1.1: Basics of Vectors

■ Notation for Euclidean space

$\mathbb{R}^n$: all points $(x_1, x_2, \ldots, x_n)$ in n-dimensional space

Examples:

1. $\mathbb{R}^1$: all points on the real number line
2. $\mathbb{R}^2$: all points (x_1, x_2) or (x, y) in the plane
3. $\mathbb{R}^3$: all points (x_1, x_2, x_3) or (x, y, z) in space

■ Notation for vectors

Printed: $\boldsymbol{v}$ (bold symbol)

Handwritten: $\underline{v}$ (underlined) or $\vec{v}$ (right arrow)

NEVER JUST v which is the scalar (number) v

■ Notation for vectors in component form Examples (in $\mathbb{R}^3$):

In the book: $\underline{v} = (1, 2, 3)$ or $\underline{v} = \underline{i} + 2\underline{j} + 3\underline{k}$

Alternatives: $\underline{v} = \langle 1, 2, 3 \rangle$ or $\underline{v} = \hat{i} + 2\hat{j} + 3\hat{k}$

1.1: Equation of a Line

■ Vector equation of a line: $\underline{l}(t) = \underline{a} + t\underline{v} \qquad -\infty < t < \infty$

Needed: a point $\underline{a}$ and a direction vector $\underline{v}$

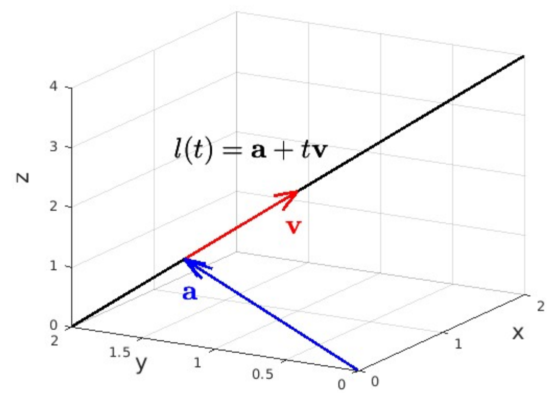
Example: Equation of the line through $(4, 5, 6)$ in the direction of $(1, 2, 3)$

Point $\underline{a} = (4, 5, 6)$ and direction $\underline{v} = (1, 2, 3)$

Eqn of the line: $\underline{l}(t) = \underline{a} + t\underline{v} \quad \Longrightarrow \quad \underline{l}(t) = (4, 5, 6) + t(1, 2, 3)$

■ Geometrical interpretation

Any point on the line can be reached when you start at a position $\underline{a}$ on the line and go in the direction of the direction vector $\underline{v}$ of the line



1.2: Inner Product

■ **Definition: inner product in $\mathbb{R}^n$** (dot product)

$$\underline{a} \cdot \underline{b} = (a_1, a_2, \ldots, a_n) \cdot (b_1, b_2, \ldots, b_n) = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n$$

Example: $(1, 2, 3) \cdot (4, 5, 6) = (1)(4) + (2)(5) + (3)(6) = 32$

■ **Definition: length of a vector**

$$\|\underline{a}\| = \sqrt{\underline{a} \cdot \underline{a}} = \sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}$$

Example: $\|(1, 2, 3)\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$

■ **Definition: unit vector $\underline{u}$** in direction of vector $\underline{v} \neq \underline{0}$.

$$\underline{u} = \frac{\underline{v}}{\|\underline{v}\|}$$

A unit vector has length 1

Example: Unit vector $\underline{u}$ in direction of vector $\underline{v} = (1, 2, 3)$

$$\underline{u} = \frac{\underline{v}}{\|\underline{v}\|} = \frac{(1, 2, 3)}{\|(1, 2, 3)\|} = \frac{1}{\sqrt{14}}(1, 2, 3)$$

1.2: Inner Product

- **Angle** between two vectors: $\underline{a} \cdot \underline{b} = \|\underline{a}\| \; \|\underline{b}\| \cos\theta$ with $\theta \in [0, \pi]$

  Example: Angle between $\underline{a} = (0, -1, 1)$ and $\underline{b} = (0, 2, 0)$.

  $$(0, -1, 1) \cdot (0, 2, 0) = \sqrt{0^2 + (-1)^2 + 1^2} \sqrt{0^2 + 2^2 + 0^2} \cos\theta$$

  $$\cos\theta = \frac{0(0) + (-1)2 + 1(0)}{2\sqrt{2}} = -\frac{1}{\sqrt{2}} \qquad \Longrightarrow \qquad \theta = 3\pi/4$$

- **Perpendicular** (orthogonal)

  Two non-zero vectors $\underline{a}$ and $\underline{b}$ are perpendicular if and only if $\underline{a} \cdot \underline{b} = 0$.

