15.6: Triple Integrals in Rectangular Coordinates

■ Evaluation of triple integrals

Example: $$\int_{z=1}^{z=3} \int_{y=0}^{y=2} \left[\int_{x=0}^{x=z^2} 3y \; \mathrm{d}x \right] \mathrm{d}y \; \mathrm{d}z$$

First partial integration over x, keeping y and z constant

Then partial integration over y, keeping z constant

Then integration over z

■ Remarks

- Start with inner inner integral and work outwards
- The result is a number, NOT a function of x, y, or z
- Integrate from low to high x, y, and z

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- **Example**: Compute $\displaystyle\int_{z=1}^{z=3}\int_{y=0}^{y=2}\left[\int_{x=0}^{x=z^2} 3y\,dx\right] dy\,dz$

$$= \int_{z=1}^{z=3}\int_{y=0}^{y=2} 3yx\Big|_{x=0}^{x=z^2} dy\,dz$$

$$= \int_{z=1}^{z=3}\int_{y=0}^{y=2} (3yz^2 - 0)\,dy\,dz$$

$$= \int_{z=1}^{z=3} \frac{3}{2}y^2z^2\Big|_{y=0}^{y=2} dz$$

$$= \int_{z=1}^{z=3} (6z^2 - 0)\,dz = 2z^3\Big|_{z=1}^{z=3} = 2(27 - 1)$$

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Applications for integrals over solids

- Volume $$V = \iiint_E 1 \, \mathrm{d}V$$
- Average value $$\overline{f} = \frac{1}{V} \iiint_E f(x, y, z) \, \mathrm{d}V$$
- Mass $$M = \iiint_E \delta(x, y, z) \, \mathrm{d}V$$
- Center of mass $$\overline{x} = \frac{1}{M} \iiint_E x\delta(x, y, z) \, \mathrm{d}V$$

$$\overline{y} = \frac{1}{M} \iiint_E y\delta(x, y, z) \, \mathrm{d}V$$

$$\overline{z} = \frac{1}{M} \iiint_E z\delta(x, y, z) \, \mathrm{d}V$$

15.6: Triple Integrals

■ **Six orders:** $dx\,dy\,dz$, $dx\,dz\,dy$, $dy\,dx\,dz$, $dy\,dz\,dx$, $dz\,dx\,dy$, $dz\,dy\,dx$

■ **Procedure to set up triple integrals** $\iiint_E f\ dV$

- **Make a sketch of solid E and projection D and determine easiest set up**
  - **One upper and one lower surface**
  - **One double integral for projection D**
- **Find integral bounds using sketch**
  - **Find curves of intersection and points of intersection**
  - **Integrate from low to high x, y, and z**

■ **Evaluation of triple intgrals**

- **Choose an easier order if evaluation of integrals is hard**