_i(\hat{x}) \\ \displaystyle\quad+\sum_{i=i_{\hat{\alpha}}+1}^{p+1}\frac{\partial R^+_{p,i}}{\partial\hat{x}}\hat{\phi}_i(\hat{x}), \quad \hat{x}\in\hat{e}^+.\end{cases} \tag{3.26}$$

Proof. Differentiate both sides of (3.9) and (3.17) with respect to $\hat{x} \in \hat{e}^-$, we have

$$0 = \frac{1}{p!}\hat{u}^{(p+1)}(\hat{x})(t_i - \hat{x})^p + \frac{\partial R^-_{p,i}}{\partial \hat{x}}, \quad i = 1, 2, \ldots, i_{\hat{\alpha}},$$

$$\begin{aligned}0 &= \frac{1}{p!}\hat{u}^{(p+1)}(\hat{x})(t_i - \hat{x})^p + \left(\frac{\beta^-}{\beta^+} - 1\right)\sum_{j=1}^{p}\frac{1}{j!}\frac{1}{(p-j)!}\hat{u}^{(p+1)}(\hat{x})(\hat{\alpha} - \hat{x})^{p-j}(t_i - \hat{\alpha})^j \\ &\quad + \left(\frac{\beta^-}{\beta^+} - 1\right)\left(\sum_{j=1}^{p}\frac{1}{j!}\frac{\partial R^-_{p,i,j}}{\partial \hat{x}}\right) + \frac{\partial R^-_{p,i,0}}{\partial \hat{x}} + \frac{\partial R^-_{p,i}}{\partial \hat{x}}, \\ &\quad i = i_{\hat{\alpha}} + 1, i_{\hat{\alpha}} + 2, \ldots, p+1.\end{aligned}$$

Then applying Lemma 3.3, we have

$$\begin{aligned}0 &= \sum_{i=1}^{p+1}\left(\frac{1}{p!}\hat{u}^{(p+1)}(\hat{x})(t_i - \hat{x})^p\right)\hat{\phi}_i(\hat{x}) \\ &\quad + \sum_{i=i_\alpha+1}^{p+1}\left(\frac{\beta^-}{\beta^+} - 1\right)\sum_{j=1}^{p}\frac{1}{j!}\frac{1}{(p-j)!}\hat{u}^{(p+1)}(\hat{x})(\hat{\alpha} - \hat{x})^{p-j}(t_i - \hat{\alpha})^j\hat{\phi}_i(\hat{x}) \\ &\quad + \sum_{i=1}^{i_{\hat{\alpha}}}\frac{\partial R^-_{p,i}}{\partial \hat{x}}\hat{\phi}_i(\hat{x}) + \sum_{i=i_\alpha+1}^{p+1}\left(\left(\frac{\beta^-}{\beta^+} - 1\right)\left(\sum_{j=1}^{p}\frac{1}{j!}\frac{\partial R^-_{p,i,j}}{\partial \hat{x}}\right) + \frac{\partial R^-_{p,i,0}}{\partial \hat{x}} + \frac{\partial R^-_{p,i}}{\partial \hat{x}}\right)\hat{\phi}_i(\hat{x}) \\ &= \sum_{i=1}^{i_{\hat{\alpha}}}\frac{\partial R^-_{p,i}}{\partial \hat{x}}\hat{\phi}_i(\hat{x}) + \sum_{i=i_\alpha+1}^{p+1}\left(\left(\frac{\beta^-}{\beta^+} - 1\right)\left(\sum_{j=1}^{p}\frac{1}{j!}\frac{\partial R^-_{p,i,j}}{\partial \hat{x}}\right) + \frac{\partial R^-_{p,i,0}}{\partial \hat{x}} + \frac{\partial R^-_{p,i}}{\partial \hat{x}}\right)\hat{\phi}_i(\hat{x}),\end{aligned}$$

which leads to the first identity in (3.26). Differentiating both sides of (3.18) and (3.20) and following similar arguments, we can show the second identity in (3.26). □

Applying the preliminary results from above, we can represent the IFE interpolation error on the reference element in terms of the remainders as stated in the following lemma.