  Example: $(1, 1)$ and $(1, -1)$ are perpendicular since $(1, 1) \cdot (1, -1) = 0$.

- **Work**: $W = \underline{F} \cdot \underline{d}$ "Force times displacement"

  Example: Force $\underline{F} = (1, -1, 3)$ moves a particle along a straight line from point $(1, 0, -2)$ to point $(6, 2, 4)$. Find the work W

  $\underline{d} = (6, 2, 4) - (1, 0, -2) = (5, 2, 6)$ "Endpoint minus initial point"

  $W = \underline{F} \cdot \underline{d} = (1, -1, 3) \cdot (5, 2, 6) = 21$

1.3: Cross Product

- **Definition: cross product in $\mathbb{R}^3$** (outer product)

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \underline{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \underline{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \underline{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Example: Cross product of $\underline{a} = (1, 2, 3)$ and $\underline{b} = (4, 5, 6)$:

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = \underline{i} \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} - \underline{j} \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} + \underline{k} \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$$

$$= (2 \cdot 6 - 3 \cdot 5)\underline{i} - (1 \cdot 6 - 3 \cdot 4)\underline{j} + (1 \cdot 5 - 2 \cdot 4)\underline{k} = (-3, 6, -3)$$

- **Geometrical interpretation**: $\underline{a} \times \underline{b}$ is perpendicular to both $\underline{a}$ and $\underline{b}$

$\underline{a} \times \underline{b}$ is oriented according to the right-hand rule:

"Fingers from $\underline{a}$ to $\underline{b}$ over smaller angle, then thumb is in direction of $\underline{a} \times \underline{b}$"

- **Area of parallelogram**: $A = \|\underline{a} \times \underline{b}\|$

Example: Area of parallelogram spanned by $\underline{a} = (1, 2, 3)$ and $\underline{b} = (4, 5, 6)$.

$$A = \|(-3, 6, -3)\| = \sqrt{(-3)^2 + 6^2 + (-3)^2}$$

1.3: Equation of a Plane

■ Equation of a plane: $\underline{n} \cdot \overrightarrow{P_0P} = 0$

$P_0 = (x_0, y_0, z_0)$: (given) point on plane

$P = (x, y, z)$: any possible point on plane

$\underline{n}$: normal vector to plane

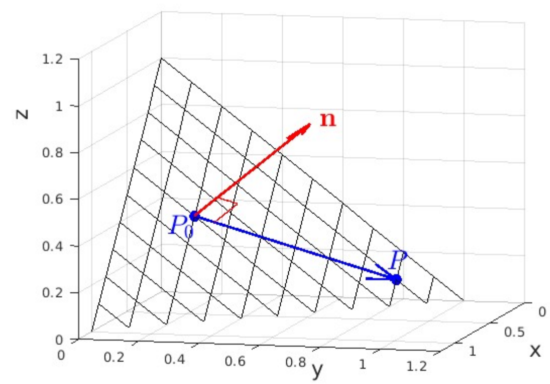
Example: Plane through $(4, 5, 6)$ and perpendicular to the vector $(1, 2, 3)$

Point on plane $P_0 = (4, 5, 6)$ and normal vector $\underline{n} = (1, 2, 3)$

$\underline{n} \cdot \overrightarrow{P_0P} = 0$ gives $(1, 2, 3) \cdot [(x, y, z) - (4, 5, 6)] = 0$

Compute dot product: $1(x - 4) + 2(y - 5) + 3(z - 6) = 0$

■ Geometrical interpretation



Normal vector $\underline{n}$ is perpendicular to any vector in the plane, $\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$

Thus the dot product is zero: $\underline{n} \cdot \overrightarrow{P_0P} = 0$

■ Sketching planes: Connect points by straight lines

- Convenient points: intercepts, vertices, or intersections