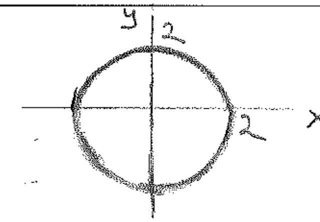
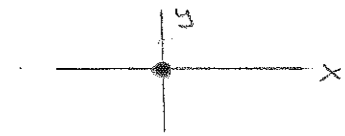
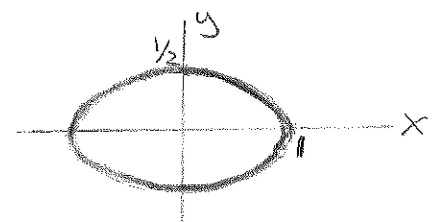
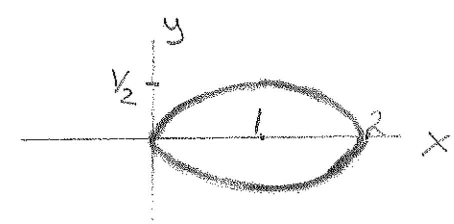
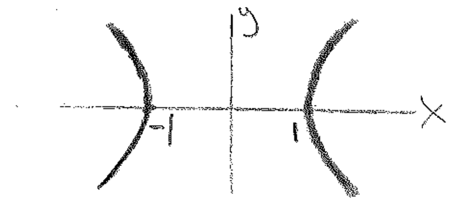
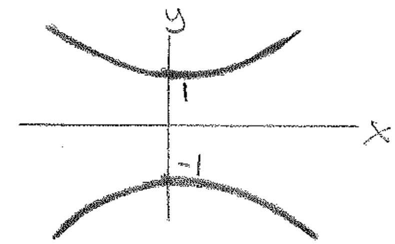
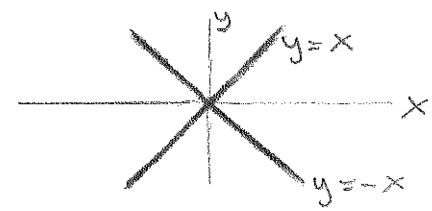
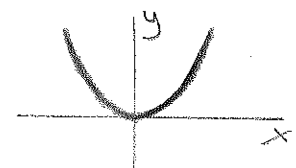
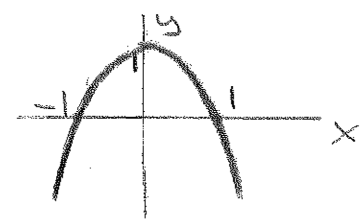
Prerequisites: Curves in 2D

Equation	Shape	Curve
$x^2 + y^2 = 4$	Circle, center $(0,0)$, radius $\sqrt{4} = 2$	
$x^2 + y^2 = 0$	Single point; $x = 0$ and $y = 0$	
$x^2 + y^2 = -4$	No solution; $x^2 + y^2 \geq 0$	
$x^2 + 4y^2 = 1$	Ellipse, center $(0,0)$, x-intercepts at $x^2 = 1$, y-intercepts at $4y^2 = 1$	
$(x-1)^2 + 4y^2 = 1$	Shifted ellipse, center $(1,0)$; x-intercepts at $(x-1)^2 = 1$; y-intercepts at $4y^2 = 1$	

Prerequisites: Curves in 2D

Equation	Shape	Curve
$x^2 - y^2 = 1$	Hyperbola, no solution for $-1 < x < 1$, x-intercepts at $x^2 = 1$	
$x^2 - y^2 = -1$	Hyperbola, no solution for $-1 < y < 1$, y-intercepts at $y^2 = 1$	
$x^2 - y^2 = 0$	Two straight lines, $y^2 = x^2$ or $y = \pm x$	
$y = x^2$	Parabola (up), minimum at $(0, 0)$	
$y = -x^2 + 1$	Parabola (down), maximum at $(0, 1)$	

2.1: Cylinders and Quadric Surfaces

I: Cylinders: Surfaces with one variable (x, y, or z) missing

- **Sketching: sketch 2D curve and extend in missing direction**

II: Quadric surfaces: only the five with elliptical traces

- **Sketching procedure**

  1. **Write in standard form: complete the square**

  2. **Determine cross-sections with elliptical traces: Find traces with**
     - **Single points**
     - **No solution**
     - **Ellipses: include the size of one ellipse**

  3. **Construct surface by connecting traces: straight or curved**
     - **Use another cross-section to determine straight or curved**
     - **Name the surface and label coordinate axes for orientation**

2.1: Graphing Quadric Surfaces

Example 2 $\quad x^2 - y^2 + 4z^2 = 0$

1. Standard form: already in standard form

2. Elliptical traces in xz: $\quad x^2 + 4z^2 = y^2$

- Single points: $y = 0 \quad \rightarrow \quad (0, 0, 0)$

- No solution: never

- Ellipses: $\quad y^2 > 0 \quad \rightarrow \quad y > 0$ or $y < 0$

  Size of one ellipse: at $y = \pm 1$, x-radius 1 and z-radius $1/2$

  Ellipse size increases when $|y|$ increases

3. Construct surface

- Other cross-section: $z = 0$ gives $x^2 = y^2$ or 2 lines $y = \pm x$, thus straight