Lemma 3.5. *For each $\hat{u} \in \tilde{H}^{p+1}(\hat{e})$, we have*

$$\begin{aligned}&I_{\hat{\alpha},p}\hat{u}(\hat{x}) - \hat{u}(\hat{x}) \\ &\quad = \begin{cases}\displaystyle\sum_{i=1}^{i_{\hat{\alpha}}} R^-_{p,i}\hat{\phi}_i(\hat{x}) + \sum_{i=i_{\hat{\alpha}}+1}^{p+1}\left(\left(\frac{\beta^-}{\beta^+} - 1\right)\left(\sum_{j=1}^{p}\frac{R^-_{p,i,j}}{j!}\right) + R^-_{p,i,0} + R^-_{p,i}\right)\hat{\phi}_i(\hat{x}), & \hat{x} \in [-1, \hat{\alpha}], \\ \displaystyle\sum_{i=1}^{i_{\hat{\alpha}}}\left(\left(\frac{\beta^+}{\beta^-} - 1\right)\left(\sum_{j=1}^{p}\frac{R^+_{p,i,j}}{j!}\right) + R^+_{p,i,0} + R^+_{p,i}\right)\hat{\phi}_i(\hat{x}) + \sum_{i=i_{\hat{\alpha}}+1}^{p+1} R^+_{p,i}\hat{\phi}_i(\hat{x}), & \hat{x} \in [\hat{\alpha}, 1],\end{cases}\end{aligned} \tag{3.27}$$

and

$$\begin{aligned}&\left(I_{\hat{\alpha},p}\hat{u}(\hat{x})\right)' - \hat{u}'(\hat{x}) \\ &\quad = \begin{cases}\displaystyle\sum_{i=1}^{i_{\hat{\alpha}}} R^-_{p,i}\hat{\phi}'_i(\hat{x}) + \sum_{i=i_{\hat{\alpha}}+1}^{p+1}\left(\left(\frac{\beta^-}{\beta^+} - 1\right)\left(\sum_{j=1}^{p}\frac{R^-_{p,i,j}}{j!}\right) + R^-_{p,i,0} + R^-_{p,i}\right)\hat{\phi}'_i(\hat{x}), & \hat{x} \in [-1, \hat{\alpha}], \\ \displaystyle\sum_{i=1}^{i_{\hat{\alpha}}}\left(\left(\frac{\beta^+}{\beta^-} - 1\right)\left(\sum_{j=1}^{p}\frac{R^+_{p,i,j}}{j!}\right) + R^+_{p,i,0} + R^+_{p,i}\right)\hat{\phi}'_i(\hat{x}) + \sum_{i=i_{\hat{\alpha}}+1}^{p+1} R^+_{p,i}\hat{\phi}'_i(\hat{x}), & \hat{x} \in [\hat{\alpha}, 1].\end{cases}\end{aligned} \tag{3.28}$$

Proof. Identity (3.27) holds for any $\hat{u} \in \tilde{C}^{p,p+1}(\hat{e})$ by (3.4), (3.5) and Lemma 3.2. Then, the fact that $\tilde{C}^{p,p+1}(\hat{e})$ is dense in $\tilde{H}^{p+1}$ further guarantees that (3.27) is true for all $\hat{u} \in \tilde{H}^{p+1}(\hat{e})$. Similarly, for any $\hat{u} \in \tilde{C}^{p,p+2}(\hat{e})$, identity (3.28) follows from differentiating both sides of (3.27) and applying Lemma 3.4. Then, using the density of $\tilde{C}^{p,p+2}(\hat{e})$ in $\tilde{H}^{p+1}(\hat{e})$, we know that this identity also holds for all $\hat{u} \in \tilde{H}^{p+1}(\hat{e})$. □

Now, we derive the bounds for the remainders.

