3.1: Higher-Order Partial Derivatives

■ Notation for second partial derivatives of $f = f(x_1, \ldots, x_n)$

1. $\mathbb{R}^2$: $f_{xx} = \dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right)$ $\quad f_{xy} = (f_x)_y = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial y \partial x}$

   $f_{yy} = \dfrac{\partial^2 f}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right)$ $\quad f_{yx} = (f_y)_x = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial x \partial y}$

2. $\mathbb{R}^n$: $\dfrac{\partial^2 f}{\partial x_1^2}, \dfrac{\partial^2 f}{\partial x_1 \partial x_n}$ etc.

■ Notation for third partial derivatives of $f = f(x_1, \ldots, x_n)$

- $\mathbb{R}^2$: $f_{xxy}, \dfrac{\partial^3 f}{\partial y \partial x^2}, \dfrac{\partial^3 f}{\partial x_2 \partial x_1^2}$.

■ Th. 1: Mixed derivative theorem

If f is of class C^2, then the mixed 2nd partial derivatives are equal.

Examples $\quad f_{xy} = f_{yx} \quad f_{xz} = f_{zx}$

3.2: Taylor Polynomials

n-th order Taylor polynomial at x_0 for $f(x)$ of class C^{n+1}

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

$$= f(x_0) + f'(x_0)[x - x_0] + \frac{f''(x_0)}{2!}[x - x_0]^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}[x - x_0]^n$$

Note

1. A Taylor polynomial of degree n consists of the first $n + 1$ terms of the Taylor series.

2. $T_1(x) = L(x)$, the linearization of f. For $n \geq 2$ you get a higher-order approximation of f.

Example: Compute $T_2(x)$ at $x_0 = 2$ for $f(x) = \mathrm{e}^x$.

$f'(x) = \mathrm{e}^x$ and $f''(x) = \mathrm{e}^x$.

$$T_2(x) = f(x_0) + f'(x_0)[x - x_0] + \frac{f''(x_0)}{2!}[x - x_0]^2 = \mathrm{e}^2 + e^2[x - 2] + \frac{\mathrm{e}^2}{2}[x - 2]^2$$

3.2: Taylor Polynomials

n-th order Taylor polynomial at (x_0, y_0) for $f(x,y)$ of class C^{n+1}

$$T_n(x,y) = f(x_0,y_0) + \left(\frac{\partial f}{\partial x}(x_0,y_0)\ [x-x_0] + \frac{\partial f}{\partial y}(x_0,y_0)\ [y-y_0]\right)$$

$$+ \frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}(x_0,y_0)\ [x-x_0]^2 + 2\frac{\partial^2 f}{\partial x \partial y}(x_0,y_0)\ [x-x_0][y-y_0] + \frac{\partial^2 f}{\partial y^2}(x_0,y_0)\ [y-y_0]^2\right)$$

$$+ \cdots + \frac{1}{n!}\sum_{j=0}^{n} \binom{n}{j} \frac{\partial^n f}{\partial x^{n-j}\partial y^{j}}(x_0,y_0)\ [x-x_0]^{n-j}\ [y-y_0]^{j}$$

Remarks

- $\binom{n}{j} = \frac{n!}{j!\,(n-j)!}$ accounts for equal mixed partial derivatives
- The order of the derivative and power of the corresponding variable are equal