- Shape: Cone (double cone); Oriented along y-axis

2.1: Graphing Quadric Surfaces

Example 3 $x^2 + 4y^2 - z^2 = 1$

1. **Standard form**: already in standard form

2. **Elliptical traces** in xy: $x^2 + 4y^2 = 1 + z^2$

- **Single points**: never

- **No solution**: never

- **Ellipses**: For all z

  Size of smallest ellipse: at $z = 0$, x-radius 1 and y-radius 1/2

  Ellipse size increases when $|z|$ increases

3. **Construct surface**

- **Other cross-section**: $y = 0$ gives hyperbola $x^2 - z^2 = 1$ thus curved

- **Shape**: Hyperboloid of 1 sheet (hour glass); Oriented along z-axis

2.1: Graphing Quadric Surfaces

Example 4 $\quad x^2 - 4y^2 - z^2 = 1$

1. Standard form: already in standard form

2. Elliptical traces in yz: $\quad 4y^2 + z^2 = x^2 - 1$

- **Single points:** $x^2 - 1 = 0$ or $x = \pm 1 \quad \rightarrow \quad (\pm 1, 0, 0)$
- **No solution:** $\quad x^2 - 1 < 0 \quad \rightarrow \quad -1 < x < 1$
- **Ellipses:** $\quad x^2 - 1 > 0 \quad \rightarrow \quad x > 1$ or $x < -1$

  Size of one ellipse: at $x = \pm\sqrt{2}$, y-radius $1/2$ and z-radius 1

  Ellipse size increases when $|x|$ increases

3. Construct surface

- **Other cross-section: $y = 0$ gives hyperbola $x^2 - z^2 = 1$, thus curved**
- **Shape: Hyperboloid of 2 sheets (2 lemon skins); Oriented along x-axis**

2.1: Graphing Quadric Surfaces

Example 5 $\quad x^2 + y^2/4 + z^2 = 1$

1. **Standard form**: already in standard form

2. **Elliptical traces** in xz: $\quad x^2 + z^2 = 1 - y^2/4$

- **Single points**: $1 - y^2/4 = 0$ or $y = \pm 2 \quad \rightarrow \quad (0, \pm 2, 0)$

- **No solution**: $\quad 1 - y^2/4 < 0 \quad \rightarrow \quad y > 2$ or $y < -2$

- **Ellipses**: $\quad 1 - y^2/4 > 0 \quad \rightarrow \quad -2 < y < 2$

  Largest circle at $y = 0$, radius 1

  Ellipse size decreases when $|y|$ increases

3. **Construct surface**

- **Other cross-section**: $z = 0$ gives ellipse $x^2 + y^2/4 = 1$; thus **curved**

- **Shape**: Ellipsoid (football); **Elongated** in y-direction

2.3: Partial Derivatives

- **Notation for partial derivatives of $f = f(x_1, \ldots, x_n)$**

  1. $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$, etc.
  2. $\dfrac{\partial f}{\partial x_1}$, $\dfrac{\partial f}{\partial x_2}$, etc.
  3. f_x, f_y, etc.

- **Notation for regular derivatives of $f = f(x)$**

  1. $\dfrac{\mathrm{d}f}{\mathrm{d}x}$
  2. f'

2.3: Partial Derivatives

- Computing partial derivatives of $f = f(x_1, \ldots, x_n)$

  $\dfrac{\partial f}{\partial x_1}$: derivative of f w.r.t. x_1, keeping other variables $x_2, \ldots, x_n$ constant

  etc.

  Example: $f(x, y) = \dfrac{x}{y} + y$

  $$\frac{\partial f}{\partial x} = \frac{1}{y} \qquad \frac{\partial f}{\partial y} = -\frac{x}{y^2} + 1.$$

- Gradient vector of $f = f(x_1, \ldots, x_n)$: $\underline{\nabla} f = \left(\dfrac{\partial f}{\partial x_1}, \ \ldots, \ \dfrac{\partial f}{\partial x_n} \right)$

  Example: $f(x, y) = \dfrac{x}{y} + y$

  $$\underline{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = \left(\frac{1}{y}, \ -\frac{x}{y^2} + 1 \right).$$

2.4: Paths and Curves

- For a path $\underline{c}(t) = (x(t),\ y(t),\ z(t))$ that is sufficiently smooth
  - Velocity: $\underline{v}(t) = \underline{c}'(t)$
  - Acceleration: $\underline{a}(t) = \underline{c}''(t)$
  - Speed: $v(t) = |\underline{c}'(t)|$
  - Newton's 2nd law: $\underline{F} = m\underline{a}$
  - Tangent vector: $\underline{c}'(t)$