Lemma 3.6. *There exists a constant C such that for any $\hat{u} \in \tilde{H}^{p+1}(\hat{e})$, we have*

$$\left|R^s_{p,i}\right| \leqslant C\frac{2^p}{p!}|\hat{u}|_{p+1}, \quad i = 1, 2, \ldots, p+1,\ s = \pm, \tag{3.29}$$

$$\left|R^-_{p,i,j}\right| \leqslant C\frac{2^p}{(p-j)!}|\hat{u}|_{p+1}, \quad i = i_{\hat{\alpha}}+1, i_{\hat{\alpha}}+2, \ldots, p+1,\ j = 0, 1, 2, \ldots, p, \tag{3.30}$$

$$\left|R^+_{p,i,j}\right| \leqslant C\frac{2^p}{(p-j)!}|\hat{u}|_{p+1}, \quad i = 1, 2, \ldots, i_{\hat{\alpha}},\ j = 0, 1, 2, \ldots, p. \tag{3.31}$$

Proof. For $s = -$, $i = 1, 2, \ldots, i_{\hat{\alpha}}$, by (3.10), we have

$$\begin{aligned}
\left|R^-_{p,i}\right| &\leqslant \frac{|t_i - \hat{x}|^{p+1}}{p!}\left|\int_0^1 \hat{u}^{(p+1)}\big(\hat{x} + t(t_i - \hat{x})\big)\,dt\right| \\
&= \frac{|t_i - \hat{x}|^{p}}{p!}\left|\int_{\hat{x}}^{t_i} \hat{u}^{(p+1)}(y)\,dy\right| \\
&\leqslant \frac{|t_i - \hat{x}|^{p}}{p!}\int_{-1}^{1} \left|\hat{u}^{(p+1)}(y)\right|dy \leqslant C\frac{2^p}{p!}|\hat{u}|_{p+1}.
\end{aligned}$$

Applying the same arguments to (3.14), (3.19), and (3.21), we can see that (3.29) holds for all the values of i and s in their specified ranges.

Estimates (3.30) and (3.31) can be shown by applying the same arguments to (3.12), (3.16), (3.22), and (3.23) accordingly. □

We are now ready to derive bounds for the IFE interpolation errors.

Theorem 3.1. *There exists a constant $C(\beta^-, \beta^+, p)$ such that for the interface element e_i and all $u \in \tilde{H}^{p+1}(e_i)$, we have*

$$\|I_{\alpha,p}u - u\|_{0,e_i} \leqslant C\big(\beta^-, \beta^+, p\big)\frac{4^p}{(p-1)!}h^{p+1}|u|_{p+1,e_i}, \tag{3.32a}$$

$$|I_{\alpha,p}u - u|_{1,e_i} \leqslant C\big(\beta^-, \beta^+, p\big)\frac{4^p}{(p-1)!}h^{p}|u|_{p+1,e_i}. \tag{3.32b}$$

Proof. We first derive estimates for $\hat{u}(\hat{x}) = u(F(\hat{x}))$ on the reference element. By Theorem 2.1, we can see that there exits a constant $C(\beta^-, \beta^+, p)$ such that

$$\|\hat{\phi}_i\|_1 \leqslant C\big(\beta^-, \beta^+, p\big), \quad i = 1, 2, \ldots, p+1.$$

Then, applying this inequality and Lemma 3.6 to the identities in Lemma 3.5 leads to estimates below

$$\|I_{\hat{\alpha},p}\hat{u} - \hat{u}\|_{0,\hat{e}} \leqslant C\big(\beta^-, \beta^+, p\big)\frac{4^p}{(p-1)!}|\hat{u}|_{p+1,\hat{e}}, \tag{3.33a}$$

$$|I_{\hat{\alpha},p}\hat{u} - \hat{u}|_{1,\hat{e}} \leqslant C\big(\beta^-, \beta^+, p\big)\frac{4^p}{(p-1)!}|\hat{u}|_{p+1,\hat{e}}. \tag{3.33b}$$

Finally, we can obtain estimates (3.32a) and (3.32b) from (3.33a) and (3.33b) through the usual scaling argument. □

Following the same reasoning as for interface elements, we can show that on a non-interface element e_i, we have *a priori* error estimates

$$\|I_{\alpha,p}u - u\|_{0,e_i} \leqslant C\frac{2^p}{(p-1)!}h^{p+1}|u|_{p+1,e_i}, \tag{3.34a}$$

$$\left\|(I_{\alpha,p}u)' - u'\right\|_{0,e_i} \leqslant C\frac{2^p}{(p-1)!}h^{p}|u|_{p+1,e_i}. \tag{3.34b}$$

Putting the local estimates on an interface element given in Theorem 3.1 together with those on regular elements given in (3.34a) and (3.34b), we can obtain estimates for the error of the IFE interpolation on the whole domain as stated below.

