4.1: Differentiation Rules for Paths

For paths $\underline{b}(t)$ and $\underline{c}(t)$ of class C^1 and scalar functions $q(t)$ of class C^1

- **Sum rule**: $$\frac{\mathrm{d}}{\mathrm{d}t}\left[\underline{b}+\underline{c}\right]=\frac{\mathrm{d}\underline{b}}{\mathrm{d}t}+\frac{\mathrm{d}\underline{c}}{\mathrm{d}t}$$

- **Product rule**: $$\frac{\mathrm{d}}{\mathrm{d}t}\left[q\ \underline{c}\right]=\frac{\mathrm{d}q}{\mathrm{d}t}\,\underline{c}+q\,\frac{\mathrm{d}\underline{c}}{\mathrm{d}t}$$

- **Chain rule**: $$\frac{\mathrm{d}}{\mathrm{d}t}\left[\underline{c}(q(t))\right]=D\left(\underline{c}\circ q\right)=\frac{\mathrm{d}\underline{c}}{\mathrm{d}t}(q(t))\,\frac{\mathrm{d}q}{\mathrm{d}t}(t)$$

  Note: Specific case of chain rule in Sec. 2.5.

- **Dot product rule**: $$\frac{\mathrm{d}}{\mathrm{d}t}\left[\underline{b}\cdot\underline{c}\right]=\frac{\mathrm{d}\underline{b}}{\mathrm{d}t}\cdot\underline{c}+\underline{b}\cdot\frac{\mathrm{d}\underline{c}}{\mathrm{d}t}$$

- **Cross product rule**: $$\frac{\mathrm{d}}{\mathrm{d}t}\left[\underline{b}\times\underline{c}\right]=\frac{\mathrm{d}\underline{b}}{\mathrm{d}t}\times\underline{c}+\underline{b}\times\frac{\mathrm{d}\underline{c}}{\mathrm{d}t}$$

4.4: Identities of Vector Analysis

For f and g scalar functions of class C^1 and c constant

1. $\underline{\nabla}(f + g) = \underline{\nabla} f + \underline{\nabla} g$

2. $\underline{\nabla}(cf) = c\underline{\nabla} f$

3. $\underline{\nabla}(fg) = f\underline{\nabla} g + g\underline{\nabla} f$

4. $\underline{\nabla}\left(\frac{f}{g}\right) = \frac{g\underline{\nabla} f - f\underline{\nabla} g}{g^2}$ at points where $g(\underline{x}) \neq 0$.

4.4: Identities of Vector Analysis

For $\underline{\boldsymbol{F}}$ and $\underline{\boldsymbol{G}}$ vector fields of class C^1 and f a scalar function of class C^1

5. $\underline{\boldsymbol{\nabla}} \cdot (\underline{\boldsymbol{F}} + \underline{\boldsymbol{G}}) = \underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{F}} + \underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{G}}$

6. $\underline{\boldsymbol{\nabla}} \times (\underline{\boldsymbol{F}} + \underline{\boldsymbol{G}}) = \underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{F}} + \underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{G}}$

7. $\underline{\boldsymbol{\nabla}} \cdot (f\underline{\boldsymbol{F}}) = f\underline{\boldsymbol{\nabla}} \cdot \underline{\boldsymbol{F}} + \underline{\boldsymbol{F}} \cdot \underline{\boldsymbol{\nabla}} f$

8. $\underline{\boldsymbol{\nabla}} \cdot (\underline{\boldsymbol{F}} \times \underline{\boldsymbol{G}}) = \underline{\boldsymbol{G}} \cdot (\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{F}}) - \underline{\boldsymbol{F}} \cdot (\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{G}})$

10. $\underline{\boldsymbol{\nabla}} \times (f\underline{\boldsymbol{F}}) = f\underline{\boldsymbol{\nabla}} \times \underline{\boldsymbol{F}} + (\underline{\boldsymbol{\nabla}} f) \times \underline{\boldsymbol{F}}$

4.4: Identities of Vector Analysis

For f and g scalar functions of class C^2 and $\underline{F}$ a vector field of class C^2

9. $\underline{\nabla} \cdot (\underline{\nabla} \times \underline{F}) = 0$ Compare $\underline{v} \cdot (\underline{v} \times \underline{w}) = 0$.

11. $\underline{\nabla} \times (\underline{\nabla} f) = \underline{0}$ Compare $\underline{v} \times \underline{v} = \underline{0}$.

12. $\underline{\nabla}^2 (fg) = f \underline{\nabla}^2 g \; + \; g \underline{\nabla}^2 f \; + \; 2 \, \underline{\nabla} f \cdot \underline{\nabla} g$

13. $\underline{\nabla} \cdot (\underline{\nabla} f \times \underline{\nabla} g) = 0$

14. $\underline{\nabla} \cdot (f \underline{\nabla} g - g \underline{\nabla} f) = f \underline{\nabla}^2 g \; - \; g \underline{\nabla}^2 f$