Chapter 1
Policy Sheet and Introduction

Course 6125 (Topics in Algebra; Homological Algebra), index 7834
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Assessment This will be based on homework which I will collect in once a week on Mondays. There will be a choice of problems: students who have not passed the Ph.D. qualifying oral exam will be required to hand in solutions to two problems; students who have passed the Ph.D. qualifying oral exam will only be required to hand in solutions to one problem! (A generous instructor!)

Late Homework Half credit (zero credit if received after I have given out the solutions in class).

Homework Policy Students are allowed to discuss homework among themselves.

Prerequisites 5125–5126 Graduate Abstract Algebra (or consent of the instructor). Also useful will be 5334 Algebraic Topology.

Books I have decided that there is no book entirely suitable for the course. If I had to choose a book, it would be "Cohomology of Groups" by K. S. Brown, Graduate Texts in Math. no. 87, Springer-Verlag, Berlin–New York, 1982, ISBN 0-387-90688-6. If you want to buy it, order from Springer direct (call toll-free 1-800-SPRINGER); assuming it is in stock you should receive it within 10 days and it would cost about $40. It is a very well written book and all the material covered is important. However it assumes rather more algebraic topology than is suitable for this course (to define CW-complex and prove the Hurewicz isomorphism theorem would take too much time, and I think most of this class are not familiar with these topics). Other books relevant to the course are as follows.

(1) "Representations and Cohomology I & II" (two books) by D. J. Benson, Cambridge Studies in Advanced Math. nos. 30 & 31, Cambridge Univ. Press, Cambridge–New York, 1991, ISBN 0-521-36134-6 and 0-521-36135-4 (call toll free 1-800-872-7423). This has a wealth of recent important material. On the other hand it is quite densely written and there is a shortage of exercises and examples. Perhaps more relevant from my point of view is that it deals only with finite groups, while I want to give finite and infinite groups equal emphasis.

(2) "Modular Representation Theory: New Trends and Methods" by D. J. Benson, Lecture Notes in Math. 1081, Springer-Verlag, Berlin–New York, 1984, ISBN 0-387-13389-5. This is the forerunner of (1), but it is not subsumed by (1) and has an appendix with many numerical examples. Also being in the Springer Lecture Note series, it should be inexpensive; I would guess about $30.

(3) "The Cohomology of Finite Groups" by L. Evens, Oxford Univ. Press, Oxford–New York, 1991, ISBN 0-19-853580-5 (call toll free 1-800-451-7556). Similar to (1) but covers less material at a slower pace.

(4) "An Introduction to Homological Algebra" by C. Weibel, Cambridge Studies in advanced Math. no. 38, Cambridge Univ. Press, Cambridge–New York, 1994, ISBN 0-521-43500-5. The material here is quite close to that which will be covered in this course. However it is rather disorganized, and there is a shortage of exercises and examples. Also only experts will find chapters 1 and 2 comprehensible, though it does become easier later on.

(5) "Homology" by S. Mac Lane, Springer-Verlag, Berlin–New York, 1975, ISBN 0-387-03823-X. A good reference for cohomology over general rings; however I would like to make explicit calculations for group cohomology: also it is somewhat out of date.

(6) "A Course in Homological Algebra" by P. J. Hilton and U. Stammbach, Graduate Texts in Math. no. 4, Springer-Verlag, Berlin–New York, 1971, ISBN 0-387-90032-2. Similar to (5), but with a little more on group cohomology.

(7) "An Introduction to Homological Algebra" by J. J. Rotman, Pure and Applied Mathematics Series no. 85, London–New York, 1979, ISBN 0-12-599250-5. Similar to (5) and (6).

(8) "Cohomology of Finite Groups" by A. Adem and R. J. Milgram, Grundlehren der Mathematischen Wissenschaften, vol. 309, Springer-Verlag, Berlin–New York, 1994, ISBN 0-387-57025-X. Concentrates on cohomology of finite groups. Assumes a fair amount of algebraic topology: for example chapter 2 assumes a knowledge of G-bundles and classifying spaces. Also chapter 7 assumes a knowledge of the Chevalley groups and Dynkin diagrams. There are no exercises.

Syllabus Unfortunately not all the students are familiar with tensor products, so that will be the starting point of the course. The textbook you used for 5125–5126 should cover tensor products, at least over commutative rings. However I will require tensor products over noncommutative rings. Since much of this is very familiar to many of the class, I will cover it quickly. For the ones who have not seen it before, it is important that you do not get lost at this point, because tensor products are fundamental to the whole of Homological Algebra. After that I will cover chain complexes and then I will define Ext and Tor.

Then I will turn to cohomology of groups. After defining the group ring RG of the group G over the commutative ring R, I will cover the Künneth formula and Universal Coefficient theorem. I will use these to determine the additive structure of $\mathrm{H}^*(G, \mathbb{Z})$ for any finitely generated abelian group G.

The next topic will be cup products. This induces a ring structure on $\mathrm{H}^*(G, R)$ for any commutative ring R, and is the reason why cohomology seems to be superior to homology, even though the two theories are dual to each other. I will calculate the cohomology ring $\mathrm{H}^*(G, k)$ for any finitely generated abelian group G and any field k. I will also consider the case $k = \mathbb{Z}$.

Notation

$\mathbb{Z}$	= integers	$\mathbb{Q}$	= rational numbers
$\mathbb{R}$	= real numbers	$\mathbb{C}$	= complex numbers
$\mathbb{N}$	= natural numbers $(0, 1, 2, \dots)$	$\mathbb{P}$	= positive integers $(1, 2, 3, \dots)$
$\forall$	= for all	$\exists$	= there exists
$\in$	= is an element of	$\notin$	= is *not* an element of
$\cup$	= union	$\cap$	= intersection

$\subseteq$ = is a subset of
$\varnothing$ = emptyset
$|A|$ = order of A (possibly infinite)
$a|b$ = a divides b
$A \subset B = A \subseteq B$ and $A \neq B$
$A \backslash B = \{a \in A \mid a \notin B\}$
$\leqslant$ = is a subgroup (or subring) of
$A < B = A \leqslant B$ and $A \neq B$
$H \lhd G = H$ is a normal subgroup of G
$\cong$ = is isomorphic to
$\ker\theta$ = kernel of the map θ
$\operatorname{im}\theta$ = image of the map θ
$\mathrm{M}_n(R)$ = $n \times n$ matrices over a ring R
$\mathrm{M}_{m,n}(R)$ = $m \times n$ matrices over a ring R
$\mathrm{GL}_n(R)$ = $\{A \in \mathrm{M}_n(R) \mid A \text{ is invertible}\}$
$\mathrm{H}_n(G,R)$ = nth homology group of the group G with coefficients in the ring R
$\mathrm{H}^n(G,R)$ = nth cohomology group of the group G with coefficients in the ring R
(a,b) = greatest common divisor of a and b
$[a,b]$ = lowest common multiple of a and b
Σ_n = symmetric group of degree n
$N_H(K)$ = normalizer in H of K (usually H and K will be subgroups)
$C_H(K)$ = centralizer in H of K (usually H and K will be subgroups)
$I \lhd R$ = I is an ideal of the ring R
$\mathfrak{P} \lhd_{\mathrm{p}} R$ = $\mathfrak{P}$ is a prime ideal of the ring R
$I \lhd_{\mathrm{l}} R$ = I is a left ideal of the ring R
$I \lhd_{\mathrm{r}} R$ = I is a right ideal of the ring R
$\mathfrak{M} \lhd\!\!\lhd R$ = $\mathfrak{M}$ is a maximal ideal of the ring R

Terminology and Assumed Elementary Results All rings will have a one, and modules may be left or right modules. However unless otherwise stated, modules will be right modules and mappings will be written on the left. Furthermore all modules will be unital modules: this means that if M is a module, then $m1 = m$ for all $m \in M$. If R is a ring and M, N are right R-modules, then $\mathrm{Hom}_R(M,N)$ will denote the R-module homomorphisms from M to N, i.e. $\{f : M \to N \mid f(mr) = (fm)r \text{ for } m \in M,\ r \in R\}$. If $\alpha\colon N \to A$ is a homomorphism of right R-modules, then we often denote the group homomorphism from $\mathrm{Hom}_R(M,N)$ to $\mathrm{Hom}_R(M,A)$ defined by $f \mapsto \alpha f$ for $f \in \mathrm{Hom}_R(M,N)$ by α_*. Similarly if $\beta\colon M \to B$ is a homomorphism of right R-modules, then we denote the group homomorphism from $\mathrm{Hom}_R(B,N) \to \mathrm{Hom}_R(M,N)$ defined by $f \mapsto f\beta$ for $f \in \mathrm{Hom}_R(B,N)$ by β^*. Then for any R-module maps α and β, we have $(\alpha\beta)_* = \alpha_*\beta_*$ and $(\alpha\beta)^* = \beta^*\alpha^*$, assuming of course that the relevant compositions are defined.

Exercise 1 Let R be a ring and let M be a right R-module. Show that we can make $\mathrm{Hom}_R(R,M)$ into a right R-module by defining $(fr)s = f(rs)$ for $f \in \mathrm{Hom}_R(R,M)$, $r, s \in R$, and then we have $M \cong \mathrm{Hom}_R(R,M)$ as R-modules.

Exercise 2 Let R be a ring and let M be a right R-module. Show that we can make $\mathrm{Hom}_R(M,R)$ into a left R-module by defining $(rf)m = r(fm)$ for $r \in R$, $f \in \mathrm{Hom}_R(M,R)$ and $m \in M$. (Often $\mathrm{Hom}_R(M,R)$ is called the dual of M and is denoted M^*.)

Exercise 3 Let R be a ring and let M be an R-module. Prove that for each $m \in M$, the formula $f \mapsto f(m)$ for $f \in M^*$ defines an element $\theta(m)$ of M^{**}. Prove further that θ is an R-module homomorphism from M to M^{**}. In the special case $R = \mathbb{Z}$, $M = \mathbb{Q}$, show that this is the zero map.

Exercise 4 Let R be a ring, let M be an R-module, and let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ be an exact sequence of R-modules. Prove that the induced sequence of groups $0 \to \mathrm{Hom}_R(M,A) \xrightarrow{\alpha_*} \mathrm{Hom}_R(M,B) \xrightarrow{\beta_*} \mathrm{Hom}_R(M,C)$ is also exact.

Exercise 5 Let R be a ring, let M be an R-module, and let $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be an exact sequence of R-modules. Prove that the induced sequence of groups $0 \to \mathrm{Hom}_R(C,M) \xrightarrow{\beta^*} \mathrm{Hom}_R(B,M) \xrightarrow{\alpha^*} \mathrm{Hom}_R(A,M)$ is also exact.

Let R be a ring, let $\mathcal{I}$ be a set, and let $\{M_i \mid i \in \mathcal{I}\}$ be a family of R-modules. Then the direct sum $\bigoplus_{i\in\mathcal{I}} M_i$ is the R-module whose elements are the sequences $\{m_i\}$ with $i \in \mathcal{I}$ and $m_i = 0$ for all but finitely many i, and the R-module structure is defined by $\{m_i\}r = \{m_i r\}$ for $r \in R$. Similarly the cartesian sum $\prod_{i\in\mathcal{I}} M_i$ is the R-module whose elements are *all* sequences $\{m_i\}$ with $i \in \mathcal{I}$, and the R-module structure is again defined by $\{m_i\}r = \{m_i r\}$ for $r \in R$. Of course if $|\mathcal{I}| < \infty$, then $\bigoplus_{i\in\mathcal{I}} M_i \cong \prod_{i\in\mathcal{I}} M_i$, but in general $\bigoplus_{i\in\mathcal{I}} M_i$ is *not* isomorphic to $\prod_{i\in\mathcal{I}} M_i$.

Exercise 6 Let $\mathcal{I}$ be a set, let R be a ring, let A be an R-module, and let $\{M_i \mid i \in \mathcal{I}\}$ be a family of R-modules. Prove

(i) $\mathrm{Hom}_R(\bigoplus_{i\in\mathcal{I}} M_i, A) \cong \prod_{i\in\mathcal{I}} \mathrm{Hom}_R(M_i, A)$,

(ii) $\mathrm{Hom}_R(A, \prod_{i\in\mathcal{I}} M_i) \cong \prod_{i\in\mathcal{I}} \mathrm{Hom}_R(A, M_i)$,

(iii) $\mathrm{Hom}_R(A, \bigoplus_{i\in\mathcal{I}} M_i) \cong \bigoplus_{i\in\mathcal{I}} \mathrm{Hom}_R(A, M_i)$ if A is finitely generated.

(iv) Show (iii) is false without the hypothesis that A is finitely generated.

Bimodules If R, S are rings, then M is an (R,S)-bimodule means

(i) M is a left R-module,
(ii) M is a right S-module,
(iii) If $r \in R$, $s \in S$, and $m \in M$, then $(rm)s = r(ms)$.

In other words, M is both a left R-module and a right S-module, and (this is the content of (iii)) the left and right actions commute (it is *not* enough to have only (i) and (ii), namely that M is both a left R-module and right S-module).

Examples

(1) If R is any ring and $n \in \mathbb{N}$, then R^n (the direct sum of n copies of R) is an (R,R)-bimodule, where the left and right R-module structures come from left and right multiplication by the elements of R respectively.

(2) If R is a commutative ring and M is a right R-module, then we can view M also as a left R-module by defining $rm = mr$ for $r \in R$ and $m \in M$. Then (i) and (ii) of the above are satisfied, and then (iii) is also satisfied and M becomes an (R, R)-bimodule.

(3) If R is any ring and M is a right R-module, then M is an $(\mathrm{End}_R(M, M), R)$-bimodule.

Exercise 7 Let R, S be rings, let M be an (R, S)-bimodule, and let N be a right S-module. Prove that $\mathrm{Hom}_S(M, N)$ becomes a right R-module by defining $(fr)m = f(rm)$ for $f \in \mathrm{Hom}_S(M, N)$, $r \in R$ and $m \in M$.

Exercise 8 Let R, S be rings, let M be a right S-module, and let N be an (R, S)-bimodule, Prove that $\mathrm{Hom}_S(M, N)$ becomes a left R-module by defining $(rf)m = r(fm)$ for $r \in R$, $f \in \mathrm{Hom}_S(M, N)$ and $m \in M$.

Projective Modules Let R be a ring and let P be an R-module. Then P is projective if and only if every short exact sequence of R-modules of the form $0 \to M \to N \to P \to 0$ splits. Then a basic result is that the following are equivalent:

(i) P is a projective R-module,
(ii) there exists an R-module Q such that $P \oplus Q$ is a free R-module,
(iii) given a short exact sequence of R-modules $M \xrightarrow{\alpha} N \to 0$, then the induced sequence $\mathrm{Hom}_R(P, M) \xrightarrow{\alpha_*} \mathrm{Hom}_R(P, N) \to 0$ is also exact, where α_* is defined to be the map $f \mapsto \alpha f$.

Exercise 9 Let R be a ring, let P be a projective right R-module, and let $e \in \mathrm{End}_R(P)$ be an idempotent (i.e. $e^2 = e$). Prove that eP is a projective R-module.

Exercise 10 Let I be a set, let R be a ring, and let $\{M_i \mid i \in \mathcal{I}\}$ be a family of projective R-modules. Prove that $\bigoplus_{i \in \mathcal{I}} M_i$ is a projective R-module.

Exercise 11 For each $i \in \mathbb{P}$, let P_i be a $\mathbb{Z}$-module isomorphic to $\mathbb{Z}$. Prove that $\prod_{i \in \mathbb{P}} P_i$ is *not* a projective $\mathbb{Z}$-module.

Injective Modules The dual notion to a projective module is an injective module. Specifically to define injective module, one reverses the arrows in the definition of projective module above, so if R is a ring and I is an R-module, then I is an injective R-module if and only if every short exact sequence of R-modules of the form $0 \to I \to M \to N \to 0$ splits. Then a basic fact is that the following are equivalent:

(i) I is an injective R-module,
(ii) given a short exact sequence of R-modules $0 \to M \xrightarrow{\alpha} N$, then the induced sequence $\mathrm{Hom}_R(N, I) \xrightarrow{\alpha^*} \mathrm{Hom}_R(M, I) \to 0$ is also exact, where α^* is defined to be the map $f \mapsto f\alpha$.

Unfortunately, there is no nice characterization of injective modules like (ii) for projective modules; on the other hand every module M can be embedded in a unique "smallest injective module", called the injective hull of M.

Exercise 12 Prove that $\mathbb{Q}$ is an injective $\mathbb{Z}$-module.

Exercise 13 Let $\mathcal{I}$ be a set, let R be a ring, and let $\{I_i \mid i \in \mathcal{I}\}$ be a family of injective R-modules. Prove that $\prod_{i\in\mathcal{I}} I_i$ is an injective R-module.

Remark In general the direct sum $\bigoplus_{i\in\mathcal{I}_i} I_i$ is not an injective module, though examples are less easy to find.

Tensor Products There are many reasons why one wants to construct tensor products, and we will describe two of them here before making the formal definition.

Let F be a field, and let U, V be vector spaces over F of dimensions m, n respectively. It is easy to construct a vector space of dimension $m+n$ over F from these two vector spaces, namely $U \oplus V$, however it would be nice to also construct a vector space of dimension mn from these two spaces in some natural way. The tensor product $U \otimes_F V$ will fulfill this purpose.

A second reason is to extend scalars. Let F be a field and let V be a vector space over F. Often we want to consider V as a vector space over some larger field E: for example one may want to compute the Jordan Canonical Form of an F-endomorphism of V, and then one wants to work in the algebraic closure of F. One way to do this is to take a basis $\{e_1, \ldots, e_n\}$ of V (at least in the case when $\dim_F V = n < \infty$), and then consider the vector space over E with basis $\{e_1, \ldots, e_n\}$. However this is non-canonical, and is also awkward when $\dim_F V = \infty$. Here the tensor product $V \otimes_F E$ is what is required.

The above two examples are tensor products over a field; however it turns out that one needs them over any ring.

Definition Let R be a ring, let M be a right R-module, and let N be a left R-module. Let F be the free abelian group with basis $\{(m,n) \mid m \in M \text{ and } n \in N\}$, and let E be the subgroup of F generated by

$$\begin{aligned}\{&(m_1+m_2,n)-(m_1,n)-(m_2,n) \mid m_1, m_2 \in M \text{ and } n \in N,\\ &(m, n_1+n_2)-(m,n_1)-(m,n_2) \mid m \in M \text{ and } n_1, n_2 \in N\\ &\text{and} \quad (mr,n)-(m,rn) \mid m \in M,\ n \in N \text{ and } r \in R\}.\end{aligned}$$

Then $M \otimes_R N$ is the abelian group F/E. It is conventional to denote the element $E+(m,n)$ of $M \otimes_R N$ by $m \otimes n$, and then we can define a map $\tau\colon M \times N \to M \otimes_R N$ by $\tau(m,n) = m \otimes n$. This map is in general neither a group homomorphism nor onto: however the image of τ does generate $M \otimes_R N$ as an abelian group. Then we have for $m, m_1, m_2 \in M$ and $n \in N$, $(m_1+m_2) \otimes n = m_1 \otimes n + m_2 \otimes n$ and $0 \otimes n = 0$. We note that τ satisfies the following:

$$\begin{aligned}&\text{(i)} \quad \tau(m_1+m_2,n) = \tau(m_1,n)+\tau(m_2,n) && \text{for all } m_1, m_2 \in M \text{ and } n \in N,\\ &\text{(ii)} \quad \tau(m, n_1+n_2) = \tau(m,n_1)+\tau(m,n_2) && \text{for all } m \in M \text{ and } n_1, n_2 \in N,\\ &\text{(iii)} \quad \tau(mr,n) = \tau(m,rn) && \text{for all } m \in M, n \in N \text{ and } r \in R.\end{aligned}$$

A map τ satisfying (i), (ii) and (iii) above is called a *balanced* (or R-balanced) map.

This is not a very enlightening definition: it can be difficult to comprehend because the free abelian group F is very large, and then we factor out by the very large subgroup E, so

what do we finish with? Also without further conditions on M and N, $M \otimes_R N$ is only an abelian group; however if N is an (R, R)-bimodule (in particular if R is commutative — see the section on bimodules), then we shall see that $M \otimes_R N$ is a right R-module.

It is much better to use the universal property of tensor products than to use the definition above.

Universal Property of Tensor Products Let R be a ring, let M be a right R-module, let N be a left R-module, and let $\tau\colon M \times N \to M \otimes_R N$ denote the balanced map $(m, n) \mapsto m \otimes n$. Suppose we are given an abelian group G and a balanced map $\theta\colon M \times N \to G$. Then there exists a unique group homomorphism $\phi\colon M \otimes_R N \to G$ such that $\theta = \phi\tau$.