Corollary 3.1. *There exists a constant* $C(\beta^-, \beta^+, p)$ *such that for all* $u \in \tilde{H}^{p+1}(\Omega)$, *we have*

$$\|I_{\alpha,p}u - u\|_0 \leqslant C\left(\beta^-, \beta^+, p\right)\frac{4^p}{(p-1)!}h^{p+1}|u|_{p+1}, \tag{3.35a}$$

$$|I_{\alpha,p}u - u|_1 \leqslant C\left(\beta^-, \beta^+, p\right)\frac{4^p}{(p-1)!}|u|_{p+1}. \tag{3.35b}$$

As for the IFE solution of model interface problem (2.25), we follow the standard procedure by applying the Céa's Lemma and the Aubin–Nitsche argument to obtain the *a priori* error estimates as stated in the following corollary

Corollary 3.2. *There exists a constant* $C(\beta^-, \beta^+, p)$ *such that the IFE solution* u_h *given by* (2.25) *satisfies*

$$\|u_h - u\|_0 \leqslant C\left(\beta^-, \beta^+, p\right)\frac{4^p}{(p-1)!}h^{p+1}|u|_{p+1}, \tag{3.36a}$$

$$|u_h - u|_1 \leqslant C\left(\beta^-, \beta^+, p\right)\frac{4^p}{(p-1)!}h^{p}|u|_{p+1}. \tag{3.36b}$$

provided that the exact solution u *to the interface problem* (1.1a) *is in* $\tilde{H}^{p+1}(\Omega)$.

It is easy to check that all conditions for the Aubin–Nitsche trick are satisfied for the model interface problem (1.1) where the true solution satisfies

$$|u|_2 \leqslant C\|f\|_0. \tag{3.37}$$

The estimates (3.36a) and (3.36b) indicate that a fixed p-th degree IFE solution has the optimal convergence rate in terms the degrees of freedom provided that the exact solution has enough smoothness.

Although, our theory does not enable us to show that the IFE solution has an exponential convergence rate by either increasing the polynomial degree p with fixed h (p-refinement) or simultaneously increasing p and decreasing h (hp-refinement) [17], numerical results presented in the next section exhibit exponential convergence rates under p refinement. We plan to investigate the p-IFE method in the future.

4. Numerical examples

In this section we present several numerical examples to demonstrate the features of IFE spaces and the numerical solution to the model interface problem generated from the IFE spaces.

Example 1. We use this example to demonstrate the accuracy of the Lagrange type p-th degree IFE interpolation of the following function:

$$u(x) = \begin{cases} e^x, & x \in [0, \alpha), \\ \left[(x-\alpha)^m + \frac{\sigma^-}{\sigma^+}\right]e^x + \left(1 - \frac{\sigma^-}{\sigma^+}\right)e^{\alpha}, & x \in (\alpha, 1], \end{cases} \tag{4.1}$$

Table 1
Errors in IFE interpolations in L^2 norm. N is the number uniform elements used in the interpolation. $\alpha = \pi/6$, $\sigma^- = 1$, $\sigma^+ = 20$

N	1st deg. IFE	2nd deg. IFE	3rd deg. IFE	4th deg. IFE	5th deg. IFE
10	4.0312e − 3	9.8955e − 5	4.4295e − 6	9.8452e − 08	3.2806e − 09
14	1.8102e − 3	2.5693e − 5	4.5159e − 7	9.1624e − 09	2.0990e − 10
18	1.1681e − 3	1.5410e − 5	3.4566e − 7	3.9808e − 09	7.7302e − 11
22	7.3562e − 4	6.6425e − 6	7.4319e − 8	9.6001e − 10	1.4003e − 11
26	5.4666e − 4	4.9228e − 6	6.6493e − 8	5.6417e − 10	7.6191e − 12
30	3.9621e − 4	2.6231e − 6	2.1522e − 8	2.0389e − 10	2.1810e − 12
34	3.1580e − 4	2.1558e − 6	1.6276e − 8	1.3783e − 10	1.4130e − 12
r_p	2.0636e + 0	3.1110e + 0	4.4448e + 0	5.3043e + 0	6.2809e + 0