    For $\underline{c}'(t) \neq \underline{0}$
  - Tangent line: $\underline{l}(t) = \underline{c}(t_0) + t\ \underline{c}'(t_0)$

    For $\underline{c}'(t_0) \neq \underline{0}$

- Example: Path $\underline{c}(t) = (t^3,\ t^2,\ t)$.
  - Velocity: $\underline{c}'(t) = (3t^2,\ 2t,\ 1)$
  - Acceleration: $\underline{c}''(t) = (6t, 2, 0)$
  - Speed: $|\underline{c}'(t)| = \sqrt{(3t^2)^2 + (2t)^2 + 1^2}$
  - Tangent vector at $(8, 4, 2)$: $\underline{c}(t_0) = (t_0^3,\ t_0^2,\ t_0) = (8, 4, 2)$ gives $t_0 = 2$.

    Tangent vector: $\underline{c}'(2) = (12, 4, 1)$
  - Tangent line at $(8, 4, 2)$: $\underline{l}(t) = \underline{c}(2) + t\ \underline{c}'(2) = (8, 4, 2) + t(12, 4, 1)$

2.5: Differentiation Rules

For vector functions of several variables of class C^1

- **Constant multiple rule** for $\underline{f} : \mathbb{R}^n \to \mathbb{R}^m$ and $c \in \mathbb{R}$: $\quad D(c\underline{f}) = cD\underline{f}$

  Example: $D(2x^2, 2x + 2y) = 2D(x^2, x + y)$

- **Sum rule** for $\underline{f}, \underline{g} : \mathbb{R}^n \to \mathbb{R}^m$: $\quad D(\underline{f} + \underline{g}) = D\underline{f} + D\underline{g}$

  Example: $D\left[(x^2, y) + (1, x^2)\right] = D(x^2, y) + D(1, x^2)$

Note: $D\underline{f}$ etc. are $m \times n$ matrices.

2.5: Differentiation Rules

For scalar functions of several variables of class C^1

- **Product rule** for $f, g : \mathbb{R}^n \to \mathbb{R}$: $\quad D(fg) = fDg + gDf$

  Example: $D\left[x^2(xy)^3\right] = x^2 D(xy)^3 + (xy)^3 Dx^2$

- **Quotient rule** for $f, g : \mathbb{R}^n \to \mathbb{R}$ and $g \neq 0$: $\quad D\left(\dfrac{f}{g}\right) = \dfrac{gDf - fDg}{g^2}$

  Example: $D\left(\dfrac{x^2 + y}{e^{xy}}\right) = \dfrac{e^{xy} D(x^2 + y) - (x^2 + y) De^{xy}}{e^{2xy}}$

Note: Df etc. are $1 \times n$ matrices.

2.5: Chain Rule

■ Chain rule for scalar-valued functions

- **Example 1**: $z(u,v) = u + 2v$, $u(x) = x^2$, and $v(x) = x^3$.

  Compute $\mathrm{d}z/\mathrm{d}x$ using the chain rule.

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\partial z}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\partial z}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}x} = (1)(2x) + (2)(3x^2)$$

- **Example 2**: $z(u,v) = u + 2v$, $u(x,y) = 3x + 4y$, and $v(x,y) = xy$.

  Compute $\partial z/\partial x$ and $\partial z/\partial y$ using the chain rule.

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = (1)(3) + (2)(y)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = (1)(4) + (2)(x)$$

2.6/2.3: Tangent Planes

■ **Tangent plane to a level surface $F(x, y, z) =$ constant at (x_0, y_0, z_0)**

$$\underline{n}\cdot(x - x_0, y - y_0, z - z_0) = 0 \text{ with } \underline{n} = \underline{\nabla} F(x_0, y_0, z_0) \qquad \text{if } \underline{\nabla} F(x_0, y_0, z_0) \neq \underline{0}$$

Example: Tangent plane to $x + y^2 + z^3 = 8$ at $(3, 2, 1)$.

$F(x, y, z) = x + y^2 + z^3$ and $\underline{\nabla} F = (1,\ 2y,\ 3z^2)$.

Normal $\underline{n} = \underline{\nabla} F(3, 2, 1) = (1, 4, 3)$.

Tangent plane: $(1, 4, 3) \cdot (x - 3, y - 2, z - 1) = 0$

$$1(x - 3) + 4(y - 2) + 3(z - 1) = 0.$$

■ **Special case**: Tangent plane to a graph $z = f(x, y)$ at (x_0, y_0)

$$F(x, y, z) = f(x, y) - z \qquad \Longrightarrow \qquad \underline{\nabla} F = (f_x,\ f_y,\ 1) \text{ and } z_0 = f(x_0, y_0)$$

$$z = f(x_0, y_0) + f_x(x_0, y_0)[x - x_0] + f_y(x_0, y_0)[y - y_0]$$