Proof That ϕ is unique is clear because the image of τ generates $M \otimes_R N$ as an abelian group. To prove the existence of ϕ, first define a group homomorphism $\psi\colon F \to G$ by $\psi(m, n) = \theta(m, n)$; this is a good definition because the elements $\{(m, n) \mid m \in M \text{ and } n \in N\}$ are a $\mathbb{Z}$-basis for the free abelian group F. Since θ is a balanced map, we have for $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$ and $r \in R$

$$\begin{aligned} \psi(m_1 + m_2, n) &= \psi(m_1, n) + \psi(m_2, n) \\ \psi(m, n_1 + n_2) &= \psi(m, n_1) + \psi(m, n_2) \\ \text{and} \quad \psi(mr, n) &= \psi(m, rn). \end{aligned}$$

Thus ψ kills a generating set for E and it follows that $E \subseteq \ker\psi$. Therefore there exists a group homomorphism $\phi\colon F/E \to G$ such that $\phi(m \otimes n) = \psi(m, n)$ for all $m \in M$ and $n \in N$, which is what is required.

It is routine to show that $M \otimes_R N$ is determined up to isomorphism by this universal property. Precisely if $(M \otimes_R N)'$ and $\tau'\colon M \times N \to (M \otimes_R N)'$ satisfy the above for $M \otimes_R N$ and $\tau\colon M \times N \to M \otimes_R N$ respectively, then there exists a unique group isomorphism $\alpha\colon M \otimes_R N \to (M \otimes_R N)'$ such that $\alpha\tau = \tau'$. It is also easy to verify that $(M_1 \oplus M_2) \otimes_R N \cong M_1 \otimes_R N \oplus M_2 \otimes_R N$. We now establish the following.

Module Structure on $M \otimes_R N$. Let R, S be rings, let M be an R-module and let N be an (R, S)-bimodule. Then there exists a unique S-module structure on $M \otimes_R N$ such that $(m \otimes n)s = m \otimes (ns)$ for all $m \in M, n \in N$ and $s \in S$.

Proof Fix $s \in S$ and define $\sigma\colon M \times N \to M \otimes_R N$ by $\sigma(m, n) = m \otimes ns$. Since $(m, n) \mapsto m \otimes n$ is a balanced map, we have for $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$ and $r, s \in R$

$$\begin{aligned} \sigma(m_1 + m_2, n) &= (m_1 + m_2) \otimes ns = m_1 \otimes ns + m_2 \otimes ns = \sigma(m_1, n) + \sigma(m_2, n), \\ \sigma(m, n_1 + n_2) &= m \otimes (n_1 s + n_2 s) = m \otimes n_1 s + m \otimes n_2 s = \sigma(m, n_1) + \sigma(m, n_2), \\ \sigma(mr, n) &= mr \otimes ns = m \otimes rns = \sigma(m, rn), \end{aligned}$$

which shows that σ is an R-balanced map. Therefore by the universal property of tensor products, there exists a unique group homomorphism $\hat{s}\colon M \otimes_R N \to M \otimes_R N$ such that $(m \otimes n)\hat{s} = m \otimes ns$ for all $m \in M, n \in N$. This establishes the uniqueness part of the S-module

structure on $M \otimes_R N$, and we now define $us = u\hat{s}$ for $u \in M \otimes_R N$ and $s \in S$. It remains to check that this makes $M \otimes_R N$ into an S-module. Since $\hat{s}$ is a group homomorphism, all we need is that $u(s+t) = us+ut$ for $u \in M \otimes_R N$ and $s, t \in S$. Since the elements $m \otimes n$ generate $M \otimes_R N$ as an abelian group, it is sufficient to do this when u is of the form $m \otimes n$. But

$$\begin{aligned}(m \otimes n)(s+t) &= m \otimes n(s+t) = m \otimes (ns+nt) \\ &= m \otimes ns + m \otimes nt = (m \otimes n)s + (m \otimes n)t\end{aligned}$$

as required.

Of course if instead of N being an (R, S)-bimodule, M is an (S, R)-bimodule, then we can make $M \otimes_R N$ into a left S-module by the rule $s(m \otimes n) = sm \otimes n$.

As a special case, consider the case when $R = S$ and is commutative. Then we can view N as an (R, R)-bimodule (see the section on bimodules). Thus $M \otimes_R N$ is an R-module via $(m \otimes n)r = m \otimes nr$, and in fact it is determined by the following universal property, which we state without proof because it is very similar to the previous case. Before proceeding, we recall the definition of a bilinear map.

Definition of Bilinear Map Let R be a commutative ring, and let M, N, P be R-modules. Then a map $\beta\colon M \times N \to P$ is R-bilinear means for $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$ and $r \in R$

(i) $\beta(m_1 + m_2, n) = \beta(m_1, n) + \beta(m_2, n)$,
(ii) $\beta(m, n_1 + n_2) = \beta(m, n_1) + \beta(m, n_2)$,
(iii) $\beta(mr, n) = \big(\beta(m, n)\big)r$,
(iv) $\beta(m, nr) = \big(\beta(m, n)\big)r$.

Note that when R is a commutative ring, then the map $\tau : M \times N \to M \otimes_R N$ defined by $\tau(m, n) = m \otimes n$ is R-bilinear.

Universal Property of Tensor Products Over Commutative Rings Let R be a commutative ring, let M and N be R-modules, and let $\tau\colon M \times N \to M \otimes_R N$ denote the R-bilinear map $(m, n) \mapsto m \otimes n$. Suppose we are given an R-module P and an R-bilinear map $\beta\colon M \times N \to P$. Then there exists a unique R-map $\theta\colon M \otimes_R N \to P$ such that $\beta = \theta\tau$.

Examples In these examples, you will find that $M \otimes_R N$ behaves in a similar way to $\mathrm{Hom}_R(M, N)$.

(1) Let K be a field, and let U, V be vector spaces over K with bases $\{u_1, \ldots, u_m\}$ and $\{v_1, \ldots, v_n\}$ respectively. Then $U \otimes_K V$ is the vector space with basis $\{u_i \otimes v_j \mid 1 \le i \le m \text{ and } 1 \le j \le n\}$.

Proof To see this, let us consider U as $m \times 1$ column vectors and V as $1 \times n$ row vectors. Then we can define a K-bilinear map $\tau\colon U \times V \to \mathrm{M}_{m,n}(K)$ by $\tau(u, v) = uv$ (matrix multiplication), which is well defined because the matrices u, v have compatible size. We want to show that $\mathrm{M}_{m,n}(K)$ has the universal property for tensor products over the commutative ring K, so suppose W is a K-module and we have a K-bilinear map $\beta\colon U \times V \to W$. We may assume that $\{u_1, \ldots, u_m\}$ and $\{v_1, \ldots, v_n\}$ are the standard K-bases for U and V respectively. Then

$\{u_iv_j \mid 1 \le i \le m\}$ and $\{1 \le j \le n\}$ is a K-basis for $\mathrm{M}_{m,n}(K)$, so we can define a K-map $\theta\colon \mathrm{M}_{m,n}(K) \to W$ by $\theta(u_iv_j) = \beta(u_i, v_j)$. If $u = \sum a_iu_i \in U$ and $v = \sum b_iv_i \in V$, then

$$\begin{aligned}\theta\tau(u,v) &= \theta\tau\big(\sum_{i,j}(a_iu_i, b_jv_j)\big) = \theta\sum_{i,j} a_ib_ju_iv_j = \sum_{i,j} a_ib_j\beta(u_i,v_j)\\ &= \sum_{i,j}\beta(a_iu_i, b_jv_j) \quad \text{(because } \beta \text{ is bilinear)}\\ &= \beta(u,v),\end{aligned}$$

hence $\beta = \theta\tau$. Also if $\theta'\colon \mathrm{M}_{m,n} \to W$ is another K-map satisfying $\beta = \theta'\tau$, then $\theta' = \theta$ because θ and θ' agree on the K-basis $\{u_iv_j\}$ of $\mathrm{M}_{m,n}(K)$. This establishes the universal property and it follows that $M \otimes_K N \cong \mathrm{M}_{m,n}(K)$.

(2) Let R be a ring, and let M be a right R-module. Then $M \otimes_R R \cong M$ as right R-modules.

Proof Define $\tau\colon M \times R \to M$ by $\tau(m,r) = mr$ for $m \in M$ and $r \in R$. Then it is easily checked that τ is a balanced map. Suppose P is a right R-module and $\beta\colon M \times R \to P$ is a balanced map. Then we define $\theta\colon M \to P$ by $\theta m = \beta(m,1)$. Then it is easily checked that θ is a group homomorphism satisfying $\beta = \theta\tau$. Also if $\theta'\colon M \to P$ is another group homomorphism satisfying $\beta = \theta'\tau$, then it is clear that $\theta' = \theta$. Thus the universal property for tensor products is verified and it follows that $M \otimes_R R \cong M$ as right R-modules.

(3) $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} = 0$.

Proof The elements of the form $m \otimes n$ for $m \in \mathbb{Z}/2\mathbb{Z}$ and $n \in \mathbb{Z}/3\mathbb{Z}$ generate $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z}$ as a $\mathbb{Z}$-module, so we need to prove that all these elements are zero. Now

$$(m \otimes n)2 = m \otimes n2 = m \otimes 2n = m2 \otimes n = 0 \otimes n = 0,$$

and similarly $(m \otimes n)3 = 0$. Therefore $m \otimes n = (m \otimes n)3 - (m \otimes n)2 = 0 - 0 = 0$ as required.

(4) Let R, S be a rings, let M be a right R-module, let N be an (R,S)-bimodule, and let P be a left S-module. Then $(M \otimes_R N) \otimes_S P \cong M \otimes_R (N \otimes_S P)$ as abelian groups.

Proof (sketch) For $p \in P$, define $f_p\colon M \times N \to M \otimes_S (N \otimes_R P)$ by $f_p(m,n) = m \otimes (n \otimes p)$. Then it is easily checked that f_p is an R-balanced map, hence it induces a group homomorphism $\hat{f}_p\colon M \otimes_R N \to M \otimes_R (N \otimes_S P)$. Since $m \otimes (ns \otimes p) = m \otimes (n \otimes sp)$ for $s \in S$ and the elements $m \otimes n$ generate $M \otimes_R N$, it follows that $\hat{f}_p(us) = \hat{f}_{sp}(u)$ for all $u \in M \otimes_R N$. We can now define an S-balanced map $f\colon (M \otimes_R N) \times P \to M \otimes_R (N \otimes_S P)$ by $f(u,p) = f_p(u)$ for $u \in M \otimes_R N$, and this induces a group homomorphism $f : (M \otimes_R N) \otimes_S P \to M \otimes_R (N \otimes_S P)$ satisfying $f((m \otimes n) \otimes p) = m \otimes (n \otimes p)$. Similarly there exists a group homomorphism $g\colon M \otimes_R (N \otimes_S P) \to (M \otimes_R N) \otimes_S P$ such that $g(m \otimes (n \otimes p)) = (m \otimes n) \otimes p$. Then fg and gf are the identity maps (because they agree with the identity on a generating set), and it follows that f (and g) is an isomorphism as required.

Exercise 14 Let $F \subseteq E$ be fields, let U be an F-vector space with basis $\mathcal{B}$, and let V be an E-vector space also with basis $\mathcal{B}$. Prove that $U \otimes_F E \cong V$ as E-vector spaces.

Exercise 15 Let R be a commutative ring, and let M, N be R-modules. Prove that the mapping $m \otimes n \mapsto n \otimes m$ defines an R-isomorphism $M \otimes_R N \to N \otimes_R M$.

Exercise 16 Let R be a commutative ring, let S be a multiplicatively closed subset of R, and let M be an R-module. Prove that $M \otimes_R S^{-1}R \cong S^{-1}M$ as $S^{-1}R$-modules. (Recall that $S^{-1}M$ is the $S^{-1}R$-module obtained from M by inverting all the elements in S. It consists of elements $s^{-1}m$ with $s^{-1}m = s_1^{-1}m_1$ if and only if there exists $t \in S$ such that $(ms_1 - m_1 s)t = 0$. The $S^{-1}R$-module structure is given by $(s^{-1}m)(t^{-1}r) = (st)^{-1}(mr)$.)

Exercise 17 Let a, b be nonzero integers, and let $l = (a, b)$. Prove that $\mathbb{Z}/a\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/b\mathbb{Z} \cong \mathbb{Z}/l\mathbb{Z}$.

If R is a ring, A, M are right R-modules, B, N are left R-modules, $\alpha\colon A \to M$ is a right R-map and $\beta\colon B \to N$ is a left R-map, then it is easily checked that the formula $a \otimes b \mapsto \alpha(a) \otimes \beta(b)$ (for $a \in A, b \in B$) well defines a group homomorphism $A \otimes_R B \to M \otimes_R N$: this is usually denoted $\alpha \otimes \beta$. If $\theta\colon M \to U$ is a right R-map and $\phi\colon N \to V$ is a left R-map, then $\alpha\theta \otimes \beta\phi = (\alpha \otimes \beta)(\theta \otimes \phi)\colon A \otimes_R B \to U \otimes_R V$.

Example Define $\mu\colon \mathbb{Z}/27\mathbb{Z} \to \mathbb{Z}/9\mathbb{Z}$ by $\mu[n] = [3n]$ where for $n \in \mathbb{Z}$, $[n]$ denotes the residue class modulo 27 or 9. Let ι denote the identity map on $\mathbb{Z}/9\mathbb{Z}$. Then $\mu \otimes \iota\colon \mathbb{Z}/27\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/9\mathbb{Z} \to \mathbb{Z}/9\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/9\mathbb{Z}$ can be identified with the map "multiplication by 3" from $\mathbb{Z}/9\mathbb{Z} \to \mathbb{Z}/9\mathbb{Z}$.

Proof This is because there are isomorphisms $\theta\colon \mathbb{Z}/27\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/9\mathbb{Z} \to \mathbb{Z}/9\mathbb{Z}$, $\phi : \mathbb{Z}/9\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/9\mathbb{Z} \to \mathbb{Z}/9\mathbb{Z}$ such that $\theta[1]\otimes[1] = [1]$, $\phi[1]\otimes[1] = [1]$, and $(\mu\otimes\iota)[1]\otimes[1] = [3]\otimes[1] = 3[1]\otimes[1]$.

The following is the corresponding result to Exercises 4 and 5 for tensor products.

Theorem Let R be a ring, let M be a left R-module, and let $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be an exact sequence of right R-modules. Then the induced sequence of abelian groups

$$A \otimes_R M \xrightarrow{\alpha\otimes 1} B \otimes_R M \xrightarrow{\beta\otimes 1} C \otimes_R M \longrightarrow 0$$

(where 1 denotes the identity map on M) is also exact.

Proof The image of $\beta \otimes 1$ contains all elements of the form $\beta b \otimes m$ ($b \in B, m \in M$), and hence all elements of the form $c \otimes m$ ($c \in C, m \in M$) because β is surjective. Since these elements generate $C \otimes_R M$, it follows that $\beta \otimes 1$ is also surjective and hence we have exactness at $C \otimes_R M$. Since $(\beta \otimes 1)(\alpha \otimes 1) = \beta\alpha \otimes 1 = 0 \otimes 1 = 0$, it follows that $\operatorname{im}\alpha \otimes 1 \subseteq \ker\beta \otimes 1$ and it remains to prove that $\ker\beta \otimes 1 \subseteq \operatorname{im}\alpha \otimes 1$.

Let $I = \operatorname{im}\alpha\otimes 1$, and choose a function $f\colon C \to B$ such that $\beta f = 1_C$ where 1_C denotes the identity map on C. Define $\theta\colon C\times M \to (B\otimes_R M)/I$ by $\theta(c, m) = I + f(c)\otimes m$. Then it is easily checked that θ is a balanced map and does not depend on the choice of f, so it induces a group homomorphism $\phi\colon C\otimes_R M \to (B\otimes_R M)/I$ such that $\phi c\otimes m = I + fc\otimes m$. Now $\beta\otimes 1 I = 0$, so $\beta\otimes 1$ induces a group homomorphism $\psi\colon (B\otimes_R M)/I \to C\otimes_R M$ such that $\psi(I+x) = (\beta\otimes 1)x$ for all $x \in (B \otimes_R M)/I$. Therefore $\phi\psi(I + b \otimes m) = \phi\beta b \otimes m = I + f\beta b \otimes m = I + b \otimes m$ (because θ does not depend on the choice of f, so we could have chosen f to send βb to b). Since the elements of the form $I + b \otimes m$ generate $(B \otimes_R M)/I$ as an abelian group, it follows that $\phi\psi$ is the identity map. Thus if $y \in \ker\beta \otimes 1$, then $\psi(I + y) = 0$ and $\phi\psi(I + y) = I + y$, hence $y \in I$ and the result is proven.

In the language of category theory (don't worry if you don't know any category theory — the meaning of the following terminology should be clear) Exercise 4 says that the functor $\mathrm{Hom}_R(M, \_)$ is left exact, Exercise 5 says that the functor $\mathrm{Hom}_R(\_, M)$ is left exact, and the above theorem says that the functor $\_ \otimes M$ is right exact. However, the following exercise says that if the words "left" and "right" are interchanged in the above statements, then they all become false.

Exercise 18

(i) Give an example of a ring R, an R-module M, and a short exact sequence of R-modules $A \xrightarrow{\theta} B \to 0$ such that the induced sequence $\mathrm{Hom}_R(M, A) \xrightarrow{\theta_*} \mathrm{Hom}_R(M, B) \to 0$ is not exact.

(ii) Give an example of a ring R, an R-module M, and a short exact sequence of R-modules $0 \to A \xrightarrow{\theta} B$ such that the induced sequence $\mathrm{Hom}_R(B, M) \xrightarrow{\theta^*} \mathrm{Hom}_R(A, M) \to 0$ is not exact.

(iii) Give an example of a ring R, an R-module M, and a short exact sequence of R-modules $0 \to A \xrightarrow{\theta} B$ such that the induced sequence $0 \to A \otimes_R M \xrightarrow{\theta \otimes 1} B \otimes_R M$ is not exact.

The above can be considered the starting point of Homological Algebra: the failure of the exactness of the functors $\mathrm{Hom}_R(\_, M)$, $\mathrm{Hom}_R(M, \_)$, $\_ \otimes_R M$. The failure of the exactness of these functors gives rise to new infinite families of functors, namely Tor_n^R and Ext_R^n for all $n \in \mathbb{N}$. To obtain these new functors, we need to study chain complexes.

Chain Complexes Let R be a ring. Then a sequence of R-modules

$$A: \cdots \xrightarrow{\partial_{n+2}} A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} A_1 \xrightarrow{\partial_1} A_0 \xrightarrow{\partial_0} 0$$

is a chain complex means that $\partial_n \partial_{n+1} = 0$ for all $n \in \mathbb{N}$. The ∂_n are called the boundary maps, and we say that A is projective (respectively free) if and only if all the A_n are projective (respectively free). Since $\operatorname{im} \partial_{n+1} \subseteq \ker \partial_n$, we can define the nth homology group $\mathrm{H}_n(A)$ of A to be $\ker \partial_n / \operatorname{im} \partial_{n+1}$.

Similarly a cochain complex B is a sequence of R-modules $B: 0 \xrightarrow{\delta_0} B_0 \xrightarrow{\delta_1} B_1 \xrightarrow{\delta_2} \cdots$ such that $\delta_{n+1} \delta_n = 0$ for all $n \in \mathbb{N}$. In this case we define the nth cohomology group $\mathrm{H}^n(B)$ to be $\ker \delta_{n+1} / \operatorname{im} \delta_n$.

Example Let $P: \cdots \xrightarrow{\theta_2} P_1 \xrightarrow{\theta_1} P_0 \xrightarrow{\theta_0} 0$ be a sequence of $\mathbb{Z}$-modules such that for each $n \in \mathbb{N}$ we have $P_{2n} = \mathbb{Z}/4\mathbb{Z}$, $P_{2n+1} = \mathbb{Z}$, $\theta_{2n+1}(1) = [2]$ and $\theta_{2n} = 0$, where $[2]$ denotes the residue class of 2 modulo 4. Then P is a chain complex, $\mathrm{H}_{2n}(P) \cong \mathbb{Z}/2\mathbb{Z}$, and $\mathrm{H}_{2n+1}(P) \cong \mathbb{Z}$.

Suppose L is a left R-module and ι denotes the identity map on L. Then $A \otimes_R L$ will denote the chain complex (of abelian groups)

$$\cdots \xrightarrow{\partial_{n+2} \otimes \iota} A_{n+1} \otimes_R L \xrightarrow{\partial_{n+1} \otimes \iota} A_n \otimes_R L \xrightarrow{\partial_n \otimes \iota} \cdots \xrightarrow{\partial_2 \otimes \iota} A_1 \otimes_R L \xrightarrow{\partial_1 \otimes \iota} A_0 \otimes_R L \xrightarrow{\partial_0 \otimes \iota} 0.$$

We note that this *is* a chain complex because $(\partial_n \otimes \iota)(\partial_{n+1} \otimes \iota) = (\partial_n \partial_{n+1}) \otimes \iota = 0$.