Table 2
Errors in IFE interpolations in semi-H^1 norm. N is the number uniform elements used in the interpolation. $\alpha = \pi/6$, $\sigma^- = 1$, $\sigma^+ = 20$

N	1st deg. IFE	2nd deg. IFE	3rd deg. IFE	4th deg. IFE	5th deg. IFE
10	1.3811e − 1	8.6783e − 3	5.0866e − 4	1.5105e − 5	6.3744e − 007
14	8.7740e − 2	2.7844e − 3	8.0625e − 5	2.2893e − 6	6.5443e − 008
18	7.2295e − 2	2.3090e − 3	6.8461e − 5	1.1440e − 6	2.7597e − 008
22	5.6052e − 2	1.1317e − 3	2.0857e − 5	3.7700e − 7	6.8612e − 009
26	4.8969e − 2	1.0327e − 3	1.8844e − 5	2.3942e − 7	3.9809e − 009
30	4.1174e − 2	6.0946e − 4	8.2372e − 6	1.0919e − 7	1.4574e − 009
34	3.7045e − 2	5.7830e − 4	6.5755e − 6	7.7685e − 8	9.7531e − 010
r_p	1.0586e + 0	2.1795e + 0	3.4280e + 0	4.2495e + 0	5.2461e + 0

Table 3
Errors in IFE solution in L^2 norm. N is the number uniform elements used in the interpolation. $\alpha = \pi/6$, $\sigma^- = 1$, $\sigma^+ = 20$

N	1st deg. IFE	2nd deg. IFE	3rd deg. IFE	4th deg. IFE	5th deg. IFE
10	1.1801e − 3	5.4366e − 5	2.6864e − 6	9.0882e − 08	2.2080e − 09
14	6.1323e − 4	2.0220e − 5	7.0934e − 7	1.7040e − 08	2.8716e − 10
18	3.7392e − 4	9.5942e − 6	2.6111e − 7	4.8671e − 09	6.1593e − 11
22	2.5145e − 4	5.2772e − 6	1.1736e − 7	1.7877e − 09	1.8285e − 11
26	1.8063e − 4	3.2049e − 6	6.0264e − 8	7.7625e − 10	6.6354e − 12
30	1.3609e − 4	2.0895e − 6	3.4035e − 8	3.7980e − 10	2.8960e − 12
34	1.0630e − 4	1.4369e − 6	2.0645e − 8	2.0322e − 10	1.6862e − 12
r_p	1.9689e + 0	2.9706e + 0	3.9794e + 0	4.9877e + 0	5.9484e + 0

where, for the necessary smoothness across the interface point α, we always choose $m = p + 1$ with p being the degree of the polynomials used the IFE space.

The errors of the IFE interpolations of u in L^2 and semi-H^1 norms for various values of h are listed in Tables 1 and 2. The data in these tables corroborate the estimates obtained in Theorem 3.1 and obey the following relation

$$\|I_{\alpha,p}u - u\|_k \approx Ch^{r_p}, \quad k = 0, 1,$$

where $r_p \approx p + 1 - k$ and $I_{\alpha,p}u$ is the p-th degree IFE interpolation of u.

Example 2. In this example, we apply the IFE method (2.25) to the model interface (1.1) in which the $f(x)$ and the boundary condition are chosen such that $u(x)$ given in (4.1) is the exact solution. The errors in the IFE solutions $u_h(x)$ for various values of h are listed in Tables 3 and 4. The data in these tables obey the following relations:

$$\|u_h - u\|_k \approx Ch^{r_p}, \quad k = 0, 1,$$

where $r_p \approx p + 1 - k$ which is consistent with the error estimates given in Corollary 3.2. Furthermore, the errors plotted in Figs. 1 and 2 suggest that the IFE solution converges exponentially fast to the exact solution under p-refinement.

