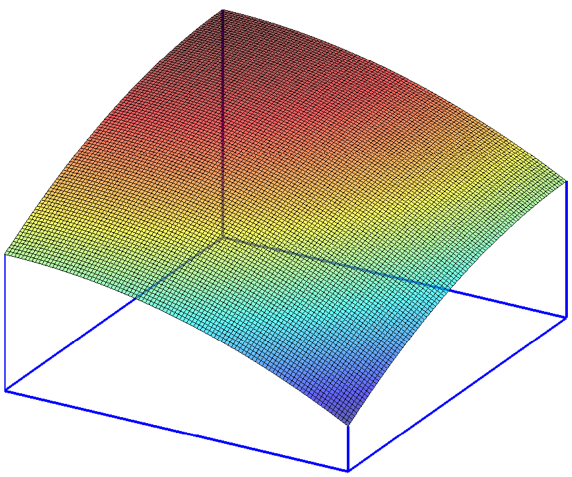
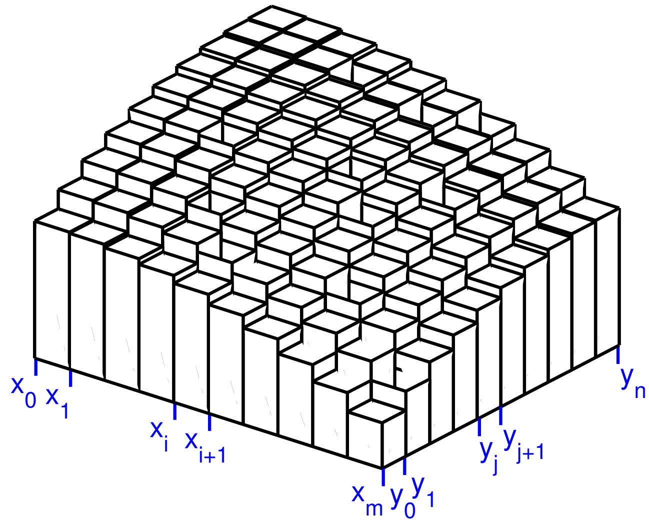
5.1: $\iint_R f(x,y)\ \mathrm{d}A$





■ Interpretation: Volume under surface $z = f(x,y)$.

■ Limit definition: $\iint_R f(x,y)\ \mathrm{d}A = \lim_{m\to\infty} \lim_{n\to\infty} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} f(x_i^*, y_j^*) \Delta A_{ij}$

- $\Delta A_{ij} = \Delta x_i \Delta y_j = (x_{i+1} - x_i)(y_{j+1} - y_j)$: Area of a subregion
- (x_i^*, y_j^*): Sample point on subregion $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$.

5.3/5.5: Double and Triple Integrals

Applications of double integrals

- **Area of a lamina**: $A = \iint_D 1 \, \mathrm{d}A$
- **Mass of a lamina**: $M = \iint_D \delta(x, y) \, \mathrm{d}A$ — Density $\delta(x, y)$
- **Volume of a solid**: $V = \iint_D h(x, y) \, \mathrm{d}A$ — Height $h(x, y)$

Applications of triple integrals

- **Volume of a solid**: $V = \iiint_E 1 \, \mathrm{d}V$
- **Mass of a solid**: $M = \iiint_E \delta(x, y, z) \, \mathrm{d}V$ — Density $\delta(x, y, z)$

5.3/5.4/6.2: Double Integrals

- **Procedure to set up double integrals** $\iint_D f \, \mathrm{d}A$
  - **Make a sketch of D and determine easiest set up**
  - **Find integral bounds using sketch**
    - **Find points of intersection**
    - **Integrate from low to high x and y**

- **Choosing $\mathrm{d}x\,\mathrm{d}y$ or $\mathrm{d}y\,\mathrm{d}x$ order may give**
  - **Easier setup: Look for a setup with only one double integral**
  - **Easier evaluation: Change order if evaluation of integrals is hard**

- **Changing to polar coordinates may give**
  - **Easier setup: Look for circular regions**
  - **Easier evaluation: functions with $\sqrt{x^2+y^2} = r$ or $x^2+y^2 = r^2$**