Similarly if N is a right R-module, then $\mathrm{Hom}_R(A, N)$ will denote the cochain complex (of abelian groups)

$$0 \xrightarrow{\partial_0^*} \mathrm{Hom}_R(A_0, N) \xrightarrow{\partial_1^*} \mathrm{Hom}_R(A_1, N) \xrightarrow{\partial_2^*} \cdots$$
$$\cdots \xrightarrow{\partial_n^*} \mathrm{Hom}_R(A_n, N) \xrightarrow{\partial_{n+1}^*} \mathrm{Hom}_R(A_{n+1}, N) \xrightarrow{\partial_{n+2}^*} \cdots .$$

As above we note that this is a cochain complex because $\partial_{n+1}^* \partial_n^* = (\partial_n \partial_{n+1})^* = 0$.

Let R be a ring and let M be a right R-module. Then a resolution of M (as an R-module) is an exact sequence of R-modules

$$(P, \epsilon)\colon \cdots \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} M \longrightarrow 0, \tag{1}$$

and then we shall write P for the chain complex

$$\cdots \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} 0. \tag{2}$$

Thus $\mathrm{H}_n(P) = 0$ for all $n \in \mathbb{P}$ and $\mathrm{H}_0(P) = M$. Also we shall write P for (2) even if (1) is only a chain complex. We can now define Ext and Tor.

Let R be a ring, let M be a right R-module, and let

$$(P, \epsilon) : \cdots \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} M \longrightarrow 0$$

be a projective resolution of M.

Definition Let L be a left R-module and let N be a right R-module.

(i) $\mathrm{Ext}_R^n(M, N) = \mathrm{H}^n(\mathrm{Hom}_R(P, N))$.

(ii) $\mathrm{Tor}_n^R(M, L) = \mathrm{H}_n(P \otimes_R L)$.

Of course we must check that $\mathrm{Ext}_R^n(M, N)$ and $\mathrm{Tor}_n^R(M, L)$ are well defined, which means that they do not depend on the choice of the resolution P.

First Homework Due 9:00 a.m., Monday, August 29.

(1) Let R be a ring, let M be a right R-module, and let $I \lhd_1 R$. Prove that $M \otimes_R R/I \cong M/MI$ as abelian groups. Prove further that if $I \lhd R$, then the above isomorphism can be taken as one of right R-modules.

(2) Let R, S be rings, let M be a right R-module, let P be a right S-module, and let N be an (R, S)-bimodule.

(i) If $f \in \mathrm{Hom}_S(M \otimes_R N, P)$, show that we can define $\hat{f}\colon M \to \mathrm{Hom}_S(N, P)$ by $(\hat{f}m)n = f(m \otimes n)$ for $m \in M, n \in N$. Then prove that the rule $f \mapsto \hat{f}$ uniquely defines a

group homomorphism from $\mathrm{Hom}_S(M \otimes_R N, P)$ to $\mathrm{Hom}_R(M, \mathrm{Hom}_S(N,P))$. (Recall the right R-moduule structure of $\mathrm{Hom}_S(N,P)$ from Exercise 7.)

(ii) If $g \in \mathrm{Hom}_R(M, \mathrm{Hom}_S(N,P))$, show that we can define a map $\tilde{g}\colon M \otimes_R N \to P$ which satisfies $\tilde{g}(m \otimes n) = (gm)n$ for all $m \in M$ and $n \in N$. Then the map $g \mapsto \tilde{g}$ uniquely defines a group homomorphism $\mathrm{Hom}_R(M, \mathrm{Hom}_S(N,P))$ to $\mathrm{Hom}_S(M \otimes_R N, P)$.

(iii) Prove that $\mathrm{Hom}_S(M \otimes_R N, P) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_S(N,P))$ as abelian groups by showing that the maps $\hat{}$ and $\tilde{}$ are inverse to each other.

(3) Prove that $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Q} \oplus \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Q}$ as $\mathbb{Z}$-modules.

(4) Let $A : \cdots \overset{\alpha_3}{\to} A_2 \overset{\alpha_2}{\to} A_1 \overset{\alpha_1}{\to} A_0 \overset{\alpha_0}{\to} 0$ be a chain complex of $\mathbb{Z}$-modules, and let $[n]$ denote the residue class of the integer n in $\mathbb{Z}/8\mathbb{Z}$ or $\mathbb{Z}/4\mathbb{Z}$. Suppose for each $r \in \mathbb{N}$, we have $A_{2r} = \mathbb{Z}/8\mathbb{Z}$, $A_{2r+1} = \mathbb{Z}/4\mathbb{Z}$, $\alpha_{2r+2}[n] = [2n]$, and $\alpha_{2r+1}[n] = [4n]$.

(i) Determine $\mathrm{H}_r(A)$ for $r \in \mathbb{N}$.

(ii) Prove that $\mathrm{H}_0(A \otimes_{\mathbb{Z}} \mathbb{Z}/4\mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z}$ and that $\mathrm{H}_r(A \otimes_{\mathbb{Z}} \mathbb{Z}/4\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ for $r \in \mathbb{P}$.

Monday, August 29

Chapter 2
Chain Complexes

We need the following lemma to show that Ext and Tor are well defined.

Lemma 1 Let M, N be R-modules, let $\theta_{-1}\colon M \to N$ be an R-map, let

$$(P, \alpha_0)\colon \cdots \xrightarrow{\alpha_2} P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} M \longrightarrow 0$$

be a chain complex of R-modules with P projective, and let

$$(Q, \beta_0)\colon \cdots \xrightarrow{\beta_2} Q_1 \xrightarrow{\beta_1} Q_0 \xrightarrow{\beta_0} N \longrightarrow 0$$

be a resolution of N.

(i) There exist R-maps $\theta_i\colon P_i \to Q_i$ such that $\theta_{i-1}\alpha_i = \beta_i\theta_i$ for all $i \in \mathbb{N}$.

(ii) If $\phi_i\colon P_i \to Q_i$ are R-maps such that $\phi_{i-1}\alpha_i = \beta_i\phi_i$ ($i \in \mathbb{N}$) and $\phi_{-1} = \theta_{-1}$, then there exist R-maps $h_i\colon P_i \to Q_{i+1}$, $h_{-1} = 0$, such that

$$\theta_i - \phi_i = h_{i-1}\alpha_i + \beta_{i+1}h_i \quad \text{for all } i \in \mathbb{N}.$$

Proof For convenience we define $\alpha_i = \beta_i = \theta_{i-1} = \phi_{i-1} = h_{i-1} = 0$ for all $i < 0$. Note that then we have $\ker \beta_{i-1} = \mathrm{im}\, \beta_i$ for all $i \in \mathbb{Z}$.

(i) We use induction. Suppose $n+1 \in \mathbb{N}$ and we have constructed R-maps $\theta_i\colon P_i \to Q_i$ such that $\theta_{i-1}\alpha_i = \beta_i\theta_i$ for all $i \le n$; we can obviously do this for $n = -1$. We now do this for $n+1$. Note that $\theta_{n-1}\alpha_n = \beta_n\theta_n$ tells us that

$$\beta_n\theta_n\alpha_{n+1} = \theta_{n-1}\alpha_n\alpha_{n+1} = 0,$$

hence $\mathrm{im}\, \theta_n\alpha_{n+1} \subseteq \ker \beta_n$. But (Q, β_0) is a resolution, so $\ker \beta_n = \mathrm{im}\, \beta_{n+1}$ and we deduce that $\mathrm{im}\, \theta_n\alpha_{n+1} \subseteq \mathrm{im}\, \beta_{n+1}$, thus we can consider $\theta_n\alpha_{n+1} \in \mathrm{Hom}_R(P_{n+1}, \mathrm{im}\, \beta_{n+1})$. Since

P_{n+1} is projective, the map $\beta_{n+1\,*}\colon \mathrm{Hom}_R(P_{n+1}, Q_{n+1}) \to \mathrm{Hom}_R(P_{n+1}, \mathrm{im}\,\beta_{n+1})$ is onto, hence there exists $\theta_{n+1} \in \mathrm{Hom}_R(P_{n+1}, Q_{n+1})$ such that $\beta_{n+1\,*}\theta_{n+1} = \theta_n\alpha_{n+1}$. Therefore $\beta_{n+1}\theta_{n+1} = \theta_n\alpha_{n+1}$, which completes the induction step as required.

(ii) Again we use induction. Suppose $n+1 \in \mathbb{N}$ and we have constructed maps $h_i\colon P_i \to Q_{i+1}$ such that $\theta_i - \phi_i = h_{i-1}\alpha_i + \beta_{i+1}h_i$ for all $i \leq n$. We can obviously do this for $n = -1$ by taking $h_{-1} = 0$. Now

$$\begin{aligned}
\theta_n - \phi_n - h_{n-1}\alpha_n &= \beta_{n+1}h_n \\
\text{yields}\quad (\theta_n - \phi_n - h_{n-1}\alpha_n)\alpha_{n+1} &= \beta_{n+1}h_n\alpha_{n+1} \\
\text{hence}\quad \theta_n\alpha_{n+1} - \phi_n\alpha_{n+1} &= \beta_{n+1}h_n\alpha_{n+1} \quad \text{because } \alpha_n\alpha_{n+1} = 0. \\
\text{Therefore}\quad \beta_{n+1}(\theta_{n+1} - \phi_{n+1} - h_n\alpha_{n+1}) &= \theta_n\alpha_{n+1} - \phi_n\alpha_{n+1} - \beta_{n+1}h_n\alpha_{n+1} = 0.
\end{aligned}$$

Since (Q, β_0) is a resolution, $\ker\beta_{n+1} = \mathrm{im}\,\beta_{n+2}$ and we deduce that $\mathrm{im}(\theta_{n+1} - \phi_{n+1} - h_n\alpha_{n+1}) \subseteq \mathrm{im}\,\beta_{n+2}$. Therefore we may view

$$(\theta_{n+1} - \phi_{n+1} - h_n\alpha_{n+1}) \in \mathrm{Hom}_R(P_{n+1}, \mathrm{im}\,\beta_{n+2}).$$

Since P_{n+1} is projective, the map $\beta_{n+2\,*}\colon \mathrm{Hom}_R(P_{n+1}, Q_{n+2}) \to \mathrm{Hom}_R(P_{n+1}, \mathrm{im}\,\beta_{n+2})$ is onto, hence there exists $h_{n+1} \in \mathrm{Hom}_R(P_{n+1}, Q_{n+2})$ such that $\beta_{n+2\,*}\,h_{n+1} = \theta_{n+1} - \phi_{n+1} - h_n\alpha_{n+1}$. Therefore $\beta_{n+2}h_{n+1} = \theta_{n+1} - \phi_{n+1} - h_n\alpha_{n+1}$, which completes the induction step as required.

We now apply Lemma 1 in the special case when P and Q are both projective resolutions to obtain uniquely defined maps between the complexes $\mathrm{Hom}_R(P, L)$ and $\mathrm{Hom}_R(Q, L)$ for any R-module L. This will establish that $\mathrm{Ext}^n_R(M, L)$ is well defined, and also show that an R-map from M to N induces a group homomorphism from $\mathrm{Ext}^n_R(N, L)$ to $\mathrm{Ext}^n_R(M, L)$ for any $n \in \mathbb{N}$. We shall further show that an R-module map $A \to B$ induces a group homomorphism $\mathrm{Ext}^n_R(M, A) \to \mathrm{Ext}^n_R(M, B)$ for all $n \in \mathbb{N}$.

Lemma 2 Let R be a ring, let L, M, N be R-modules, let $\theta_{-1}\colon M \to N$ be an R-map, and let

$$\begin{aligned}
(P, \alpha_0)&: \cdots \xrightarrow{\alpha_2} P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} M \longrightarrow 0 \\
\text{and}\quad (Q, \beta_0)&: \cdots \xrightarrow{\beta_2} Q_1 \xrightarrow{\beta_1} Q_0 \xrightarrow{\beta_0} N \longrightarrow 0
\end{aligned}$$

be projective resolutions of M and N respectively.

(i) If $\theta_i\colon P_i \to Q_i$ are R-maps with $\theta_{i-1}\alpha_i = \beta_i\theta_i$ for all $i \in \mathbb{N}$, then $\theta_i^*\colon \mathrm{Hom}_R(Q_i, L) \to \mathrm{Hom}_R(P_i, L)$ induces a well defined map, also denoted θ_i^*, from $\mathrm{H}^i(\mathrm{Hom}_R(Q, L))$ to $\mathrm{H}^i(\mathrm{Hom}_R(P, L))$ satisfying $\theta_i^*(f + \mathrm{im}\,\beta_i^*) = f\theta_i + \mathrm{im}\,\alpha_i^*$ for $f \in \ker\beta_{i+1}^*$.

(ii) If $\phi_i\colon P_i \to Q_i$ are also R-maps such that $\phi_{i-1}\alpha_i = \beta_i\phi_i$ for all $i \in \mathbb{N}$ and $\theta_{-1} = \phi_{-1}$, then $\phi_i^* = \theta_i^*\colon \mathrm{H}^i(\mathrm{Hom}_R(Q, L)) \to \mathrm{H}^i(\mathrm{Hom}_R(P, L))$.

Proof (i) First we show that the definition of θ_i^* does not depend on f; i.e. if $f + \operatorname{im}\beta_i^* = g + \operatorname{im}\beta_i^*$, then $f\theta_i + \operatorname{im}\alpha_i^* = g\theta_i + \operatorname{im}\alpha_i^*$. Set $k = f - g$. Then $k \in \operatorname{im}\beta_i^*$, so $k = k'\beta_i$ for some $k' \in \operatorname{Hom}_R(Q_{i-1}, L)$, hence $k\theta_i = k'\beta_i\theta_i = k'\theta_{i-1}\alpha_i$, so $k\theta_i \in \operatorname{im}\alpha_i^*$. It follows that $f\theta_i - g\theta_i \in \operatorname{im}\alpha_i^*$ as required.

Now we show that θ_i^* maps into $\mathrm{H}^i(\operatorname{Hom}_R(P, L))$, i.e. $f\theta_i \in \ker\alpha_{i+1}^*$ This is clear because

$$\alpha_{i+1}^*(f\theta_i) = f\theta_i\alpha_{i+1} = f\beta_{i+1}\theta_{i+1} = (\beta_{i+1}^* f)\theta_{i+1} = 0,$$

where we have used $f \in \ker\beta_{i+1}^*$.

(ii) By Lemma 1, we know there exist R-maps $h_i\colon P_i \to Q_{i+1}$ such that $\theta_i - \phi_i = h_{i-1}\alpha_i + \beta_{i+1}h_i$ for all $i \in \mathbb{N}$. Suppose $f \in \operatorname{Hom}_R(Q_i, L)$ and $\beta_{i+1}^* f = 0$ (so f represents an element of $\mathrm{H}^i(\operatorname{Hom}_R(Q, L))$). Then $f\beta_{i+1} = 0$, so $f\theta_i - f\phi_i = fh_{i-1}\alpha_i = \alpha_i^*(fh_{i-1})$, hence $\theta_i^* f - \phi_i^* f \in \operatorname{im}\alpha_i^*$. This shows that $\theta_i^* f$ and $\phi_i^* f$ represent the same element of $\mathrm{H}^i(\operatorname{Hom}_R(P, L))$ as required.

Now we can show that $\operatorname{Ext}_R^n(M, L)$ does not depend on the choice of the projective resolution P. Indeed suppose

$$(P, \alpha_0) : \cdots \xrightarrow{\alpha_2} P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} M \longrightarrow 0$$
$$\text{and} \quad (Q, \beta_0) : \cdots \xrightarrow{\beta_2} Q_1 \xrightarrow{\beta_1} Q_0 \xrightarrow{\beta_0} M \longrightarrow 0$$

are projective resolutions for M. Then Lemma 1 shows that there are R-maps $\theta_i\colon P_i \to Q_i$, $\phi_i\colon Q_i \to P_i$ such that

$$\theta_{i-1}\alpha_i = \beta_i\theta_i \quad \text{and} \quad \phi_{i-1}\beta_i = \alpha_i\phi_i$$

for all $i \in \mathbb{N}$. Applying Lemma 2, we see that θ_i and ϕ_i induce group homomorphisms

$$\theta_i^*\colon \mathrm{H}^i(\operatorname{Hom}_R(Q, L)) \to \mathrm{H}^i(\operatorname{Hom}_R(P, L)) \quad \text{and} \quad \phi_i^*\colon \mathrm{H}^i(\operatorname{Hom}_R(P, L)) \to \mathrm{H}^i(\operatorname{Hom}_R(Q, L))$$

respectively. Now let $\iota_i\colon P_i \to P_i$ denote the identity map and use Lemma 2(ii). Since $\phi_{i-1}\theta_{i-1}\alpha_i = \alpha_i\phi_i\theta_i$ and $\iota_{i-1}\alpha_i = \alpha_i\iota_i$, we see that

$$(\phi_i\theta_i)^* = \iota^*\colon \mathrm{H}^i(\operatorname{Hom}_R(P, L)) \to \mathrm{H}^i(\operatorname{Hom}_R(P, L))$$

and hence $\theta_i^*\phi_i^*$ is the identity map on $\mathrm{H}^i(\operatorname{Hom}_R(P, L))$. Similarly $\phi_i^*\theta_i^*$ is the identity on $\mathrm{H}^i(\operatorname{Hom}_R(Q, L))$. Therefore $\mathrm{H}^i(\operatorname{Hom}_R(P, L)) \cong \mathrm{H}^i(\operatorname{Hom}_R(Q, L))$, thus $\operatorname{Ext}_R^n(M, L)$ does not depend on the choice of P.

Suppose now that $\psi\colon M \to N$ is a homomorphism of R-modules. We shall use the notation of Lemma 2 and set $\theta_{-1} = \psi$. This shows that there exist well defined group homomorphisms $\psi_i^*\colon \operatorname{Ext}_R^i(N, L) \to \operatorname{Ext}_R^i(M, L)$ which satisfy

$$\psi_i^*(f + \operatorname{im}\beta_i^*) = f\theta_i + \operatorname{im}\alpha_i^*.$$

Perhaps we should be careful what we mean by well defined. Certainly the ψ_i^* do not depend on the choice of the θ_i, however they do depend on the choice of the resolutions P and

Q. Suppose we have different resolutions $\tilde{P}$ and $\tilde{Q}$ from which we obtain corresponding Ext groups $\widetilde{\mathrm{Ext}}^i_R(M,L)$, $\widetilde{\mathrm{Ext}}^i_R(N,L)$ with corresponding maps $\tilde{\psi}^*_i\colon \widetilde{\mathrm{Ext}}^i_R(N,L) \to \widetilde{\mathrm{Ext}}^i_R(M,L)$. From the above there are isomorphisms $\mu_i\colon \mathrm{Ext}^i_R(M,L) \to \widetilde{\mathrm{Ext}}^i_R(M,L)$, $\nu_i\colon \mathrm{Ext}^i_R(N,L) \to \widetilde{\mathrm{Ext}}_R(M,L)$. Then we have the following commutative diagram.

$$\begin{array}{ccc} \mathrm{Ext}^i_R(N,L) & \xrightarrow{\psi^*_i} & \mathrm{Ext}^i_R(M,L) \\ \nu_i \downarrow & & \downarrow \mu_i \\ \widetilde{\mathrm{Ext}}^i_R(N,L) & \xrightarrow{\tilde{\psi}^*_i} & \widetilde{\mathrm{Ext}}^i_R(M,L) \end{array}$$

Thus if, for example ψ^*_i is an isomorphism, then so is $\tilde{\psi}^*_i$. Also if $M = N$, then we can take $P = Q$ and then ψ^*_i only depends on P.

Second Homework Due 9:00 a.m., Monday, September 5.

(1) Let k be a field, let $R = k[X]/(X^4)$, and let $A : \cdots \xrightarrow{\alpha_3} A_2 \xrightarrow{\alpha_2} A_1 \xrightarrow{\alpha_1} A_0 \xrightarrow{\alpha_0} 0$ be a chain complex of R-modules. Suppose for each $n \in \mathbb{N}$ we have $A_n = R$, $\alpha_{n+1}(1 + (X^4)) = X^2 + (X^4)$, and $\alpha_0 = 0$.
   (i) Determine $\mathrm{H}_n(A)$ for all $n \in \mathbb{N}$.
   (ii) Prove that $\mathrm{H}_0(A \otimes_R k[X]/(X^3)) \cong k \oplus k$ and $\mathrm{H}_n(A \otimes_R k[X]/(X^3)) \cong k$ for all $n \in \mathbb{P}$ as k-modules.