Table 4
Errors in IFE solution in semi-H^1 norm. N is the number uniform elements used in the interpolation. $\alpha = \pi/6$, $\sigma^- = 1$, $\sigma^+ = 20$

N	1st deg. IFE	2nd deg. IFE	3rd deg. IFE	4th deg. IFE	5th deg. IFE
10	4.0826e − 2	4.1759e − 3	3.1670e − 4	1.4406e − 5	4.5408e − 07
14	2.9738e − 2	2.1835e − 3	1.1728e − 4	3.7842e − 6	7.9460e − 08
18	2.3339e − 2	1.3343e − 3	5.5549e − 5	1.3900e − 6	2.1972e − 08
22	1.9199e − 2	8.9777e − 4	3.0527e − 5	6.2414e − 7	7.9734e − 09
26	1.6310e − 2	6.4467e − 4	1.8529e − 5	3.2032e − 7	3.4286e − 09
30	1.4183e − 2	4.8512e − 4	1.2077e − 5	1.8084e − 7	1.6746e − 09
34	1.2554e − 2	3.7817e − 4	8.3027e − 6	1.0967e − 7	8.9597e − 10
r_p	9.6537e − 1	1.9646e + 0	2.9768e + 0	3.9867e + 0	5.0859e + 0

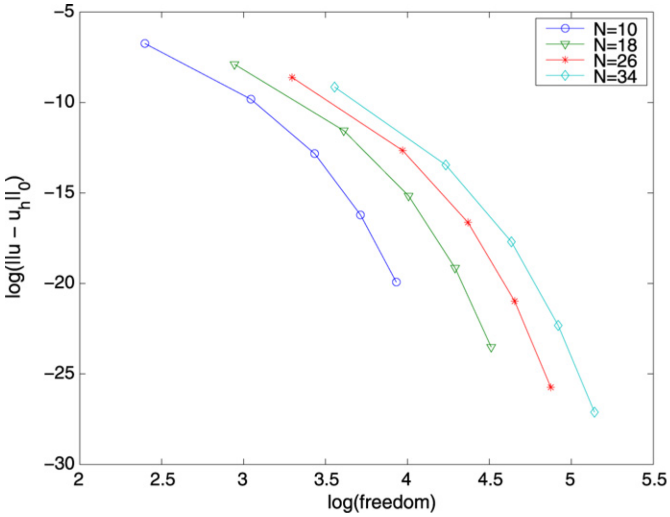

Fig. 1. IFE errors in L^2 versus the number of degree of freedom. $\alpha = \pi/6$, $\sigma^- = 1$, $\sigma^+ = 20$.

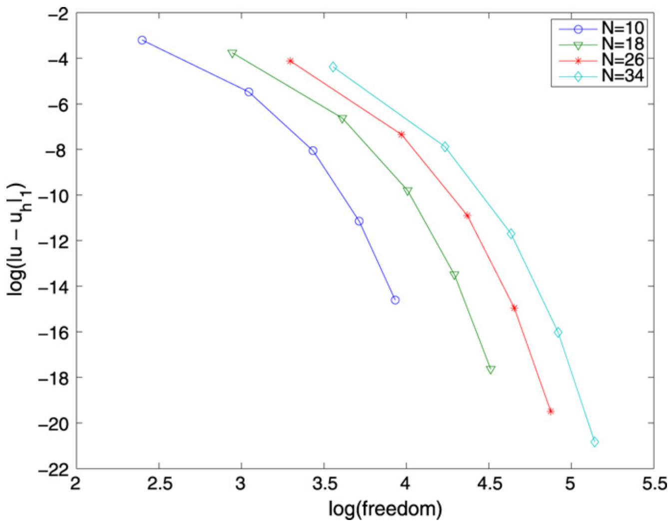

Fig. 2. IFE errors in semi-H^1 norm versus the number of degrees of freedom. $\alpha = \pi/6$, $\sigma^- = 1$, $\sigma^+ = 20$.

5. Conclusion

We have presented and analyzed a class of p-th degree immersed finite element spaces, where p is an arbitrary integer, for one-dimensional second-order elliptic problems with discontinuous coefficients. We have proved that these IFE spaces have optimal convergence rates under mesh refinement on meshes that are not necessarily aligned with material discontinuities. Computational results further suggest that these IFE spaces have exponential convergence rates under p refinement if the function to be approximated possesses suitable regularity. The error estimates obtained in the previous section are corroborated by numerical examples. We are currently working on extensions of the results in this article to multi-dimensional and higher-order problems.

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