13.2: Differentiation and Integration

- **Differentiation of vector functions**: Differentiate each component

  Example: $\underline{r}(t) = \langle t,\ t^2,\ t^3 \rangle$

$$\underline{r}'(t) = \left\langle \frac{dt}{dt},\ \frac{dt^2}{dt},\ \frac{dt^3}{dt} \right\rangle = \left\langle 1,\ 2t,\ 3t^2 \right\rangle$$

- **Second derivative** $\underline{r}''(t) = \left(\underline{r}'\right)'$

  Example: $\underline{r}(t) = \langle t,\ t^2,\ t^3 \rangle$

$$\underline{r}''(t) = \langle 0,\ 2,\ 6t \rangle$$

- **Integration of vector functions**: Integrate each component

  Example: $\underline{r}(t) = \langle t,\ t^2,\ t^3 \rangle$

$$\int \underline{r}(t)\ dt = \left\langle \int t\ dt,\ \int t^2\ dt, \int t^3\ dt \right\rangle = \left\langle \frac{t^2}{2} + C_1,\ \frac{t^3}{3} + C_2,\ \frac{t^4}{4} + C_3 \right\rangle$$

13.2: Differentiation Rules for Vector Functions

For sufficiently smooth vector functions $\underline{u}(t)$ and $\underline{v}(t)$ and scalar functions $f(t)$

1. **Sum rule**: $$\frac{d}{dt}\left[\underline{u}+\underline{v}\right] = \frac{d\underline{u}}{dt} + \frac{d\underline{v}}{dt}$$

2. **Product rule**: $$\frac{d}{dt}\left[f\,\underline{u}\right] = \frac{df}{dt}\,\underline{u} + f\,\frac{d\underline{u}}{dt}$$

3. **Dot product rule**: $$\frac{d}{dt}\left[\underline{u}\cdot\underline{v}\right] = \frac{d\underline{u}}{dt}\cdot\underline{v} + \underline{u}\cdot\frac{d\underline{v}}{dt}$$

4. **Cross product rule**: $$\frac{d}{dt}\left[\underline{u}\times\underline{v}\right] = \frac{d\underline{u}}{dt}\times\underline{v} + \underline{u}\times\frac{d\underline{v}}{dt}$$

5. **Chain rule**: $$\frac{d}{dt}\left[\underline{u}(f(t))\right] = \frac{d\underline{u}}{dt}(f(t))\,\frac{df}{dt}(t)$$

13.2: Tangent Vector and Tangent Line

Let C be the curve defined by $\underline{r}(t)$

- Tangent vector: $\underline{r}'(t)$ is tangent to curve C
  - $\underline{r}'(t)$ is in direction of the curve (not the opposite direction)
  - $\underline{r}'(t)$ needs to exist and $\underline{r}'(t) \neq \underline{0}$ ($\underline{0}$ has no direction)

- Unit tangent vector: $\underline{T}(t) = \dfrac{\underline{r}'(t)}{|\underline{r}'(t)|}$ if $\underline{r}'(t) \neq \underline{0}$

- Tangent line to curve C at point (x_0, y_0, z_0)
  - Vector eqn: $\underline{l}(t) = \underline{r}_0 + t\underline{v}$ with $\underline{r}_0 = \underline{r}(t_0)$ and $\underline{v} = \underline{r}'(t_0)$

    Or: $\langle x(t), y(t), z(t)\rangle = \langle x_0, y_0, z_0\rangle + t\langle x'(t_0), y'(t_0), z'(t_0)\rangle$
  - Meaning: Equation of a line with tangent vector as direction vector
  - t_0 is the t-value corresponding to (x_0, y_0, z_0)

13.2: Intersections and Collisions

- **Intersection of curve and surface**: Position on curve and surface is equal
  - Typically a set of points
  - See 12.5 for intersections of lines and planes

- **Intersection of 2 curves**: positions are equal $\underline{r}_1(t) = \underline{r}_2(s)$
  - Rare in real life, in 2204 typically a point
  - Note time t and s do not need to be equal

- **Collisions**: position and time are equal $\underline{r}_1(t) = \underline{r}_2(t)$
  - Very rare in real life, in 2204 typically a point or no collision
  - Note both position and time are equal