(2) Let R be a ring, let M, N be R-modules, and let $\theta_{-1}\colon M \to N$ be an R-map. Suppose

$$(Q, \alpha_0) : 0 \longrightarrow M \xrightarrow{\alpha_0} Q_0 \xrightarrow{\alpha_1} Q_1 \xrightarrow{\alpha_2} Q_2 \xrightarrow{\alpha_3} \cdots$$
$$\text{and} \quad (I, \beta_0) : 0 \longrightarrow N \xrightarrow{\beta_0} I_0 \xrightarrow{\beta_1} I_1 \xrightarrow{\beta_2} I_2 \xrightarrow{\beta_3} \cdots$$

are cochain complexes with (Q, α_0) exact (i.e. $\ker \alpha_{i+1} = \operatorname{im} \alpha_i$ for all $i \in \mathbb{N}$ and α_0 a monomorphism) and (I, β_0) injective (i.e. I_i is an injective R-module for all $i \in \mathbb{N}$).
   (i) Prove that there exist R-maps $\theta_i\colon Q_i \to I_i$ such that $\beta_i\theta_{i-1} = \theta_i\alpha_i$ for all $i \in \mathbb{N}$.
   (ii) Suppose $\phi_i\colon Q_i \to I_i$ are R-maps such that $\beta_i\phi_{i-1} = \phi_i\alpha_i$ $(i \in \mathbb{N})$ and $\phi_{-1} = \theta_{-1}$. Prove that there exist R-maps $h_i\colon Q_i \to I_{i-1}$ such that

$$\theta_i - \phi_i = \beta_i h_i + h_{i+1}\alpha_{i+1}.$$

(3) Let R be a ring, let M be a right R-module, let L be a left R-module, and let

$$(P, \alpha_0) : \cdots \xrightarrow{\alpha_2} P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} M \to 0$$
$$\text{and} \quad (Q, \beta_0) : \cdots \xrightarrow{\beta_2} Q_1 \xrightarrow{\beta_1} Q_0 \xrightarrow{\beta_0} M \to 0$$

be projective resolutions for M. Prove that $\mathrm{H}_n(P \otimes_R L) \cong \mathrm{H}_n(Q \otimes_R L)$ as abelian groups for all $n \in \mathbb{N}$.

(4) Let $r \in \mathbb{N}$ and $q, n \in \mathbb{P}$. If $q|n$ and $l = (n/q, q)$, prove that $\mathrm{Ext}^0_{\mathbb{Z}/n\mathbb{Z}}(\mathbb{Z}/q\mathbb{Z}, \mathbb{Z}/q\mathbb{Z}) \cong \mathbb{Z}/q\mathbb{Z}$ and $\mathrm{Ext}^r_{\mathbb{Z}/n\mathbb{Z}}(\mathbb{Z}/q\mathbb{Z}, \mathbb{Z}/q\mathbb{Z}) \cong \mathbb{Z}/l\mathbb{Z}$ as abelian groups.

(5) Let $\mathcal{I}$ be a set, let R be a ring, let A be an R-module, let $n \in \mathbb{N}$, and let $\{M_i \mid i \in \mathcal{I}\}$ be a family of R-modules. Prove that $\mathrm{Ext}^n_R(\bigoplus_{i\in\mathcal{I}} M_i, A) \cong \prod_{i\in\mathcal{I}} \mathrm{Ext}^n_R(M_i, A)$.

Friday, September 2

Chapter 3
Ext and Tor

Remarks and Exercises

(i) Let R be a ring and let M, N be R-modules. Then $\mathrm{Ext}^0_R(M, N) \cong \mathrm{Hom}_R(M, N)$. Proof: let

$$(P, \alpha_0)\colon \cdots \xrightarrow{\alpha_2} P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} M \longrightarrow 0$$

be a projective resolution of M. Then $\mathrm{Ext}^0_R(M, N) = \mathrm{H}^0(\mathrm{Hom}_R(P, N))$, so $\mathrm{Ext}^0_R(M, N) \cong \ker \alpha_1^*$. Furthermore (see Exercise 5 from chapter 1)

$$0 \longrightarrow \mathrm{Hom}_R(M, N) \xrightarrow{\alpha_0^*} \mathrm{Hom}_R(P_0, N) \xrightarrow{\alpha_1^*} \mathrm{Hom}_R(P_1, N)$$

is exact, hence $\ker \alpha_1^* = \mathrm{im}\, \alpha_0^* \cong \mathrm{Hom}_R(M, N)$ as required.

(ii) Let R be a ring, let M be a right R-module, and let N be a left R-module. Then $\mathrm{Tor}^R_0(M, N) \cong M \otimes_R N$. Proof: exercise.

(iii) In general $\mathrm{Ext}^n_R(M, N)$ and $\mathrm{Tor}^R_n(M, N)$ only have the structure of an abelian group. However if M is a right R-module and N is an (R, S)-bimodule, then $\mathrm{Tor}^R_n(M, N)$ is a right S-module for all $n \in \mathbb{N}$. To see this, take a projective resolution

$$(P, \alpha_0)\colon \cdots \xrightarrow{\alpha_2} P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} M \longrightarrow 0$$

of M. Then

$$\mathrm{Tor}^R_n(M, N) = \frac{\ker(\alpha_n \otimes 1)}{\mathrm{im}(\alpha_{n+1} \otimes 1)}.$$

Since $P_i \otimes_R N$ is a right S-module and $\alpha_i \otimes 1$ is an S-map for all $i \in \mathbb{N}$, it follows that $\ker(\alpha_i \otimes 1)$ and $\mathrm{im}(\alpha_i \otimes 1)$ are right S-modules and hence $\mathrm{Tor}^R_n(M, N)$ is a right S-module. This means that in (ii), $\mathrm{Tor}^R_0(M, N) \cong M \otimes_R N$ as right S-modules.

Similarly if M is a right R-module and N is an (S, R)-bimodule, then $\mathrm{Ext}^R_n(M, N)$ is a left S-module. Thus in (i) above, we have $\mathrm{Ext}^0_R(M, N) \cong \mathrm{Hom}_R(M, N)$ as left S-mmodules. Furthermore if M is an (S, R)-bimodule and N is a left R-module, then $\mathrm{Tor}^R_n(M, N)$ is a left S-module, and if M is an (S, R)-bimodule and N is a right R-module, then $\mathrm{Ext}^n_R(M, N)$ is a right S-module. However we cannot apply the above arguments to obtain this because we

cannot in general choose a resolution of M consisting of bimodules and bimodule maps; we will demonstrate the S-module structures of this paragraph later.

(iv) Let $R = \mathbb{Z}[X]/(X^2)$, let k denote the R-module $\mathbb{Z}/2\mathbb{Z}$ with X acting trivially (so $aX = 0$ for $a \in k$), let $\mathbb{Z}$ be the R-module also with X acting trivially (so $zX = 0$ for all $z \in \mathbb{Z}$), and let us calculate $\operatorname{Tor}_n^R(\mathbb{Z}, k)$ for $n \in \mathbb{N}$. First we obtain a projective resolution for $\mathbb{Z}$; one possibility is

$$(P, \alpha_0)\colon \cdots \xrightarrow{\alpha_3} R \xrightarrow{\alpha_2} R \xrightarrow{\alpha_1} R \xrightarrow{\alpha_0} \mathbb{Z} \longrightarrow 0$$

where $P_i = R$ for all $i \in \mathbb{N}$, $\alpha_0 1 = 1$, $\bar{X} = X + (X^2)$ (the image of X in R), and $\alpha_i 1 = \bar{X}$ for all $i \in \mathbb{P}$. Then $P \otimes_R k$ is the complex

$$\cdots \xrightarrow{\alpha_3 \otimes 1} R \otimes_R k \xrightarrow{\alpha_2 \otimes 1} R \otimes_R k \xrightarrow{\alpha_1 \otimes 1} R \otimes_R k \longrightarrow 0$$

where 1 is the identity map. Since $R \otimes_R k \cong k$ and $\bar{X} \otimes a = 1 \otimes Xa = 1 \otimes 0 = 0$ for all $a \in k$, it follows that the above complex is

$$B : \cdots \xrightarrow{\beta_3} k \xrightarrow{\beta_2} k \xrightarrow{\beta_1} k \xrightarrow{\beta_0} 0$$

where $B_i = k$ and $\beta_i = 0$ for all $i \in \mathbb{N}$. Then $\mathrm{H}_i(B) = \ker \beta_i / \operatorname{im} \beta_{i+1} = k/0 \cong k$ for all $i \in \mathbb{N}$. It follows that $\operatorname{Tor}_n^R(\mathbb{Z}, k) \cong k$ for all $n \in \mathbb{N}$.

(v) Let R be a ring, let $\mathcal{I}$ be a set, let A be a left R-module, and let $\{M_i \mid i \in I\}$ be a family of right R-modules. Then $\operatorname{Tor}_n^R(\bigoplus_{i\in\mathcal{I}} M_i, A) \cong \bigoplus_{i\in\mathcal{I}} \operatorname{Tor}_n^R(M_i, A)$ for all $n \in \mathbb{N}$. Proof: For each $i \in \mathcal{I}$, let

$$(P^i, \alpha_0^i) : \cdots \xrightarrow{\alpha_3^i} P_2^i \xrightarrow{\alpha_2^i} P_1^i \xrightarrow{\alpha_1^i} P_0^i \xrightarrow{\alpha_0^i} M_i \longrightarrow 0$$

be a projective resolution for the R-module M_i Then

$$(\bigoplus_{i\in\mathcal{I}} P^i, \bigoplus_{i\in\mathcal{I}} \alpha_0^i) : \cdots \xrightarrow{\oplus\alpha_3^i} \bigoplus_{i\in\mathcal{I}} P_2^i \xrightarrow{\oplus\alpha_2^i} \bigoplus_{i\in\mathcal{I}} P_1^i \xrightarrow{\oplus\alpha_1^i} \bigoplus_{i\in\mathcal{I}} P_0^i \xrightarrow{\oplus\alpha_0^i} \bigoplus_{i\in\mathcal{I}} M_i \longrightarrow 0$$

is a projective resolution of $\bigoplus_{i\in\mathcal{I}} M_i$ (the resolution is projective because the direct sum of an arbitrary number of projectives is projective), so

$$\operatorname{Tor}_n^R(\bigoplus_{i\in\mathcal{I}} M_i, A) = \mathrm{H}_n((\bigoplus_{i\in\mathcal{I}} P^i) \otimes_R A) = \frac{\ker(\oplus_{i\in\mathcal{I}} \alpha_n^i) \otimes 1}{\operatorname{im}(\oplus_{i\in\mathcal{I}} \alpha_{n+1}^i) \otimes 1}.$$

Now define a homomorphism $\theta_n\colon (\bigoplus_{i\in\mathcal{I}} P_n^i) \otimes_R A \to \bigoplus_{i\in\mathcal{I}} (P_n^i \otimes_R A)$ as follows. If $u \in \bigoplus P_n^i$ has components $u_i \in P_n^i$ (where $u_i = 0$ for all but finitely many $i \in \mathcal{I}$) and $a \in A$, then set $\theta_n(u \otimes a)$ to be the element of $\bigoplus_{i\in\mathcal{I}} (P_n^i \otimes_R A)$ whose ith component is $u_i \otimes a$. It is then routine to check that θ_n is a well defined group homomorphism. We want to construct a map which is inverse to θ_n, so for $j \in \mathcal{I}$, let $\phi_n^j\colon P_n^j \to \bigoplus_{i\in\mathcal{I}} P_n^i$ denote the natural monomorphism, and set $\phi_n = \bigoplus_{i\in\mathcal{I}} \phi_i \otimes 1$. Then it is easily checked that ϕ_n is a well defined

group homomorphism $\bigoplus_{i\in\mathcal{I}} P_n^i \otimes_R A \to (\bigoplus_{i\in\mathcal{I}} P_n^i)\otimes_R A$, and that $\theta_n\phi_n$ and $\phi_n\theta_n$ are the identity maps. It follows that θ_n is an isomorphism from $(\bigoplus_{i\in\mathcal{I}} P_n^i)\otimes_R A \to \bigoplus_{i\in\mathcal{I}} P_n^i \otimes_R A$.

Continuing with the notation that $u \in \bigoplus_{i\in\mathcal{I}} P_n^i$ has components u_i, let $v \in \bigoplus_{i\in\mathcal{I}} P_{n-1}^i$ have components $v_i = \alpha_n^i u_i$. Then we have for $a \in A$

$$\theta_{n-1}(\oplus_{i\in\mathcal{I}}\,\alpha_n^i u)\otimes a = \theta_{n-1}v\otimes a = \oplus_{i\in\mathcal{I}}\,(\alpha_n^i\otimes 1)\theta_n u\otimes a,$$

hence $\theta_{n-1}(\oplus_{i\in\mathcal{I}}\,\alpha_n^i)\otimes 1 = \oplus_{i\in\mathcal{I}}\,(\alpha_n^i\otimes 1)\theta_n$ because the elements of the form $u\otimes a$ generate $(\bigoplus_{i\in\mathcal{I}} P_n^i)\otimes_R A$ as an abelian group. It follows that θ_n maps $\ker(\oplus_{i\in\mathcal{I}}\,\alpha_n^i)\otimes 1$ to $\ker\oplus_{i\in\mathcal{I}}\,(\alpha_n^i\otimes 1)$. Similarly θ_{n+1} maps $\operatorname{im}(\oplus_{i\in\mathcal{I}}\,\alpha_{n+1}^i)\otimes 1$ to $\operatorname{im}\oplus_{i\in\mathcal{I}}\,(\alpha_{n+1}^i\otimes 1)$. Therefore

$$\frac{\ker(\oplus_{i\in\mathcal{I}}\,\alpha_n^i)\otimes 1}{\operatorname{im}(\oplus_{i\in\mathcal{I}}\,\alpha_{n+1}^i)\otimes 1} \cong \frac{\ker\oplus_{i\in\mathcal{I}}\,(\alpha_n^i\otimes 1)}{\operatorname{im}\oplus_{i\in\mathcal{I}}\,(\alpha_{n+1}^i\otimes 1)} = \bigoplus_{i\in\mathcal{I}} \frac{\ker\alpha_n^i\otimes 1}{\operatorname{im}\alpha_{n+1}^i\otimes 1} = \bigoplus_{i\in\mathcal{I}} \mathrm{H}_n(P_n^i\otimes_R A).$$

This establishes $\operatorname{Tor}_n^R(\bigoplus_{i\in\mathcal{I}} M_i, A) \cong \bigoplus_{i\in\mathcal{I}} \operatorname{Tor}_n^R(M_i, A)$ as required.

(vi) Exercise: let R be a ring, let $\mathcal{I}$ be a set, let A be a left R-module, and let $\{M_i \mid i\in I\}$ be a family of right R-modules. Then in (v) it was proved that $(\bigoplus_{i\in\mathcal{I}} M_i)\otimes_R A \cong \bigoplus_{i\in\mathcal{I}}(M_i\otimes_R A)$ and the proof depended on a number of routine verifications. By the same proof we have $(\prod_{i\in\mathcal{I}} M_i)\otimes_R A \cong \prod_{i\in\mathcal{I}}(M_i\otimes_R A)$ (where we need to do similar routine verifications), or do we? Give an example with $M_i\otimes_R A = 0$ for all $i\in\mathcal{I}$, yet $(\prod_{i\in\mathcal{I}} M_i)\otimes_R A \neq 0$. Where does the proof in (v) go wrong here?

(vii) Let R be a ring, let $\mathcal{I}$ be a set, let A be an R-module, and let $\{M_i \mid i\in I\}$ be a family of R-modules. Then $\operatorname{Ext}_R^n(A, \prod_{i\in\mathcal{I}} M_i) \cong \prod_{i\in\mathcal{I}} \operatorname{Ext}_R^n(A, M_i)$ for all $n\in\mathbb{N}$. Proof: Let

$$(P,\alpha_0) : \cdots \xrightarrow{\alpha_3} P_2 \xrightarrow{\alpha_2} P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} A \longrightarrow 0$$

be a projective resolution for the R-module A. Then

$$\operatorname{Ext}_R^n(A, \prod_{i\in\mathcal{I}} M_i) = \mathrm{H}^n(\operatorname{Hom}_R(P_n, \prod_{i\in\mathcal{I}} M_i)).$$

Now (see Exercise 6(ii) from the chapter 1) $\operatorname{Hom}_R(P_n, \prod_{i\in\mathcal{I}} M_i) \cong \prod_{i\in\mathcal{I}} \operatorname{Hom}_R(P_n, M_i)$: if this isomorphism is called θ_n, then it is given as follows. For $j\in\mathcal{I}$ let $\pi_j\colon \prod_{i\in\mathcal{I}} M_i \to M_j$ denote the projection onto M_j (i.e. pick out the jth component). Then for

$$f \in \operatorname{Hom}_R(P_n, \prod_{i\in\mathcal{I}} M_i),$$

$\theta_n f$ is the element whose components are $\pi_i f$. For $n\in\mathbb{N}$, let $\alpha_n^* : \operatorname{Hom}_R(P_n, \prod_{i\in\mathcal{I}} M_i) \to \operatorname{Hom}_R(P_{n+1}, \prod_{i\in\mathcal{I}} M_i)$ and

$$\alpha_n^{j\,*}\colon \operatorname{Hom}_R(P_n, M_j) \to \operatorname{Hom}_R(P_{n+1}, M_j)$$

denote the maps induced by α_n. Then it is not difficult to check that θ_n maps $\ker\alpha_n^*$ to $\prod_{i\in\mathcal{I}} \ker\alpha_n^{i\,*}$ and $\operatorname{im}\alpha_{n+1}^*$ to $\prod_{i\in\mathcal{I}} \operatorname{im}\alpha_{n+1}^{i\,*}$. Since

$$\mathrm{H}^n(\mathrm{Hom}_R(P_n, \prod_{i\in\mathcal{I}} M_i)) = \frac{\ker \alpha_n^*}{\mathrm{im}\, \alpha_{n+1}^*} \quad \text{and} \quad \mathrm{H}^n(\mathrm{Hom}_R(P_n, M_i)) = \frac{\ker \alpha_n^{i*}}{\mathrm{im}\, \alpha_{n+1}^{i*}},$$

we see that $\mathrm{H}^n(\mathrm{Hom}_R(P_n, \prod_{i\in\mathcal{I}} M_i)) = \prod_{i\in\mathcal{I}} \mathrm{H}^n(\mathrm{Hom}_R(P, M_i))$ and the result follows.

(viii) Exercise: Let R be a ring, let $\mathcal{I}$ be a set, let A be a right R-module, and let $\{M_i \mid i \in I\}$ be a family of left R-modules. Then $\mathrm{Tor}_n^R(A, \bigoplus_{i\in\mathcal{I}} M_i) \cong \bigoplus_{i\in\mathcal{I}} \mathrm{Tor}_n^R(A, M_i)$ for all $n \in \mathbb{N}$.

(ix) Let R be a ring, let P be a projective R-module, and let M be an R-module. Then $\mathrm{Ext}_R^n(P, M) = 0$ for all $n \in \mathbb{P}$. Proof: exercise.

(x) Let R be a ring, let P be a projective left R-module, and let $0 \to A \xrightarrow{\alpha} B$ be an exact sequence of right R-modules. Prove that

$$0 \to A \otimes_R P \xrightarrow{\alpha\otimes 1} B \otimes_R P$$

is an exact sequence of abelian groups.

(xi) Let R be a ring, let $n \in \mathbb{P}$, let P, M be right R-modules, and let Q, N be left R-modules. If P and Q are projective, prove that $\mathrm{Tor}_n^R(P, N) = \mathrm{Tor}_n^R(M, Q) = 0$.

Tor for abelian groups One may ask the reason for the name Tor; the following is an explanation. It turns out that $\mathrm{Tor}_n^{\mathbb{Z}}(A, B)$ is a torsion group (i.e. all the elements have finite order) for all abelian groups (i.e. $\mathbb{Z}$-modules) A, B, and for all $n \in \mathbb{P}$. First we prove the following result.

Proposition Let $q \in \mathbb{P}$ and let B be an abelian group. Then

(i) $\mathrm{Tor}_0^{\mathbb{Z}}(\mathbb{Z}/q\mathbb{Z}, B) \cong B/qB$.
(ii) $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/q\mathbb{Z}, B) \cong \{b \in B \mid bq = 0\}$.
(iii) $\mathrm{Tor}_n^{\mathbb{Z}}(\mathbb{Z}/q\mathbb{Z}, B) = 0$ for all $n \geq 2$.

