7.3: Parametrization of Surfaces

Three important types of parametrizations

I. Graph parametrization

II. Surface of revolution parametrization

III. Sphere parametrization

7.3: Surface Parametrization $\underline{\Phi}(u,v) = \Big(x(u,v), y(u,v), z(u,v)\Big)$

I. Graph parametrization

a) For $z = f(x,y)$

Set $x = u$ and $y = v \quad \Longrightarrow \quad z = f(u,v)$

Parametrization: $\underline{\Phi}(u,v) = \Big(u,\ v,\ f(u,v)\Big)$

b) For $y = g(x,z)$

Set $x = u$ and $z = v \quad \Longrightarrow \quad y = g(u,v)$

Parametrization: $\underline{\Phi}(u,v) = \Big(u,\ g(u,v),\ v\Big)$

c) For $x = h(y,z)$

Set $y = u$ and $z = v \quad \Longrightarrow \quad x = h(u,v)$

Parametrization: $\underline{\Phi}(u,v) = \Big(h(u,v),\ u,\ v\Big)$

7.3: Surface Parametrization $\underline{\Phi}(u, v) = \Big(x(u, v), y(u, v), z(u, v)\Big)$

II. Surface of revolution parametrization

a) For surfaces with circular traces in xy

Set $x = R(u)\cos\theta, \quad y = R(u)\sin\theta, \quad$ and $z = u$

Parametrization $\underline{\Phi}(u, \theta) = \Big(R(u)\cos\theta,\ R(u)\sin\theta,\ u\Big)$

b) For surfaces with circular traces in xz

Set $x = R(u)\cos\theta, \quad z = R(u)\sin\theta, \quad$ and $y = u$

Parametrization $\underline{\Phi}(u, \theta) = \Big(R(u)\cos\theta,\ u,\ R(u)\sin\theta\Big)$

c) For surfaces with circular traces in yz

Set $y = R(u)\cos\theta, \quad z = R(u)\sin\theta, \quad$ and $x = u$

Parametrization $\underline{\Phi}(u, \theta) = \Big(u,\ R(u)\cos\theta,\ R(u)\sin\theta\Big)$

7.3: Surface Parametrization $\underline{\Phi}(u, v) = \Big(x(u, v), y(u, v), z(u, v)\Big)$

III. Sphere parametrization

For (parts of) spheres $x^2 + y^2 + z^2 = R^2$ with easy ϕ and θ bounds

Set $x = (R \sin\phi \cos\theta,\ R \sin\phi \sin\theta,\ R \cos\phi)$

Parametrization: $\underline{\Phi}(\phi, \theta) = (R \sin\phi \cos\theta,\ R \sin\phi \sin\theta,\ R \cos\phi)$

- R: radius of the sphere; R is a constant
- ϕ: angle with the positive z-axis; ϕ is in $[0, \pi]$
- θ: angle with the positive x-axis in the xy-plane projection; θ is in $[0, 2\pi]$

Remark: ϕ and θ are exactly the same as in spherical coordinates

The variable ρ in spherical coordinates is now a constant R

7.3: Parametrized Surfaces

For a surface S with parametrization $\underline{\Phi}(u, v)$ of class C^1

- Tangent vectors to S: $\underline{T}_u = \dfrac{\partial \underline{\Phi}}{\partial u}$ $\qquad$ $\underline{T}_v = \dfrac{\partial \underline{\Phi}}{\partial v}$

- S is called regular if $\underline{T}_u \times \underline{T}_v \neq \underline{0}$ for all (u, v)
    - S is regular at a point if $\underline{T}_u \times \underline{T}_v \neq \underline{0}$ for the (u_0, v_0) at that point

For a regular surface S

- Normal vector to S: $\underline{n} = \underline{T}_u \times \underline{T}_v$

- Tangent plane to S at point $(x_0, y_0, z_0) = \underline{\Phi}(u_0, v_0)$:

  $\underline{n} \cdot (x - x_0, y - y_0, z - z_0) = 0$ $\quad$ with $\quad$ $\underline{n} = \underline{T}_u \times \underline{T}_v$ at (u_0, v_0)