5.5/6.2: Triple Integrals

- **Procedure to set up triple integrals** $\iiint_E f \, \mathrm{d}V$
  - **Make a sketch of solid E and determine easiest set up**
    - **Look for one upper and one lower surface**
    - **Look for an easy projection: circular, triangular, rectangular**
  - **Find integral bounds using sketch**
    - **Find curves and points of intersection**
    - **Integrate from low to high x, y, and z**

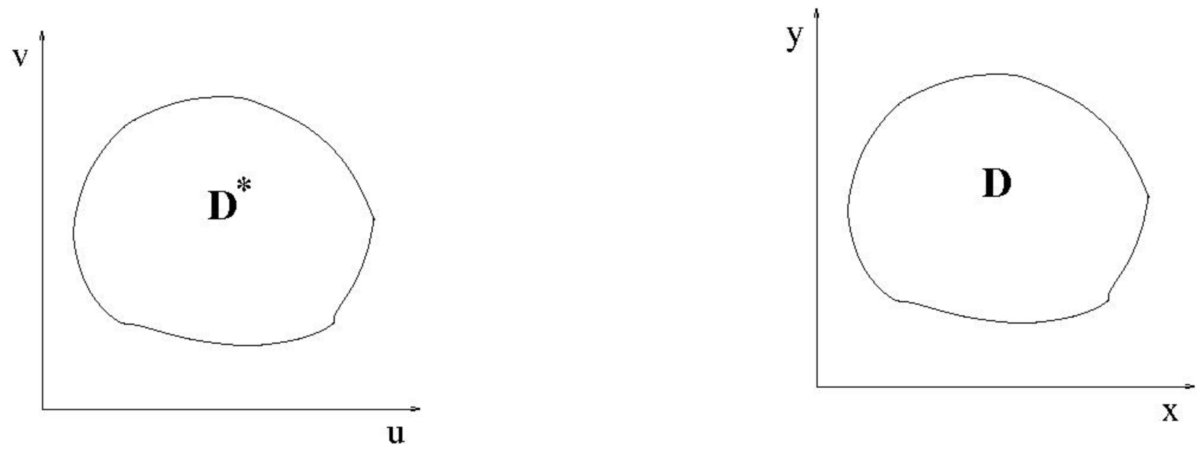
- **Choosing a different order $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$, $\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$ etc. may give**
  - **Easier setup: Look for a setup with only one triple integral**
  - **Easier evaluation: Change order if evaluation of integrals is hard**

5.5/6.2: Triple Integrals

- Changing to cylindrical coordinates may give
  - Easier setup: Look for circular cross-sections in xy
  - Easier evaluation: functions with $\sqrt{x^2+y^2} = r$ or $x^2+y^2 = r^2$

- Changing to spherical coordinates may give
  - Easier setup: Look for (parts of) spheres
  - Easier evaluation: functions with $\sqrt{x^2+y^2+z^2} = \rho$ or $x^2+y^2+z^2 = \rho^2$

6.1/6.2: Change of Variables Theorem



How to write $\iint_D$ dx dy as $\iint_{D^*}$ du dv?

■ Three main parts

- **Mapping $\underline{T}(u, v)$: How to map D^* to D?**
- **Properties of $\underline{T}(u, v)$: Can mapping be used?**
- **Change of Variables Theorem: How are $\iint_{D^*}$ du dv and $\iint_D$ dx dy related?**

6.2: Applications of Change of Variables

- **Evaluate otherwise hard/impossible integrals**
  - **Easier evaluation**: Look for simpler integrand
  - **Easier setup**: Typically not a sufficient reason

- **Numerical approximation of integrals** (quadrature)
  - **Idea: Map to an easy region**. Example: D^* a unit square
  - **Advantage**: Sample points used for integration are well defined

    Example: Midpoint is well defined for a unit square.
  - **Solving Partial Differential Equations numerically**: Finite Element Method

6.2: Change of Variables Theorem

■ Triple integrals Change of Variables Theorem

$$\iiint_E f(x, y, z)\ \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z = \iiint_{E^*} f\Big(x(u, v, w), y(u, v, w), z(u, v, w)\Big)\ |J|\ \mathrm{d}u\ \mathrm{d}v\ \mathrm{d}w$$

- Jacobian determinant: $J = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix}$

- $|J|$: absolute value of J

- The change of variables theorem holds when

(1) $\underline{T}$ is of class C^1

(2) $\underline{T}$ is one-to-one on the interior of D^*

(3) $\underline{T}$ is onto D

(4) $J \neq 0$ on the interior of D^*