Proof Let $\mu\colon \mathbb{Z} \to \mathbb{Z}$ denote multiplication by q, and let $\pi\colon \mathbb{Z} \to \mathbb{Z}/q\mathbb{Z}$ denote the natural epimorphism. Then

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\mu} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/q\mathbb{Z} \longrightarrow 0$$

is a projective resolution for $\mathbb{Z}/q\mathbb{Z}$. Since $\mathbb{Z} \otimes_{\mathbb{Z}} B \cong B$, it follows that $\mathrm{Tor}_n^{\mathbb{Z}}$ is H_n of the chain complex

$$0 \longrightarrow B \xrightarrow{\nu} B \longrightarrow 0$$

where ν denotes multiplication by q. This proves the result.

Corollary Let A be a finitely generated abelian group and let B be any abelian group. Then

(i) $\mathrm{Tor}_1^{\mathbb{Z}}(A, B)$ is a torsion group.

(ii) $\operatorname{Tor}_n^{\mathbb{Z}}(A,B) = 0$ for all $n \geq 2$.

Proof Since A is finitely generated, it is a direct sum of cyclic groups. Also $\operatorname{Tor}_n^{\mathbb{Z}}(\mathbb{Z},B) = 0$ for all $n \in \mathbb{P}$ by (xi) above. Now apply (v) and the above proposition.

This shows that $\operatorname{Tor}_n^{\mathbb{Z}}(A,B)$ is a torsion group for all $n \in \mathbb{P}$, provided that A is finitely generated. To establish that $\operatorname{Tor}_n^{\mathbb{Z}}(A,B)$ is a torsion group for all abelian groups A is just beyond the techniques so far developed. It will follow from the fact that Tor commutes with direct limits, to be covered later.

Monday, September 5

Chapter 4
Long Exact Sequences

Let R be a ring, let L, M, N be R-modules, and let $n \in \mathbb{N}$. We have already shown that an R-map $\theta\colon M \to N$ induces a well defined group homomorphism $\theta_n^*\colon \operatorname{Ext}_R^n(N,L) \to \operatorname{Ext}_R^n(M,L)$. We now show that θ also induces a well defined group homomorphism $\theta_{n*}\colon \operatorname{Ext}_R^n(L,M) \to \operatorname{Ext}_R^n(L,N)$ as follows: as usual, let

$$(P,\alpha_0)\colon \cdots \xrightarrow{\alpha_2} P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} L \longrightarrow 0$$

be a projective resolution of the R-module L. For $n \in \mathbb{N}$, let

$$\alpha_n^*\colon \operatorname{Hom}_R(P_{n-1},M) \to \operatorname{Hom}_R(P_n,M) \quad \text{and} \quad \beta_n^*\colon \operatorname{Hom}_R(P_{n-1},N) \to \operatorname{Hom}_R(P_n,N)$$

denote the maps induced by α_n (where $\alpha_0^* = \beta_0^* = 0$). Then

$$\operatorname{Ext}_R^n(L,M) = \mathrm{H}^n(\operatorname{Hom}_R(P,M)) = \frac{\ker \alpha_{n+1}^*}{\operatorname{im} \alpha_n^*}.$$

Now for each $n \in \mathbb{N}$, θ induces a map $\theta_{n*} : \operatorname{Hom}_R(P_n,M) \to \operatorname{Hom}_R(P_n,N)$ (defined by $\theta_{n*}f = \theta f$). Since $\alpha_{n+1}^*\theta_{n*} = \theta_{n+1\,*}\,\alpha_{n+1}^*$, it follows that θ_{n*} maps $\ker \alpha_{n+1}^*$ to $\ker \beta_{n+1}^*$ and $\operatorname{im} \alpha_n^*$ to $\operatorname{im} \beta_n^*$. Therefore θ_{n*} induces a well defined map (which we will also call θ_{n*}) from $\operatorname{Ext}_R^n(L,M)$ to $\operatorname{Ext}_R^n(L,N)$. It is easy to see that if $\phi\colon N \to K$ is an R-module homomorphism, then $(\phi\theta)_{n*} = \phi_{n*}\theta_{n*}$. In the same sense as with the maps θ_n^*, the θ_{n*} do not depend ("up to isomorphism") on the choice of the resolution (P,α_0) of L, in a way which we make precise as follows. Suppose

$$(Q,\beta_0)\colon \cdots \xrightarrow{\beta_2} Q_1 \xrightarrow{\beta_1} Q_0 \xrightarrow{\beta_0} L \longrightarrow 0$$

is another projective resolution for L. Then as in chapter 2 (Chain Complexes), we obtain Ext groups $\widetilde{\operatorname{Ext}}_R^i(L,M) = \mathrm{H}^i(\operatorname{Hom}_R(Q,M))$ and $\widetilde{\operatorname{Ext}}_R^i(L,N) = \mathrm{H}^i(\operatorname{Hom}_R(Q,N))$ with corresponding group homomorphisms $\tilde{\theta}_{i*}\colon \widetilde{\operatorname{Ext}}_R^i(L,M) \to \widetilde{\operatorname{Ext}}_R^i(L,N)$. There will also exist isomorphisms $\mu_i\colon \operatorname{Ext}_R^i(L,M) \to \widetilde{\operatorname{Ext}}_R^i(L,M)$ and $\nu_i\colon \operatorname{Ext}_R^i(L,N) \to \widetilde{\operatorname{Ext}}_R^i(L,N)$. Then we will have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Ext}^i_R(L,M) & \xrightarrow{\theta_{i*}} & \mathrm{Ext}^i_R(L,N) \\ \mu_i \downarrow & & \downarrow \nu_i \\ \widetilde{\mathrm{Ext}}^i_R(L,M) & \xrightarrow{\tilde{\theta}_{i*}} & \widetilde{\mathrm{Ext}}^i_R(L,N), \end{array}$$

and we will have properties such as θ_i^* is onto if and only if $\tilde{\theta}_i^*$ is onto.

We now show how bimodule structures on the modules M, N, induce module structures on the Ext and Tor groups. We will just consider one case. Suppose R, S are rings, M is an (S, R)-bimodule and N is a right R-module. If $s \in S$, let $\hat{s}\colon M \to M$ denote the map "left multiplication by s". Then $\hat{s}$ is a right R-map, so for $n \in \mathbb{N}$, it induces a group homomorphism $s_n^*\colon \mathrm{Ext}^n_R(M,N) \to \mathrm{Ext}^n_R(M,N)$. It is easily checked that if $t \in S$, then $s_n^* t_n^* = (t_n s_n)^*$. Therefore $\mathrm{Ext}^n_R(M,N)$ is a right S-module. (Perhaps this last step needs a little explanation: we are writing our mappings on the left: the fact that $s_n^* t_n^* = (t_n s_n)^*$ means that things are twisted round, so we do not get a left S-module structure. However if we write our mappings on the right, then things are OK: we would have $t_n^* s_n^* = (ts)_n^*$ and then we obtain a right S-module structure.)

The alert student would have noticed that in some of the cases we have obtained two ways to obtain a module structure on Ext and Tor: the obvious question is whether these are the same. The answer is yes (see exercises below).

Exercises/Examples

(1) Let R be a ring, let M be a right R-module, let N be an (S, R)-bimodule, and let $n \in \mathbb{N}$. Then there are two ways we can make $\mathrm{Ext}^n_R(M,N)$ into a left S-module: either by using the method of (iii) of the previous chapter (where we let (P, α_0) be a projective resolution of M and then use the fact that the chain complex $\mathrm{Hom}_R(P,N)$ has a left S-module structure), or for $s \in S$ considering the R-endomorphism $\hat{s}$ of N "left multiplication by s" which will induce a group homomorphism s_{n*} of $\mathrm{Ext}^n_R(M,N)$. Prove that the two S-module structures on $\mathrm{Ext}^n_R(M,N)$ are the same.

(2) Let R be a ring, let L be an (S, R)-bimodule, let M, N be a left R-modules, and let $n \in \mathbb{N}$.

(i) Prove that $\mathrm{Tor}^R_n(L,M)$ and $\mathrm{Tor}^R_n(L,N)$ have well defined left S-module structures.

(ii) If $\theta\colon M \to N$ is a left R-module map, prove that it induces a map $\theta_{n,*}\colon \mathrm{Tor}^R_n(L,M) \to \mathrm{Tor}^R_n(L,N)$ of left S-modules.

(iii) Prove that $\mathrm{Tor}^R_n(L,M)$ has the structure of an $(S, \mathrm{End}_R(M))$-bimodule. (Remark: since M is a *left* R-module, it is conventional to write elements of $\mathrm{End}_R(M)$ on the *right*.)

(3) Let R be a ring, let M, N be R-modules, and let $q \in \mathbb{Z}$. Let $\mu\colon M \to M$ denote "left multiplication by q". Prove that the induced maps $\mu_n^*\colon \mathrm{Ext}^n_R(M,N) \to \mathrm{Ext}^n_R(M,N)$ are also "left multiplication by q".

(4) Let R be a commutative ring, let $r \in R$, and let M, N be R-modules. Suppose $Mr = 0$ and $Nr = N$. Give an example with $Tor^R_i(M,N) \neq 0$.

Long Exact Homology Sequences We now come to a very important tool for computing the Ext and Tor groups, namely if we are given an exact sequence of R-modules $0 \to A \to$

$B \to C \to 0$, we get long exact sequences connecting the Ext^n_R and Tor^R_n groups. This will enable us to calculate many of the Ext and Tor groups. Theses sequences are very like the exact homology sequence and Mayer-Vietoris sequence in Algebraic Topology. Recall the exact homology sequence of Algebraic Topology. It says that if $A \subseteq X$ are topological spaces, then there is an infinite exact sequence

$$\cdots \longrightarrow \mathrm{H}_q(A) \xrightarrow{\alpha_q^*} \mathrm{H}_q(X) \longrightarrow \mathrm{H}_q(X, A) \xrightarrow{\partial_q} \mathrm{H}_{q-1}(A) \longrightarrow \cdots,$$

where α_q^* denotes the map induced by the inclusion of A in X and ∂_q is called the connecting homomorphism.

Let me state corresponding theorems for Ext and Tor.

Theorem Let R be a ring, let M be an R-module, and let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be an exact sequence of R-modules. Then there exist long exact sequences

$$0 \longrightarrow \mathrm{Ext}^0_R(M, A) \xrightarrow{\alpha_{0*}} \mathrm{Ext}^0_R(M, B) \xrightarrow{\beta_{0*}} \mathrm{Ext}^0_R(M, C) \xrightarrow{\partial_1} \mathrm{Ext}^1_R(M, A) \xrightarrow{\alpha_{1*}} \mathrm{Ext}^1_R(M, B) \xrightarrow{\beta_{1*}} \cdots$$
$$\cdots \xrightarrow{\partial_n} \mathrm{Ext}^n_R(M, A) \xrightarrow{\alpha_{n*}} \mathrm{Ext}^n_R(M, B) \xrightarrow{\beta_{n*}} \mathrm{Ext}^n_R(M, C) \xrightarrow{\partial_{n+1}} \mathrm{Ext}^{n+1}_R(M, A) \xrightarrow{\alpha_{n+1}{}_*} \cdots$$

and

$$0 \longrightarrow \mathrm{Ext}^0_R(C, M) \xrightarrow{\beta_0^*} \mathrm{Ext}^0_R(B, M) \xrightarrow{\alpha_0^*} \mathrm{Ext}^0_R(A, M) \xrightarrow{\partial_1} \mathrm{Ext}^1_R(C, M) \xrightarrow{\beta_1^*} \mathrm{Ext}^1_R(B, M) \xrightarrow{\alpha_1^*} \cdots$$
$$\cdots \xrightarrow{\partial_n} \mathrm{Ext}^n_R(C, M) \xrightarrow{\beta_n^*} \mathrm{Ext}^n_R(B, M) \xrightarrow{\alpha_n^*} \mathrm{Ext}^n_R(A, M) \xrightarrow{\partial_{n+1}} \mathrm{Ext}^{n+1}_R(C, A) \xrightarrow{\beta_{n+1}^*} \cdots$$

Similarly we have long exact sequences for Tor:

Theorem Let M be a right R-module, and let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be an exact sequence of left R-modules. Then there is a long exact sequence

$$\cdots \xrightarrow{\partial_{n+1}} \mathrm{Tor}^R_n(M, A) \xrightarrow{\alpha_{n*}} \mathrm{Tor}^R_n(M, B) \xrightarrow{\beta_{n*}} \mathrm{Tor}^R_n(M, C) \xrightarrow{\partial_n} \mathrm{Tor}^R_{n-1}(M, A) \xrightarrow{\alpha_{n-1}{}_*} \cdots$$
$$\cdots \xrightarrow{\alpha_{1*}} \mathrm{Tor}^R_1(M, B) \xrightarrow{\beta_{1*}} \mathrm{Tor}^R_1(M, C) \xrightarrow{\partial_1} \mathrm{Tor}^R_0(M, A) \xrightarrow{\alpha_{0*}} \mathrm{Tor}^R_0(M, B) \xrightarrow{\beta_{0*}} \mathrm{Tor}^R_0(M, C) \longrightarrow 0.$$

Theorem Let M be a left R-module, and let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be an exact sequence of right R-modules. Then there is an long exact sequence

$$\cdots \xrightarrow{\partial_{n+1}} \mathrm{Tor}^R_n(A, M) \xrightarrow{\alpha_{n*}} \mathrm{Tor}^R_n(B, M) \xrightarrow{\beta_{n*}} \mathrm{Tor}^R_n(C, M) \xrightarrow{\partial_n} \mathrm{Tor}^R_{n-1}(A, M) \xrightarrow{\alpha_{n-1}{}_*} \cdots$$
$$\cdots \xrightarrow{\alpha_{1*}} \mathrm{Tor}^R_1(B, M) \xrightarrow{\beta_{1*}} \mathrm{Tor}^R_1(C, M) \xrightarrow{\partial_1} \mathrm{Tor}^R_0(A, M) \xrightarrow{\alpha_{0*}} \mathrm{Tor}^R_0(B, M) \xrightarrow{\beta_{0*}} \mathrm{Tor}^R_0(C, M) \longrightarrow 0.$$

Thus, for example in the last sequence, given $\mathrm{Tor}_n^R(A,M)$ and $\mathrm{Tor}_n^R(C,M)$ for all n, one can hope to calculate with the aid of this sequence $\mathrm{Tor}_n^R(B,M)$ for all n. For a quick application, we will prove the following.

Proposition Let R be a ring and let P be an R-module. If $\mathrm{Ext}_R^1(P,M) = 0$ for all R-modules M, then P is a projective R-module.

Proof Let $0 \to A \to B \to C \to 0$ be an exact sequence of R-modules. Then using the first long exact sequence involving the Ext groups, we obtain an exact sequence of abelian groups

$$0 \to \mathrm{Hom}_R(P,A) \to \mathrm{Hom}_R(P,B) \to \mathrm{Hom}_R(P,C) \to 0$$

because $\mathrm{Ext}_R^0(P,A) \cong \mathrm{Hom}_R(P,A)$ and $\mathrm{Ext}_R^1(P,A) = 0$. This shows that P is a projective R-module (see (iii) in the paragraph on projective modules on the policy sheet).

We shall just establish the second sequence involving the Ext's. For this we need the Horseshoe Lemma (using injective resolutions, one can avoid the use of this lemma).

Horseshoe Lemma Let R be a ring, let $0 \to A \overset{\theta_{-1}}{\to} C \overset{\phi_{-1}}{\to} B \to 0$ be a short exact sequence of R-modules, and let

$$(P,\alpha_0) : \cdots \xrightarrow{\alpha_2} P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} A \longrightarrow 0,$$
$$(Q,\beta_0) : \cdots \xrightarrow{\beta_2} Q_1 \xrightarrow{\beta_1} Q_0 \xrightarrow{\beta_0} B \longrightarrow 0$$

be projective resolutions for A, B respectively. For each $n \in \mathbb{N}$ set $K_n = P_n \oplus Q_n$, let $\theta_n \colon P_n \to K_n$ denote the natural inclusion, and let $\phi_n \colon K_n \to Q_n$ denote the natural projection with kernel P_n. Then there exist R-maps $\gamma_n \colon K_n \to K_{n-1}$ for $n \in \mathbb{P}$ and $\gamma_0 \colon K_0 \to C$ such that (K,γ_0) is a projective resolution for C, $\theta_{n-1}\alpha_n = \gamma_n\theta_n$, and $\phi_{n-1}\gamma_n = \beta_n\phi_n$.

Proof (sketch) We use induction, so suppose the maps γ_i have been constructed for all $i \le n$. We show that we can construct γ_{n+1}.

First we show that ϕ_n maps $\ker\gamma_n$ onto $\mathrm{im}\,\beta_{n+1}$. Indeed suppose $u \in \mathrm{im}\,\beta_{n+1}$. Write $u = \phi_n v$ where $v \in K_n$. Then $\phi_{n-1}\gamma_n v = \beta_n\phi_n v = \beta_n u = 0$, so $\gamma_n v = \theta_{n-1}w$ for some $w \in P_{n-1}$. Since $\gamma_{n-1}\gamma_n v = 0$, it follows that $\theta_{n-2}\alpha_{n-1}w = 0$ so $\alpha_{n-1}w = 0$ and hence we may write $w = \alpha_n x$ for some $x \in P_n$. Set $y = \theta_n x$. Then $\phi_n(v-y) = \phi_n v = u$ and $\gamma_n(v-y) = \gamma_n v - \theta_{n-1}\alpha_n x = 0$, as required.

Since Q_{n+1} is projective and ϕ_n maps $\ker\gamma_n$ onto $\mathrm{im}\,\beta_{n+1}$, it follows that there exists an R-map $\beta'_{n+1} \colon Q_{n+1} \to \ker\gamma_n$ such that $\phi_n\beta'_{n+1} = \beta_{n+1}$. Now set $\gamma_{n+1} = (\alpha_{n+1}, \beta'_{n+1})$. Then clearly $\theta_n\alpha_{n+1} = \gamma_{n+1}\theta_{n+1}$, $\phi_n\gamma_{n+1} = \beta_{n+1}\phi_{n+1}$, so it remains to show that $\ker\gamma_n = \mathrm{im}\,\gamma_{n+1}$. Since $\gamma_n\gamma_{n+1}\theta_{n+1} = \theta_{n-1}\alpha_n\alpha_{n+1} = 0$ it follows that γ_{n+1} maps the first coordinate P_{n+1} of K_{n+1} to $\ker\gamma_n$, and since β'_{n+1} maps into $\ker\gamma_n$, it follows that γ_{n+1} maps the second coordinate Q_{n+1} of K_{n+1} also into $\ker\gamma_n$. Therefore $\mathrm{im}\,\gamma_{n+1} \subseteq \ker\gamma_n$.

Let $u \in K_n$. Then

$$\begin{aligned}\gamma_n u = 0 &\Longrightarrow \phi_{n-1}\gamma_n u = 0 \Longrightarrow \beta_n\phi_n u = 0\\ &\Longrightarrow \phi_n u = \beta_{n+1}v \quad \text{for some } v \in Q_{n+1}\\ &\Longrightarrow \phi_n u = \beta_{n+1}\phi_{n+1}w \quad \text{for some } w \in K_{n+1}\\ &\Longrightarrow \phi_n u = \phi_n\gamma_{n+1}w \quad \text{for some } w \in K_{n+1}\\ &\Longrightarrow u = \gamma_{n+1}w + \theta_n x \quad \text{for some } x \in P_n.\end{aligned}$$

Since $\gamma_n u = 0$, it follows that $\gamma_n\theta_n x = 0$, so $\theta_{n-1}\alpha_n x = 0$ and hence $x = \alpha_{n+1}y$ for some $y \in P_{n+1}$. Therefore $u = \gamma_{n+1}(w + \theta_{n+1}y)$ as required.

This completes the induction step, and it is not difficult to show that the induction starts.

Third Homework Due 9:00 a.m., Monday, September 12.

(1) Let R be a ring.
   (i) Let $n \in \mathbb{P}$. If I is injective R-module, prove that $\operatorname{Ext}^n_R(M, I) = 0$ for all R-modules M.
   (ii) Let I be an R-module. If $\operatorname{Ext}^1_R(M, I) = 0$ for all R-modules M, prove that I is an injective R-module.

(2) Let $q \in \mathbb{P}$ and let A be an abelian group. Prove:
   (i) $\operatorname{Ext}^0_{\mathbb{Z}}(\mathbb{Z}/q\mathbb{Z}, A) \cong \{a \in A \mid aq = 0\}$.
   (ii) $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/q\mathbb{Z}, A) \cong A/qA$.
   (iii) $\operatorname{Ext}^n_{\mathbb{Z}}(\mathbb{Z}/q\mathbb{Z}, A) = 0$ for all $n \geq 2$.

(3) Let A be a finitely generated abelian group and let B be any abelian group.
   (i) Prove that $\operatorname{Ext}^1_{\mathbb{Z}}(A, B)$ is a torsion group.
   (ii) Prove that $\operatorname{Ext}^n_{\mathbb{Z}}(A, B) = 0$ for all $n \geq 2$.
   (iii) By considering the group $\bigoplus_{q=2}^{\infty} \mathbb{Z}/q\mathbb{Z}$, prove that $\operatorname{Ext}^1_{\mathbb{Z}}(C, B)$ can have elements of infinite order.

(4) Let A, B be abelian groups. Prove that $\operatorname{Ext}^n_{\mathbb{Z}}(A, B) = \operatorname{Tor}^{\mathbb{Z}}_n(A, B) = 0$ for all $n \geq 2$.

(5) Let R be a commutative ring, let $I, J \lhd R$, let M, N be R-modules, and let $n \in \mathbb{N}$.
   (i) If $MI = 0$, prove that $\operatorname{Ext}^n_R(M, N)I = 0$.
   (ii) If $NJ = 0$, prove that $\operatorname{Ext}^n_R(M, N)J = 0$.
   (iii) If $MI = NJ = 0$ and $I + J = R$, prove that $\operatorname{Ext}^n_R(M, N) = 0$.

(6) Let R, S be rings, let M, N be S-modules, and let $\theta\colon R \to S$ be a ring homomorphism.
   (i) Prove that M, N become R-modules by defining $mr = m\theta r$ for $m \in M$ or N, $r \in R$ and $s \in S$. Prove also that if $f\colon M \to N$ is an S-module map, then it is also an R-module map.

(ii) Let

$$(P, \alpha_0) : \cdots \xrightarrow{\alpha_2} P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} M \longrightarrow 0$$
$$\text{and} \quad (Q, \beta_0) : \cdots \xrightarrow{\beta_2} Q_1 \xrightarrow{\beta_1} Q_0 \xrightarrow{\beta_0} M \longrightarrow 0$$

be projective resolutions for M as R and S-modules respectively. Viewing (Q, β_0) as a sequence of R-modules (not necessarily projective), prove that there exist R-maps $\theta_n \colon P_n \to Q_n$ such that $\beta_n \theta_n = \theta_{n-1} \alpha_n$ for all $n \in \mathbb{N}$, where θ_{-1} is the identity map on M.

(iii) Prove that the θ_n induce well defined group homomorphisms $\theta_n^* \colon \operatorname{Ext}_S^n(M, N) \to \operatorname{Ext}_R^n(M, N)$. (You will need to set $\theta_n^*(f + \operatorname{im} \beta_n^*) = f\theta_n + \operatorname{im} \alpha_n^*$ for $f \in \ker \beta_{n+1}^*$.)

(iv) Prove that if $\phi_n \colon P_n \to Q_n$ are R-maps satisfying $\beta_n \phi_n = \phi_{n-1} \alpha_n$, then $\phi_n^* = \theta_n^* \colon \operatorname{Ext}_S^n(M, N) \to \operatorname{Ext}_R^n(M, N)$.

(This is a very important result which will be used later. Its proof is very similar to the results we have been doing in class that module homomorphisms induce homomorphisms in Ext and Tor. There is an analogous result for Tor: namely the following. Let R, S be rings, let M be a right S-module, let N be a left S-module, and let $\theta \colon R \to S$ be a ring homomorphism. Then there exist well defined group homomorphisms $\theta_{n*} \colon \operatorname{Tor}_n^R(M, N) \to \operatorname{Tor}_n^S(M, N)$ for all $n \in \mathbb{N}$.)

(7) Let R be a ring, let $I \lhd_{\mathrm{r}} R$, and let $J \lhd_{\mathrm{l}} R$.

(i) Prove that the map $i \mapsto i \otimes (1 + J)$ defines a group epimorphism $I \to I \otimes_R R/J$ with kernel IJ.

(ii) Prove that $\operatorname{Tor}_1^R(R/I, R/J)$ is isomorphic to the kernel of the map $I \otimes_R R/J \to R \otimes_R R/J$ defined by $i \otimes (1 + J) \mapsto i \otimes (1 + J)$. (Use the long exact homology sequence.)

(iii) Prove that $\operatorname{Tor}_1^R(R/I, R/J) \cong \dfrac{I \cap J}{IJ}$.

Monday, September 12

Chapter 5
Commutative Diagrams

We now come to the two fundamental lemmas for obtaining the long exact sequences involving Ext and Tor. There is one for chain complexes, and one for cochain complexes.

Lemma 1 Let R be a ring, and let $0 \to A \xrightarrow{\theta} B \xrightarrow{\phi} C \to 0$ be an exact sequence of R-chain complexes; i.e. a commutative diagram of R-modules and R-maps with exact rows (but not necessarily exact columns)

$$\begin{array}{ccccccc}
 & & \alpha_3\downarrow & \beta_3\downarrow & \gamma_3\downarrow & & \\
0 & \longrightarrow & A_2 & \xrightarrow{\theta_2} & B_2 & \xrightarrow{\phi_2} & C_2 \longrightarrow 0 \\
 & & \alpha_2\downarrow & \beta_2\downarrow & \gamma_2\downarrow & & \\
0 & \longrightarrow & A_1 & \xrightarrow{\theta_1} & B_1 & \xrightarrow{\phi_1} & C_1 \longrightarrow 0 \\
 & & \alpha_1\downarrow & \beta_1\downarrow & \gamma_1\downarrow & & \\
0 & \longrightarrow & A_0 & \xrightarrow{\theta_0} & B_0 & \xrightarrow{\phi_0} & C_0 \longrightarrow 0 \\
 & & \downarrow & \downarrow & \downarrow & & \\
 & & 0 & 0 & 0 & &
\end{array}$$

Then there exists a long exact sequence (of R-modules)

$$\cdots \longrightarrow \mathrm{H}_{n+1}(C) \xrightarrow{\partial_{n+1}} \mathrm{H}_n(A) \xrightarrow{\theta_{n*}} \mathrm{H}_n(B) \xrightarrow{\phi_{n*}} \mathrm{H}_n(C) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(A) \xrightarrow{\theta_{n-1*}} \cdots$$
$$\cdots \xrightarrow{\partial_1} \mathrm{H}_0(A) \xrightarrow{\theta_{0*}} \mathrm{H}_0(B) \xrightarrow{\phi_{0*}} \mathrm{H}_0(C) \longrightarrow 0.$$

Proof (Sketch) The proof is very similar in spirit to the Horseshoe Lemma, so we will no more than define the relevant maps: the only maps which are not easy to define are the ∂'s. These are often termed the connecting homomorphisms.

The θ_* and ϕ_* maps are induced by the θ and ϕ. Specifically, $\theta_{n*}a = \theta_n a + \operatorname{im}\beta_{n+1}$ for $a \in \ker\alpha_n$, while $\phi_{n*}b = \phi_n b + \operatorname{im}\gamma_{n+1}$ for $b \in \ker\beta_n$. To define ∂_n, suppose $\bar{c} \in \mathrm{H}_n(C)$ is the element $c + \operatorname{im}\gamma_{n+1}$ where $c \in \ker\gamma_n$. Since ϕ_n is onto, we may choose $b \in B_n$ such that $\phi_n b = c$. Then $\phi_{n-1}\beta_n b = \gamma_n\phi_n b = \gamma_n c = 0$, hence there exists $a \in A_{n-1}$ such that $\theta_{n-1}a = \beta_n b$, because $0 \to A_{n-1} \xrightarrow{\theta_{n-1}} B_{n-1} \xrightarrow{\phi_{n-1}} C_{n-1} \to 0$ is exact. It is easy to check that $a \in \ker\alpha_{n-1}$, so $a + \operatorname{im}\alpha_n$ defines an element $\bar{a} \in \mathrm{H}_{n-1}(A)$. Then it is not difficult to show that the rule $\partial_n\bar{c} = \bar{a}$ yields a well defined R-map $\mathrm{H}_n(C) \to \mathrm{H}_{n-1}(A)$, and the resulting sequence is exact.

We have a similar result for cochain complexes.

Lemma 2 Let R be a ring, and let $0 \to A \xrightarrow{\theta} B \xrightarrow{\phi} C \to 0$ be an exact sequence of R-cochain complexes; i.e. a commutative diagram of R-modules and R-maps with exact rows (but not necessarily exact columns)

$$\begin{array}{ccccccccc}
 & & \alpha_3\uparrow & & \beta_3\uparrow & & \gamma_3\uparrow & & \\
0 & \longrightarrow & A_2 & \xrightarrow{\theta_2} & B_2 & \xrightarrow{\phi_2} & C_2 & \longrightarrow & 0 \\
 & & \alpha_2\uparrow & & \beta_2\uparrow & & \gamma_2\uparrow & & \\
0 & \longrightarrow & A_1 & \xrightarrow{\theta_1} & B_1 & \xrightarrow{\phi_1} & C_1 & \longrightarrow & 0 \\
 & & \alpha_1\uparrow & & \beta_1\uparrow & & \gamma_1\uparrow & & \\
0 & \longrightarrow & A_0 & \xrightarrow{\theta_0} & B_0 & \xrightarrow{\phi_0} & C_0 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Then there exists a long exact sequence (of R-modules)

$$\begin{aligned}
0 \longrightarrow \mathrm{H}^0(A) \xrightarrow{\theta_{0*}} \mathrm{H}^0(B) \xrightarrow{\phi_{0*}} \mathrm{H}^0(C) \xrightarrow{\partial_1} \mathrm{H}^1(A) \xrightarrow{\theta_{1*}} \mathrm{H}^1(B) \xrightarrow{\phi_{1*}} \cdots \\
\cdots \xrightarrow{\partial_n} \mathrm{H}^n(A) \xrightarrow{\theta_{n*}} \mathrm{H}^n(B) \xrightarrow{\phi_{n*}} \mathrm{H}^n(C) \xrightarrow{\partial_{n+1}} \cdots .
\end{aligned}$$

Proof The proof of this is a homework exercise.

We now apply the above Lemmas to obtain the long exact sequences for Ext.

Corollary 3 Let R be a ring, let M be an R-module, and let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be an exact sequence of R-modules. Then there is a long exact sequence

$$\begin{aligned}
0 \longrightarrow \mathrm{Ext}^0_R(M,A) \xrightarrow{\alpha_{0*}} \mathrm{Ext}^0_R(M,B) \xrightarrow{\beta_{0*}} \mathrm{Ext}^0_R(M,C) \xrightarrow{\partial_1} \mathrm{Ext}^1_R(M,A) \xrightarrow{\alpha_{1*}} \mathrm{Ext}^1_R(M,B) \xrightarrow{\beta_{1*}} \cdots \\
\cdots \xrightarrow{\partial_n} \mathrm{Ext}^n_R(M,A) \xrightarrow{\alpha_{n*}} \mathrm{Ext}^n_R(M,B) \xrightarrow{\beta_{n*}} \mathrm{Ext}^n_R(M,C) \xrightarrow{\partial_{n+1}} \mathrm{Ext}^{n+1}_R(M,A) \xrightarrow{\alpha_{n+1}{}^*} \cdots .
\end{aligned}$$

Proof Let

$$(P,\mu_0)\colon \cdots \xrightarrow{\mu_2} P_1 \xrightarrow{\mu_1} P_0 \xrightarrow{\mu_0} M \longrightarrow 0$$

be a projective resolution for M. Then we have an exact sequence of cochain complexes

$$0 \longrightarrow \mathrm{Hom}_R(P,A) \xrightarrow{\theta} \mathrm{Hom}_R(P,B) \xrightarrow{\phi} \mathrm{Hom}_R(P,C) \longrightarrow 0.$$

Here $\theta_n = \alpha_*\colon \mathrm{Hom}_R(P_n,A) \to \mathrm{Hom}_R(P_n,B)$, $\phi_n = \beta_*\colon \mathrm{Hom}_R(P_n,B) \to \mathrm{Hom}_R(P_n,C)$, $\alpha_n = \mu_n^*\colon \mathrm{Hom}_R(P_{n-1},A) \to \mathrm{Hom}_R(P_n,A)$, $\beta_n = \mu_n^*\colon \mathrm{Hom}_R(P_{n-1},B) \to \mathrm{Hom}_R(P_n,B)$, and $\gamma_n = \mu_n^*\colon \mathrm{Hom}_R(P_{n-1},C) \to \mathrm{Hom}_R(P_n,C)$. Note that

$$0 \longrightarrow \mathrm{Hom}_R(P_n,A) \xrightarrow{\theta_n} \mathrm{Hom}_R(P_n,B) \xrightarrow{\phi_n} \mathrm{Hom}_R(P_n,C) \longrightarrow 0.$$

is exact because P_n is projective; thus the sequence of cochain complexes is exact. Now apply Lemma 2.

Corollary 4 Let R be a ring, let M be an R-module, and let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be an exact sequence of R-modules. Then there is a long exact sequence

$$0 \longrightarrow \mathrm{Ext}^0_R(C,M) \xrightarrow{\beta_0^*} \mathrm{Ext}^0_R(B,M) \xrightarrow{\alpha_0^*} \mathrm{Ext}^0_R(A,M) \xrightarrow{\partial_1} \mathrm{Ext}^1_R(C,M) \xrightarrow{\beta_1^*} \mathrm{Ext}^1_R(B,M) \xrightarrow{\alpha_1^*} \cdots$$
$$\cdots \xrightarrow{\partial_n} \mathrm{Ext}^n_R(C,M) \xrightarrow{\beta_n^*} \mathrm{Ext}^n_R(B,M) \xrightarrow{\alpha_n^*} \mathrm{Ext}^n_R(A,M) \xrightarrow{\partial_{n+1}} \mathrm{Ext}^{n+1}_R(C,A) \xrightarrow{\beta_{n+1}^*} \cdots .$$

Proof Let

$$(P,\alpha_0) : \cdots \xrightarrow{\alpha_2} P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} A \longrightarrow 0,$$
$$(T,\gamma_0) : \cdots \xrightarrow{\gamma_2} T_1 \xrightarrow{\gamma_1} T_0 \xrightarrow{\gamma_0} C \longrightarrow 0$$

be projective resolutions for A, C respectively. By the Horseshoe Lemma, we obtain a commutative diagram with exact rows and the outer columns exact

$$\begin{array}{ccccccccc}
 & & \alpha_3\downarrow & & \beta_3\downarrow & & \gamma_3\downarrow & & \\
0 & \longrightarrow & P_2 & \xrightarrow{\theta_2} & Q_2 & \xrightarrow{\phi_2} & T_2 & \longrightarrow & 0 \\
 & & \alpha_2\downarrow & & \beta_2\downarrow & & \gamma_2\downarrow & & \\
0 & \longrightarrow & P_1 & \xrightarrow{\theta_1} & Q_1 & \xrightarrow{\phi_1} & T_1 & \longrightarrow & 0 \\
 & & \alpha_1\downarrow & & \beta_1\downarrow & & \gamma_1\downarrow & & \\
0 & \longrightarrow & P_0 & \xrightarrow{\theta_0} & Q_0 & \xrightarrow{\phi_0} & T_0 & \longrightarrow & 0 \\
 & & \alpha_0\downarrow & & \beta_0\downarrow & & \gamma_0\downarrow & & \\
0 & \longrightarrow & A & \xrightarrow{\theta_{-1}} & B & \xrightarrow{\phi_{-1}} & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where (Q,β_0) is a projective resolution for B. Since T is projective, the sequence $0 \to \mathrm{Hom}_R(T_n,M) \xrightarrow{\phi_n^*} \mathrm{Hom}_R(Q_n,M) \xrightarrow{\theta_n^*} \mathrm{Hom}_R(P_n,M) \to 0.$ is exact for all $n \in \mathbb{N}$. Now apply Lemma 2.

Let us give another application of these long exact sequences. For this we will assume the well known result (mentioned in the first chapter) that every module can be embedded in an injective module; this result is proved in nearly any book on homological algebra, and may be in the book you used for 5000 Algebra.

Example Let R be a ring, let M be an R-module, and let $n \in \mathbb{N}$. If $\mathrm{Ext}^n_R(M, A) = 0$ for all R-modules A, then $\mathrm{Ext}^t_R(M, B) = 0$ for all $t \geq n$ and for all R-modules B.

Proof By induction on t, we need only consider the case $t = n+1$. We may embed B in an injective R-module I, and then there exists a short exact sequence $0 \to B \to I \to A \to 0$ for some R-module A. Applying the long exact sequence for Ext_R, we obtain an exact sequence of abelian groups

$$\cdots \longrightarrow \mathrm{Ext}^n_R(M, I) \longrightarrow \mathrm{Ext}^n_R(M, A) \longrightarrow \mathrm{Ext}^{n+1}_R(M, B) \longrightarrow \mathrm{Ext}^{n+1}_R(M, I) \longrightarrow \cdots .$$

By Exercise 1 of the third homework, $\mathrm{Ext}^{n+1}_R(M, I) = 0$ because I is injective and $n+1 \geq 1$. The result follows.

Change of Rings There are many results concerning the affect of Ext_R and Tor^R when the ring R is changed. There has already been one along these lines, namely problem 6 from the previous (third) homework. In that problem it showed that a homomorphism $R \to S$ of rings gives a corresponding natural homomorphism $\theta_{M,N} \colon \mathrm{Ext}^n_S(M, N) \to \mathrm{Ext}^n_R(M, N)$. The adjective natural here means that the maps induced in Ext by module homomorphisms will commute with $\theta_{M,N}$. Also the $\theta_{M,N}$ will commute with the connecting homomorphism in long exact sequences. Thus for example, if $0 \to A \to B \to C \to 0$ is an exact sequence of R-modules, then there exists a commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^n_R(M, C) & \stackrel{\partial_{n+1}}{\longrightarrow} & \mathrm{Ext}^{n+1}_R(M, A) \\ \theta_{M,C} \downarrow & & \downarrow \theta_{M,A} \\ \mathrm{Ext}^n_R(M, C) & \stackrel{\partial_{n+1}}{\longrightarrow} & \mathrm{Ext}^{n+1}_R(M, A). \end{array}$$

In this case there is no assertion that the θ's are isomorphisms; in fact usually they will not be isomorphisms, but the existence of these natural homomorphisms can be a powerful tool. We shall now consider some of these change of rings theorems where we in fact do have isomorphisms.

Let R be a ring and let M be an R-module. Recall that M is flat means that if $0 \to A \to B \to C \to 0$ is an exact sequence of R-modules, then the induced sequence

$$0 \longrightarrow A \otimes_R M \longrightarrow B \otimes_R M \longrightarrow C \otimes_R M \longrightarrow 0$$

is also exact. Since taking tensor product is always right exact, an equivalent formulation of this is that given an exact sequence $0 \to A \to B$, then the induced sequence $0 \to A \otimes_R M \to B \otimes_R M$ is also exact. By definition, all projective modules are flat, though the converse is not true: this will become apparent in what follows. An important example of a flat module is that of a ring obtained by localization. Let R be a commutative ring and let S be a multiplicatively closed subset of R. Then $S^{-1}R$ is a flat R-module. To see this let $0 \to A \stackrel{\alpha}{\to} B$ be an exact sequence of R-modules. By Exercise 16 of the first chapter, $A \otimes_R S^{-1}R \cong S^{-1}A$; the isomorphism is given by $m \otimes s^{-1}r \mapsto ms^{-1}r$. It is easy to see that this isomorphism is natural, so we have a commutative diagram

$$\begin{array}{ccc} A \otimes_R S^{-1}R & \stackrel{\alpha\otimes 1}{\longrightarrow} & B \otimes_R S^{-1}R \\ \downarrow & & \downarrow \\ S^{-1}A & \stackrel{S^{-1}\alpha}{\longrightarrow} & S^{-1}B \end{array}$$

where the vertical arrows are isomorphisms, and $S^{-1}\alpha$ denotes the map $s^{-1}a \mapsto s^{-1}\alpha a$. Since $A \to B \to C$ exact implies $S^{-1}A \to S^{-1}B \to S^{-1}C$ is also exact, it now follows that $0 \to A \otimes_R S^{-1} \stackrel{\alpha\otimes 1}{\to} B \otimes_R S^{-1}$ is exact as required.

It should be noted that the above remains true even when R is noncommutative, provided that S is contained in the center of R. To see this, simply let C denote the center of R and apply $\_ \otimes_C S^{-1}C$ instead of $\_ \otimes_R S^{-1}R$.

Exercise The purpose of this exercise is to show that Tor can be computed using flat resolutions. In detail, let R be a ring, let A be a right R-module, let B be a left R-module, and let $(P, \alpha_0)\colon \cdots \stackrel{\alpha_2}{\to} P_1 \stackrel{\alpha_1}{\to} P_0 \stackrel{\alpha_0}{\to} A \to 0$ be a flat resolution of A (i.e. (P, α_0) is a resolution and all the P_i are flat).

(i) Using the fact that $\_ \otimes_R B$ is right exact, prove that $\mathrm{Tor}_0^R(A, B) \cong \mathrm{H}_0(P \otimes_R B)$.

(ii) Apply the long exact sequence for Tor to $0 \to P_1/\operatorname{im}\alpha_2 \to P_0 \to P_0/\operatorname{im}\alpha_1 \to 0$ to deduce that $\mathrm{Tor}_1^R(A, B) \cong \mathrm{H}_1(P \otimes_R B)$.

(iii) Use the long exact sequence for Tor to show that $\mathrm{Tor}_n^R(A, B) \cong \mathrm{Tor}_1^R(\operatorname{im}\alpha_{n-1}, B)$ for $n \geq 1$. Deduce that $\mathrm{Tor}_n^R(A, B) \cong \mathrm{H}_n(P \otimes_R B)$ as required.

To make further progress, we need the following easy but important lemma.

Lemma 5 Let R be a ring, let P be a chain complex of right R-modules, and let M be a flat left R-module. Then $\mathrm{H}_n(P \otimes_R M) \cong \mathrm{H}_n(P) \otimes_R M$.

Proof Let the boundary maps of P be α_n, let $Z_n = \ker\alpha_n$, let $B_n = \operatorname{im}\alpha_{n+1}$, and let $\iota\colon Z_n \to P_n$ denote the natural inclusion. Since $0 \to Z_n \stackrel{\iota}{\to} P_n \stackrel{\alpha_n}{\to} B_{n-1} \to 0$ is exact and M is a flat R-module, we see that $\iota \otimes 1$ maps $Z_n \otimes_R M$ isomorphically onto $\ker(\alpha_n \otimes 1)$, and $B_n \otimes_R M$ isomorphically onto $\operatorname{im}(\alpha_{n+1} \otimes 1)$. Using the fact that M is flat again, tensoring $0 \to B_n \to Z_n \to Z_n/B_n \to 0$ with M yields the required result.

We can now state the following.

Lemma 6 Let R be a commutative ring, let $n \in \mathbb{N}$, let A, B be a R-modules, and let T be a flat R-module. Then $\mathrm{Tor}_n^R(A, B) \otimes_R T \cong \mathrm{Tor}_n^R(A, B \otimes_R T)$.

Proof Let $(P, \epsilon)\colon \cdots \to P_2 \to P_1 \to P_0 \stackrel{\epsilon}{\to} A \to 0$ be a projective resolution of A. Then $\mathrm{Tor}_n^R(A, B) = \mathrm{H}_n(P \otimes_R B)$ and $\mathrm{Tor}_n^R(A, B \otimes_R T) = \mathrm{H}_n(P \otimes_R (B \otimes_R T))$. Since $(P_n \otimes_R B) \otimes_R T \cong P_n \otimes_R (B \otimes_R T)$ (see Example 4 on Tensor Products on the first chapter), the result follows from Lemma 5.

Lemma 7 Let R, T be a rings, let $\theta\colon R \to T$ be a ring homomorphism which makes T into a flat left R-module, let $n \in \mathbb{N}$, let A be a right R-module, and let B be a left T-module. Then $\mathrm{Tor}_n^R(A, B) \cong \mathrm{Tor}_n^T(A \otimes_R T, B)$.

Proof Let $(P, \epsilon)\colon \cdots \to P_2 \to P_1 \to P_0 \xrightarrow{\epsilon} A \to 0$ be a projective resolution of A. Since T is flat as a left R-module, it follows that $(P \otimes_R T, \epsilon \otimes 1)$ is a projective resolution of $A \otimes_R T$ as T-modules. Thus $\operatorname{Tor}_n^R(A, B) = \operatorname{H}_n(P \otimes_R B)$ and $\operatorname{Tor}_n^T(A \otimes_R T, B) = \operatorname{H}_n((P \otimes_R T) \otimes_T B)$. Since

$$(P_n \otimes_R T) \otimes_T B \cong P_n \otimes_R (T \otimes_T B) \cong P_n \otimes_R B$$

naturally (see (2) and (4) on the section on Tensor Products on the first chapter), the result follows.

Note that the above result is true even when R is noncommutative. This will be useful when we do group cohomology: in that situation, T will be the group ring of a group G, and R will be the group ring of a subgroup of G.

Theorem 8 Let R be a commutative ring, let T be a flat R-algebra, let $n \in \mathbb{N}$, and let A, B be R-modules. Then $\operatorname{Tor}_n^R(A, B) \otimes_R T \cong \operatorname{Tor}_n^T(A \otimes_R T, B \otimes_R T)$ as R-modules.

Proof To say that T is an R-algebra means that we are given a ring homomorphism $\theta\colon R \to T$ such that $\operatorname{im}\theta$ lies in the center of T. This means that T can be considered as an R-bimodule over the commutative ring R, and the left and right actions are the same. The proof of this result is immediate from Lemmas 6 and 7.

There are two immediate applications of the above result.

Corollary 9 Let R be a ring, let S be a multiplicatively closed subset in the center of R, let $n \in \mathbb{N}$, and let A, B be R-modules. Then $S^{-1} \operatorname{Tor}_n^R(A, B) \cong \operatorname{Tor}_n^{S^{-1}R}(S^{-1}A, S^{-1}B)$.

Proof This follows from Theorem 8 and the fact that $S^{-1}R$ is a flat R-module.

Corollary 10 Let $k \subseteq K$ be fields, let R be a k-algebra, let $n \in \mathbb{N}$, and let A, B be R-modules. Then $\operatorname{Tor}_n^R(A, B) \otimes_k K \cong \operatorname{Tor}_n^{R \otimes_k K}(A \otimes_k K, B \otimes_k K)$.

Proof This is because $R \otimes_k K$ is a flat (even free) K-module.

We can apply Corollary 10 to prove that $\operatorname{Tor}_n^{\mathbb{Z}}(A, B) = 0$ for all $n \in \mathbb{P}$ and for all $\mathbb{Z}$-modules A, B. To see this, let $S = \mathbb{Z}\backslash 0$. Then $S^{-1}\mathbb{Z} \cong \mathbb{Q}$ as rings. It is easy to prove that $\operatorname{Tor}_n^k(A, B) = 0$ for all $n \in \mathbb{P}$ if k is a field (though this requires the fact that every vector space has a basis, i.e. every k-vector space is a free k-module): thus $\operatorname{Tor}_n^{S^{-1}Z}(S^{-1}A, S^{-1}B) = 0$ for all $n \in \mathbb{P}$ and for all $\mathbb{Z}$-modules A, B. Now apply Corollary 10 to deduce that $S^{-1} \operatorname{Tor}_n^{\mathbb{Z}}(A, B) = 0$, and then it is easy to prove that $\operatorname{Tor}_n^{\mathbb{Z}}(A, B)$ is a torsion group for all $n \in \mathbb{P}$ and for all $\mathbb{Z}$-modules A, B.

Similar results hold for Ext, though some restriction on the rings are required. We will just state the following result, whose proof is similar to the results on Tor which we have just been doing. Recall that a ring R is right Noetherian if every right ideal is finitely generated.

Theorem Let R be a right Noetherian ring, let S be a multiplicatively closed subset contained in the center of R, let $n \in \mathbb{N}$, and let A, B be R-modules with A finitely generated. Then

$$S^{-1}\operatorname{Ext}^n_R(A,B) \cong \operatorname{Ext}^n_{S^{-1}R}(S^{-1}A, S^{-1}B).$$

Fourth Homework Due 9:00 a.m., Monday, September 19.

(1) Prove (in detail) Lemma 2.

(2) Prove the two long exact sequences for Tor of the last chapter.

(3) Let k be a field, let $R = k[X,Y]/(X^2, Y^2, XY)$, and let $n \in \mathbb{N}$. Thus $\dim_k R = 3$ and R has a nilpotent ideal of dimension 2. Let $\bar{X}$, $\bar{Y}$ denote the images of X, Y in R respectively, and let k denote the R-module with $a\bar{X} = a\bar{Y} = 0$ for all $a \in R$. Prove that $\operatorname{Ext}^n_R(k,k) \cong k^{(2^n)}$ as k-modules.

(4) Let k be a field, let R denote the 2×2 upper triangular matrices over k, let $\{e_{ij} \mid 1 \leq i \leq j \leq 2\}$ denote the matrix units of R (i.e. matrices with one entry 1 and 0's elsewhere). Thus R is the set of matrices $\{(a_{ij}) \mid a_{21} = 0\}$. Let U denote the irreducible R-module with e_{22} acting trivially (i.e. $ue_{22} = 0$ and $ue_{11} = u$ for all $u \in U$), and let V denote the irreducible R-module with e_{11} acting trivially. If $n \in \mathbb{N}$, prove that $\operatorname{Ext}^n_R(U,V) = 0$ if $n \neq 1$, and $\operatorname{Ext}^n_R(U,V) \cong k$ as k-modules.

(5) The purpose of this problem is to give a proof of the well known fact that $\mathbb{Q}$ is an injective $\mathbb{Z}$-module, assuming the result that every subspace of a vector space has a direct complement. In the following, M denotes a $\mathbb{Z}$-module which is isomorphic to $\mathbb{Q}$.
   (i) Let $S = \mathbb{Z}\backslash 0$, and suppose that M is a $\mathbb{Z}$-submodule of the torsion free $\mathbb{Z}$-module A. By considering the natural ring homomorphism from A to $S^{-1}A$ ($a \mapsto a/1$), prove that there exists a $\mathbb{Z}$-submodule B of A which is a direct complement of M in A.
   (ii) Prove that M is an injective $\mathbb{Z}$-module.

(6) For this problem, assume the result of the previous problem, namely that $\mathbb{Q}$ is an injective $\mathbb{Z}$-module. Let C_{p^∞} denote the $\mathbb{Z}$-module which consists of the elements of p-power order of $\mathbb{Q}/\mathbb{Z}$. One can say that C_{p^∞} is the Sylow p-subgroup of $\mathbb{Q}/\mathbb{Z}$ (though it will of course be an infinite group), or one can consider it as $\bigcup_{n=1}^{\infty} \mathbb{Z}/p^n\mathbb{Z}$.
   (i) Prove that $\operatorname{Ext}^1_{\mathbb{Z}}(C_{p^\infty}, \mathbb{Z})$ is an infinite torsion free group.
   (ii) Show that if R is a commutative ring, S is a multiplicatively closed subset of R, and A, B are R-modules, then $S^{-1}\operatorname{Ext}^1_R(A,B) \cong \operatorname{Ext}^1_{S^{-1}R}(S^{-1}A, S^{-1}B)$ is not true in general.
   (iii) Using the fact that $\mathbb{Q}$ is an injective $\mathbb{Z}$-module, prove that C_{p^∞} is an injective $\mathbb{Z}$-module.

(7) Let K be an integral domain (i.e. a commutative ring with no zero divisors) and let $R = K[X]$. Prove that $\operatorname{Tor}^R_n(A,B)$ is a torsion K-module for all $n \geq 2$ and for all R-modules A, B.

Monday, September 19

Chapter 6
Limits

Remarks on last chapter

(1) Let R be a ring, let A be a right R-module and let B be a flat left R-module. Then $\mathrm{Tor}_n^R(A,B) = 0$ for all $n \in \mathbb{P}$. This is easy to prove directly; it can also be seen from Lemma 5 of chapter 5.

(2) Let R be a ring, let A be a flat right R-module and let B be a left R-module. Then $\mathrm{Tor}_n^R(A,B) = 0$ for all $n \in \mathbb{P}$. To see this, take a short exact sequence $0 \to K \to F \to B \to 0$ where F is a free R-module and apply the long exact sequence for Tor in the second variable. By induction on n and the fact that $\mathrm{Tor}_n^R(A,F) = 0$ for all $n \in \mathbb{P}$ (see (1)), we immediately reduce to the case $n = 1$. Now use the fact that $0 \to A \otimes_R K \to A \otimes_R F \to A \otimes_R B \to 0$ is exact because A is flat.

(3) All these change of ring isomorphisms are natural. Thus for example if

$$\theta_{A,B}\colon\ \mathrm{Tor}_n^R(A,B) \otimes_R T \longrightarrow \mathrm{Tor}_n^T(A \otimes_R T, B \otimes_R T)$$

is the isomorphism of Theorem 8 of the fifth (i.e. previous chapter) and $\beta\colon B \to C$ is an R-module homomorphism, then there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Tor}_n^R(A,B) \otimes_R T & \overset{\beta_{n*}\otimes 1}{\longrightarrow} & \mathrm{Tor}_n^R(A,C) \otimes_R T \\ \theta_{A,B} \downarrow & & \downarrow \theta_{A,C} \\ \mathrm{Tor}_n^T(A \otimes_R T, B \otimes_R T) & \overset{(\beta\otimes 1)_{n*}}{\longrightarrow} & \mathrm{Tor}_n^T(A \otimes_R T, C \otimes_R T). \end{array}$$

(4) Let R, T be rings, and let $\theta\colon R \to T$ be a ring homomorphism which makes T into a flat right R-module, let A be a right T-module and let B be a left R-module. Then there is a natural isomorphism between $\mathrm{Tor}_n^R(A,B)$ and $\mathrm{Tor}_n^T(A, T \otimes_R B)$.

(5) Let R, T be rings, let $\theta\colon R \to T$ be a ring homomorphism which makes T into a flat left R-module, let A be a right R-module and let T be a right T-module. Then there is a natural isomorphism between $\mathrm{Ext}_R^n(A,B)$ and $\mathrm{Ext}_T^n(A \otimes_R T, B)$.

(6) By a similar argument to the exercise of the previous chapter, we can prove that Tor can be computed by using a flat resolution of the second argument. Specifically, let R be a ring, let M be a right R-module, and let A be a left R-module. Let $(P, \alpha_0)\colon \cdots \overset{\alpha_2}{\to} P_1 \overset{\alpha_1}{\to} P_0 \overset{\alpha_0}{\to} A \to 0$ be a flat resolution of A with flat left R-modules. Then $\mathrm{H}_n(M \otimes_R P) \cong \mathrm{Tor}_n^R(M,A)$.

(7) We can use (6) to demonstrate the balancing of Tor ($\mathrm{Tor}(A,B) = \mathrm{Tor}(B,A)$). Specifically, let R be a ring, and use the superscript op to denote the opposite ring and the corresponding opposite modules. Thus if R is ring, then R^{op} will denote the opposite ring which has the same underlying set as R and with new multiplication given by $r \cdot s = sr$ for $r, s \in R$. If M is a right R-module, then M^{op} will denote the left R^{op} module which has the same underlying set as R, and in which $rm = mr$ for $m \in M$ and $r \in R$. Then we have

Theorem Let R be a ring, let A be a right R-module, let B be a left R-module, and let $n \in \mathbb{N}$. Then $\mathrm{Tor}_n^R(A,B) \cong \mathrm{Tor}_n^{R^{\mathrm{op}}}(B^{\mathrm{op}}, A^{\mathrm{op}})$ as abelian groups.

Remark In the special case R is commutative, we do not have to bother with opposite rings. In fact then we have the following result: let R be a commutative ring, let A, B be R-modules, and let $n \in \mathbb{N}$. Then $\mathrm{Tor}_n^R(A,B) \cong \mathrm{Tor}_n^R(B,A)$ as R-modules.

Proof of Theorem Let $(P,\alpha_0)\colon \cdots \stackrel{\alpha_2}{\to} P_1 \stackrel{\alpha_1}{\to} P_0 \stackrel{\alpha_0}{\to} A \to 0$ be a projective resolution for A. Then $(P^{\mathrm{op}},\alpha_0)\colon \cdots \stackrel{\alpha_2}{\to} P_1^{\mathrm{op}} \stackrel{\alpha_1}{\to} P_0^{\mathrm{op}} \stackrel{\alpha_0}{\to} A^{\mathrm{op}} \to 0$ is a projective resolution for P^{op}, $\mathrm{Tor}_n^R(A,B) \cong \mathrm{H}_n(P \otimes_R B)$ and $\mathrm{Tor}_n^{R^{\mathrm{op}}}(B^{\mathrm{op}}, A^{\mathrm{op}}) \cong \mathrm{H}_n(B^{\mathrm{op}} \otimes_{R^{\mathrm{op}}} A^{\mathrm{op}})$. However the map $x \otimes b \mapsto b \otimes x$ induces a group isomorphism $P_n \otimes_R B \to B^{\mathrm{op}} \otimes_{R^{\mathrm{op}}} P_n^{\mathrm{op}}$, which in turn induces an isomorphism $\mathrm{H}_n(P \otimes_R B) \to \mathrm{H}_n(B^{\mathrm{op}} \otimes_{R^{\mathrm{op}}} A^{\mathrm{op}})$, and the result follows.

Direct and Inverse Limits The notions of direct limit and inverse limit are in a certain sense dual to each other. Let us start off with direct limits, because they are somewhat easier to understand. They are very useful for dealing with modules which are not finitely generated. We will prove the result mentioned earlier that Tor commutes with direct limits: this will reduce the calculation of Tor down to modules which are finitely generated. First let us recall what a directed set is.

Definition Let $\mathcal{I}$ be a partially ordered set. This means that $\mathcal{I}$ is a set with a partial order $\leq$; by definition $\leq$ is a binary relation with the following properties.

(i) $i \leq i$ for all $i \in \mathcal{I}$.
(ii) If $i \leq j$ and $j \leq i$, then $i = j$.
(iii) If $i \leq j \leq k$, then $i \leq k$.

We say that $\mathcal{I}$ is a directed set if for any $i, j \in \mathcal{I}$, there exists $k \in \mathcal{I}$ such that $i, j \leq k$. Thus $\mathbb{Z}$ is a directed set where $\leq$ is as usual. Another example of a directed set is the set of all finite subsets of a set S, where $A \leq B$ means A is contained in B.

Let $\mathcal{I}$ be a directed set. Then a direct system of sets (indexed by $\mathcal{I}$, or over the set $\mathcal{I}$) means a family of sets $\{M_i \mid i \in \mathcal{I}\}$ such that for each $i \leq j$, there exists a map $f_i^j\colon M_i \to M_j$ satisfying

$$f_i^i = \mathrm{id}_{M_i} \quad \text{and} \quad f_j^k f_i^j = f_i^k$$

whenever $i \leq j \leq k$. Similarly if R is a ring, then a direct system of R-modules is a directed family of sets M_i such that the M_i are R-modules and the maps f_i^j above are R-module maps. Similarly we could define a direct system of groups to mean a directed family of sets M_i such that the M_i are groups and the f_i^j are group homomorphisms. We shall sometimes write (M_i) or (M_i, f_i^j) to stand for the direct system of R-modules together with their R-maps f_i^j.

Let R be a ring and $\{M_i \mid i \in \mathcal{I}\}$ be a direct system of R-modules with R-maps $f_i^j\colon M_i \to M_j$. Then the *direct limit* $\varinjlim M_i$ of the M_i is an R-module M and R-maps $f_i\colon M_i \to M$ for $i \in \mathcal{I}$ such that $f_j f_i^j = f_i$ whenever $i \leq j$, and with the following universal property: if N is an R-module and $g_i\colon M_i \to N$ are R-maps such that $g_j f_i^j = g_i$, then there exists a unique R-map $\theta\colon M \to N$ such that $\theta f_i = g_i$ for all $i \in \mathcal{I}$.

As usual it is easy to check that if $\varinjlim M_i$ exists, then it is unique up to isomorphism; specifically if M and N are two direct limits with corresponding maps $f_i\colon M_i \to M$ and

$g_i\colon M_i \to N$, then there is an isomorphism $\theta\colon M \to N$ such that $\theta f_i = g_i$ for all $i \in \mathcal{I}$. To prove the existence of a direct limit, set $D = \bigoplus_{i\in\mathcal{I}} M_i$, and let $\pi_i\colon M_i \to D$ denote the natural injection. Now define F to be the R-submodule of D generated by all elements of the form $\pi_i m_i - \pi_j f_i^j m_i$ where $m_i \in M_i$, let $M = D/F$, and let $\sigma\colon D \to M$ denote the natural surjection with kernel F. Then it is easily checked that M together with the maps $\sigma\pi_i\colon M_i \to M$ form a direct limit for the direct system of R-modules M_i; thus direct limits exist.

Remark In a similar fashion we can also construct the direct limit of a direct system of abelian groups. Suppose (M_i) is direct system of R-modules with direct limit M. We can also view (M_i) as a direct system of abelian groups: if N is the direct limit of this system of abelian groups, then $M \cong N$ as abelian groups. Another way of phrasing this is that the abelian group N is naturally an R-module.

Examples

(i) Let R be a ring, let M be an R-module, and let $\{M_i \mid i \in \mathcal{I}\}$ be a family of R-modules indexed $\mathcal{I}$ such that $M = \bigcup_{i\in\mathcal{I}} M_i$. The partial order on $\mathcal{I}$ is characterized by the property that $i \le j$ if and only if $M_i \subseteq M_j$, and the corresponding maps $f_i^j\colon M_i \to M_j$ for $i \le j$ are just the inclusions. Then $\varinjlim M_i \cong M$, and the corresponding maps $f_i\colon M_i \to M$ are just the inclusion maps. In this example, one often takes the set $\{M_i \mid i \in \mathcal{I}\}$ to be the set of finitely generated submodules of M.

(ii) Let R be a ring, let M be an R-module, and let $\{M_i \mid i \in \mathcal{I}\}$ be a family of R-submodules of M over the set $\mathcal{I}$. The partial order on $\mathcal{I}$ is characterized by the property that $i \le j$ if and only if $M_i \subseteq M_j$ (as in (i)) and here the corresponding maps $f_i^j\colon M/M_i \to M/M_j$ for $i \le j$ are the natural epimorphisms induced by the identity map from M to M. Let $N = \bigcup_{i\in\mathcal{I}} M_i$. Then $\varinjlim M/M_i \cong M/N$, and the corresponding maps $f_i\colon M/M_i \to M/N$ are just the natural epimorphisms induced by the identity map $M \to M$.

Exercise

Let R be a ring, let $\mathcal{I}$ be a directed set, let $\{M_i \mid i \in \mathcal{I}\}$ be a system of R-modules, and let $M = \varinjlim M_i$. Let $f_i^j\colon M_i \to M_j$ and $f_i\colon M_i \to M$ be the corresponding R-module maps.

(i) Let $m \in M$. Prove that there exists $k \in \mathcal{I}$ such that $m \in \operatorname{im} f_i$ for all $i \ge k$.

(ii) Let $m_i \in M_i$. Prove that $m_i \in \ker f_i$ if and only if $m_i \in \ker f_i^j$ for some $j \ge i$.

We now want to determine what Tor of a direct limit of modules is. To proceed, we will need to study maps between direct systems of R-modules. We need the following definition.

Definition Let R be a ring, let $\mathcal{I}$ be a directed set, and let $\{M_i \mid i \in \mathcal{I}\}$ and $\{N_i \mid i \in \mathcal{I}\}$ be direct systems of R-modules with corresponding maps f_i^j and g_i^j. Then a map $\theta\colon (M_i) \to (N_i)$ is a system of R-maps $\theta_i\colon M_i \to N_i$ which commute with the maps f_i^j and g_i^j, i.e. $g_i^j\theta_i = \theta_j f_i^j$. Then it is easy to check that θ induces an R-module map (which we will still denote by θ) $\theta\colon \varinjlim M_i \to \varinjlim N_i$.

Specifically, the map θ is defined as follows: let $m \in M$. Then by the above exercise $m = f_i m_i$ for some $i \in \mathcal{I}$ and for some $m_i \in M_i$, and then we define $\theta m = g_i\theta_i m_i$. The map θ will have

the property that $g_i\theta_i = \theta f_i$. The following lemma is fundamental to studying direct limits of modules.

Lemma Let R be a ring and let $\mathcal{I}$ be a directed set. Suppose

$$0 \longrightarrow (L_i, f_i^j) \xrightarrow{\theta} (M_i, g_i^j) \xrightarrow{\phi} (N_i, h_i^j) \longrightarrow 0$$

is an exact sequence of direct systems of R-modules. Then the induced sequence $0 \to L \xrightarrow{\theta} M \xrightarrow{\phi} N \to 0$ is also exact.

Of course the sequence of direct systems of R-modules is exact means that for each $i \in \mathcal{I}$, the sequence $0 \to L_i \xrightarrow{\theta_i} M_i \xrightarrow{\phi_i} N_i \to 0$ is also exact.

Proof This all follows from the above exercise. For example, let us prove that $\ker\phi \subseteq \operatorname{im}\theta$, so suppose $m \in \ker\phi$. By the exercise, we may write $m = g_i m_i$ where $i \in \mathcal{I}$ and $m_i \in M_i$. Since $\phi m = 0$, we see that $h_i\phi_i m_i = \phi g_i m_i = 0$ so by the above exercise $h_i^j \phi_i m_i = 0$ for some $j \geq i$. Therefore $\phi_j g_i^j m_i = h_i^j \phi_i m_i = 0$, so by exactness of the direct systems, there exists $l_j \in L_j$ such that $\theta_j l_j = g_i^j m_i$. Then $\theta(f_j l_j) = g_j\theta_j l_j = g_j g_i^j m_i = g_i m_i = m$, as required.

Next we show that tensor products commute with direct limits; this together with the above Lemma will enable us to prove that Tor commutes with direct limits. Let R be a ring, let A be an R-module, and let (B_i, f_i^j) be a direct system of R-modules with direct limit B. Then it is easily checked that $(A \otimes_R B_i, 1 \otimes f_i^j)$ is a direct system of abelian groups, so $\varinjlim(A \otimes_R B_i)$ exists: let $g_i \colon A \otimes_R B_i \to \varinjlim(A \otimes_R B_i)$ be the corresponding maps, so $g_j(1 \otimes f_i^j) = g_i$. We also have maps $1 \otimes f_i \colon A \otimes_R B_i \to A \otimes_R B$ such that $(1 \otimes f_j)(1 \otimes f_i^j) = 1 \otimes f_i$, so by the universal property of direct limits there is a unique group homomorphism $\theta \colon \varinjlim(A \otimes_R B_i) \to A \otimes_R B$ such that $\theta g_i = 1 \otimes f_i$.

Now we construct a map going the other way. For each $i \in \mathcal{I}$ and $a \in A$, we define a group homomorphism $\phi_i(a) \colon B_i \to \varinjlim A \otimes_R B_i$ by $\phi_i(a)b = g_i(a \otimes b)$ for $b \in B_i$. Then $\phi_i(a) = \phi_j(a) f_i^j$, so by the universal property of direct limits, the $\phi_i(a)$ induce a unique group homomorphism $\phi(a) \colon B \to \varinjlim A \otimes_R B_i$ satisfying $\phi(a) f_i b = g_i(a \otimes b)$ for all $b \in B$. We now define a map $\phi \colon A \times B \to \varinjlim A \otimes_R B_i$ by $\phi(a, b) = \phi(a)b$. It is routine to check that ϕ is an R-balanced map, so it induces a group homomorphism $\phi \colon A \otimes_R B \to \varinjlim B_i$, and then it is easily verified that $\phi\theta$ is the identity map on $\varinjlim A \otimes_R B_i$, and that $\theta\phi$ is the identity map on $A \otimes_R B$. Thus $\varinjlim A \otimes_R B_i \cong A \otimes_R B$ as required. It is also routine to see that this is a natural isomorphism.

Thus we have shown that Tor_0 commutes with direct limits, and we now need to show that Tor_n commutes with direct limits for all $n \in \mathbb{N}$ (not just $n = 0$). This is really a consequence of the following principle (if we knew some category theory, we could just quote a theorem now; instead I will phrase things in categorical language without defining my terms precisely). A functor is exact if it maps exact sequences into exact sequences: thus taking direct limits is an exact functor. Another example of an exact functor is localization: if R is a commutative ring and S is a multiplicatively closed subset of R, then the map $M \mapsto S^{-1}M$ sending R-modules to $S^{-1}R$-modules is exact, because if $A \to B \to C$ is an exact sequence of R-modules, then $S^{-1}A \to S^{-1}B \to S^{-1}C$ is an exact sequence of $S^{-1}R$-modules. An

example of a functor which is not exact is the map $M \mapsto M \otimes_R B$ for a fixed left R-module B sending right R-modules to abelian groups, though it will be exact if B is a flat left R-module. Then any exact functor which commutes with the functor $A \otimes_R \_$ will commute with $\mathrm{Tor}_n^R(A, \_)$ for all $n \in \mathbb{N}$. Similar remarks apply to the functors $\mathrm{Tor}_n^R(\_, A)$, $\mathrm{Ext}_R^n(\_, A)$, $\mathrm{Ext}_R^n(A, \_)$, or any other functor which has derived functors ($\mathrm{Ext}_R^n(A, \_)$ is a derived functor of the functor $\mathrm{Ext}_R^0(A, \_)$). Let us now prove that $\mathrm{Tor}_n^R(A, \_)$ commutes with direct limits; we state precisely the result we are trying to prove.

Theorem Let R be a ring, let A be a right R-module, let (M_i, f_i^j) be a direct system of R-modules indexed by the set $\mathcal{I}$, and let $n \in \mathbb{N}$. Then $(\mathrm{Tor}_n^R(A, M_i), (f_i^j)_{n*})$ is a direct system of abelian groups, and $\varinjlim \mathrm{Tor}_n^R(A, M_i) \cong \mathrm{Tor}_n^R(A, \varinjlim M_i)$.

Proof First we shall define the map giving the isomorphism. For each $i \in \mathcal{I}$ we have an R-map $f_i \colon M_i \to \varinjlim M_i$, and this induces a map $(f_i)_{n*} \colon \mathrm{Tor}_n^R(A, M_i) \to \mathrm{Tor}_n^R(A, \varinjlim M_i)$, which in turn induces a map $g \colon \varinjlim \mathrm{Tor}_n^R(A, M_i) \to \mathrm{Tor}_n^R(A, \varinjlim M_i)$. Now choose an exact sequence of right R-modules $0 \to K \to F \to A \to 0$ where F is a free R-module. Then for $n \geq 1$, the long exact sequence for Tor in the first variable together with the naturality of the maps involved yields a commutative diagram

$$\begin{array}{ccccccc}
\varinjlim \mathrm{Tor}_n^R(F, M_i) & \longrightarrow & \varinjlim \mathrm{Tor}_n^R(A, M_i) & \longrightarrow & \varinjlim \mathrm{Tor}_{n-1}^R(K, M_i) & \longrightarrow & \varinjlim \mathrm{Tor}_{n-1}^R(F, M_i) \\
g_n(F) \downarrow & & g_n(A) \downarrow & & \downarrow g_{n-1}(K) & & \downarrow g_{n-1}(F) \\
\mathrm{Tor}_n^R(F, \varinjlim M_i) & \longrightarrow & \mathrm{Tor}_n^R(A, \varinjlim M_i) & \longrightarrow & \mathrm{Tor}_{n-1}^R(K, \varinjlim M_i) & \longrightarrow & \mathrm{Tor}_{n-1}^R(F, \varinjlim M_i)
\end{array}$$

in which the rows are exact (we have used the exactness of $\varinjlim$ for the top row), and $g_{n-1}(K)$ is an isomorphism by induction. Since $\mathrm{Tor}_n^R(F, \_) = 0$ when F is a free R-module, we see that $g_n(F) = 0$ for $n \geq 1$, and it has already been proved that $g_0(F)$ is an isomorphism. It follows that $g_n(A)$ is an isomorphism.

Thus we have shown that Tor commutes with direct limits in the second variable, and an exactly similar argument shows that Tor commutes with direct limits in the first variable (or we could deduce this from the above by using the balancing of Tor). Specifically the result we have is

Theorem Let R be a ring, let A be a left R-module, let $n \in \mathbb{N}$, and let (M_i) be a direct system of right R-modules. Then $\mathrm{Tor}_n^R(\varinjlim M_i, A) \cong \varinjlim \mathrm{Tor}_n^R(M_i.A)$.

Application We have already seen in the third chapter that tensor products commute with direct sums: we now have another way of proving this (assuming the result for finite direct sums). If $M = \bigoplus_{i \in \mathcal{I}} M_i$, then $M \cong \varinjlim \bigoplus_{i \in \mathcal{J}} M_i$ where $\mathcal{J}$ runs through all the finite subsets of $\mathcal{I}$.

Inverse Limits As has already been mentioned, inverse limits are the dual notion of direct limits: the definition for inverse limit is the same as for direct limit, except we reverse the arrows everywhere. Also direct limits can be considered as generalizations of direct sums, whereas inverse limits can be considered as generalizations of cartesian products. In detail, we have the following definition.

Definition Let R be a ring and let $\mathcal{I}$ be a directed set. Then an inverse system of R-modules over $\mathcal{I}$ is a family of R-modules $\{M_i \mid i \in \mathcal{I}\}$ such that for each $i \leq j$, there exists an R-map $f_i^j \colon M_j \to M_i$ satisfying

$$f_i^i = \mathrm{id}_{M_i} \quad \text{and} \quad f_i^j f_j^k = f_i^k$$

whenever $i \leq j \leq k$. As with direct limits, we shall write (M_i) or (M_i, f_i^j) to stand for the inverse system together with the R-maps f_i^j.

If (M_i, f_i^j) is an inverse system of R-modules, then the inverse limit $\varprojlim M_i$ of the system is an R-module M and R-maps $f_i \colon M \to M_i$ for $i \in \mathcal{I}$ such that $f_i^j f_j = f_i$ whenever $i \leq j$, and with the following universal property: if N is an R-module and $g_i \colon N \to M_i$ are R-maps such that $f_i^j g_j = g_i$, then there exists a unique R-map $\theta \colon N \to M$ such that $f_i \theta = g_i$ for all $i \in \mathcal{I}$.

As in the case with direct limits, if $\varprojlim M_i$ exists, then it is unique up to isomorphism: specifically if M and N are two inverse limits with corresponding maps $f_i \colon M \to M_i$ and $g_i \colon N \to M_i$, then there is an isomorphism $\theta \colon N \to M$ such that $f_i \theta = g_i$ for all $i \in \mathcal{I}$. To prove the existence of an inverse limit, let $C = \prod_{i \in \mathcal{I}} M_i$, and identify each module M_i with its canonical image in C. Now define M to be the R-submodule consisting of elements $\{(m_i) \in C \mid f_i^j m_j = m_i\}$ whenever $i \leq j$, and let $f_i \colon M \to M_i$ denote the restriction to M of the natural projection of C onto M_i. Then it is easy to check that M together with the maps f_i form an inverse limit for the inverse system (M_i).

Examples

(i) Let R be a ring, let M be an R-module, and define an inverse system (A_i, f_i^j) over $\mathbb{P}$ by $A_i = M$ for all $i \in \mathbb{P}$ and $f_i^j \colon M_j \to M_i$ to be the identity whenever $i \leq j$. Then the inverse limit of this system is just the R-module M.

(ii) Let R be a ring and let $\{M_i \mid i \in \mathcal{I}\}$ be a family of R-modules over the directed set $\mathcal{I}$. For each finite subset $\mathcal{J}$ of $\mathcal{I}$, set $M_{\mathcal{J}} = \bigoplus_{i \in \mathcal{J}} M_i$. Then if $\mathcal{J} \subseteq \mathcal{K}$, we have a projection $\pi_{\mathcal{J}}^{\mathcal{K}} \colon M_{\mathcal{K}} \to M_{\mathcal{J}}$, and it is easy to verify that $(M_{\mathcal{J}}, \pi_{\mathcal{J}}^{\mathcal{K}})$ form an inverse system over the finite subsets of $\mathcal{I}$. The inverse limit will be $\prod_{i \in \mathcal{I}} M_i$.

(iii) Let k be a field, and for $i \leq j \in \mathbb{P}$, define $f_i^j \colon k[X]/(X^j) \to k[X]/(X^i)$ to be the natural epimorphism. Then $(k[X]/(X^i), f_i^j)$ form an inverse system, the inverse limit is the power series ring $k[[X]]$, and the corresponding maps $f_i \colon k[[X]] \to k[X]/(X^i)$ are just the natural epimorphisms (note that the inclusion $k[X] \hookrightarrow k[[X]]$ induces an isomorphism $k[X]/(X^i) \cong k[[X]]/(X^i)$ for all $i \in \mathbb{P}$).

(iv) Let $p \in \mathbb{P}$ be a prime number and for $i \leq j \in \mathbb{P}$, let $f_i^j \colon \mathbb{Z}/p^j\mathbb{Z} \to \mathbb{Z}/p^i\mathbb{Z}$ denote the natural epimorphism. Then $(\mathbb{Z}/p^i\mathbb{Z}, f_i^j)$ form an inverse system: the inverse limit is the p-adic integers $\mathbb{Z}_p$ which are an integral domain, are a local ring with unique maximal ideal $p\mathbb{Z}$, and have uncountable cardinality. They are very important in many branches of Mathematics, especially number theory.

Maps between inverse systems are defined in exactly the same way as for inverse limits. Here is the formal definition.

Definition Let R be a ring, let $\mathcal{I}$ be a directed set, and let (M_i, f_i^j) and (N_i, g_i^j) be inverse systems over the directed set $\mathcal{I}$. Then a map $\theta\colon (M_i) \to (N_i)$ is a family of R-maps $\theta_i\colon M_i \to N_i$ which commute with the maps f_i^j and g_i^j, i.e. $g_i^j\theta_j = \theta_i f_i^j$ whenever $i \le j$. Then it is easy to check that θ induces an R-module map (which we will still denote by θ) $\theta\colon \varprojlim M_i \to \varprojlim N_i$.

Specifically the map θ is defined as follows: for $i \in \mathcal{I}$, define $h_i\colon M \to N_i$ by $h_i = \theta_i f_i$. Then

$$g_i^j h_j = g_i^j \theta_j f_j = \theta_i f_i^j f_j = \theta_i f_i = h_i,$$

so by the universal property of inverse limits, there exists a unique map $\theta\colon \varprojlim M_i \to \varprojlim N_i$ such that $g_i\theta = h_i = \theta_i f_i$ for all $i \in \mathcal{I}$, as required.

We now show that Hom commutes with inverse systems in the second variable. Let R be a ring, let A be an R-module, and let $((M_i), f_i^j)$ be an inverse system of R-modules. Then it is easily checked that $(\mathrm{Hom}_R(A, M_i), f_{i*}^j)$ is an inverse system of abelian groups, so $\varprojlim \mathrm{Hom}_R(A, M_i)$ exists: let $g_i\colon \varprojlim \mathrm{Hom}_R(A, M_i) \to \mathrm{Hom}_R(A, M_i)$ be the corresponding maps. We also have maps $f_{i*}\colon \mathrm{Hom}_R(A, \varprojlim M_i) \to \mathrm{Hom}_R(A, M_i)$ for $i \in \mathcal{I}$ such that $f_{i*}^j f_{j*} = f_{i*}$, so by the universal property of inverse limits, there exists a unique group homomorphism $\theta\colon \mathrm{Hom}_R(A, \varprojlim M_i) \to \varprojlim \mathrm{Hom}_R(A, M_i)$ such that $f_{i*} = g_i\theta$.

We now construct a map ϕ inverse to θ. For each $a \in A$, define a group homomorphism $\phi_i(a)\colon \varprojlim \mathrm{Hom}_R(A, M_i) \to M_i$ by $\phi_i(a)x = (g_i x)a$ for $x \in \varprojlim \mathrm{Hom}_R(A, M_i)$ and $a \in A$. Then $\phi_i(a) = f_i^j \phi_j(a)$, so by the universal property of inverse limits, the $\phi_i(a)$ induce a group homomorphism $\phi(a)\colon \varprojlim \mathrm{Hom}_R(A, M_i) \to \varprojlim M_i$ such that $f_i\phi(a) = \phi_i(a)$. A routine check shows that for $x \in \varprojlim \mathrm{Hom}_R(A, M_i)$, $a \mapsto \phi(a)(x)$ defines an element of $\mathrm{Hom}_R(A, \varprojlim M_i)$, so we can define a group homomorphism $\phi\colon \varprojlim \mathrm{Hom}_R(A, M_i) \to \mathrm{Hom}_R(A, \varprojlim M_i)$ satisfying $f_i(\phi(x)(a)) = (g_i x)a$. Then it is easily verified that $\theta\phi$ is the identity map on $\varprojlim \mathrm{Hom}_R(A, M_i)$ and that $\phi\theta$ is the identity map on $\mathrm{Hom}_R(A, \varprojlim M_i)$ as required.

Now it would seem plausible to proceed as in the case of direct limits, namely to prove that $\mathrm{Ext}_R(A, \_)$ commutes with direct limits. Since inverse limits are just direct limits with the arrows reversed, it follows that an exact sequence of inverse systems would induce an exact sequence of inverse limits. But consider the following example.

We will construct inverse systems indexed by $\mathbb{P}$. Let $p, q \in \mathbb{P}$ be such that $p, q \ge 2$ and $q \equiv 1 \mod p$. Define $A_i = B_i = \mathbb{Z}$ for all $i \in \mathbb{Z}$, and $C_i = \mathbb{Z}/p\mathbb{Z}$. We make (A_i, α_i^j) into an inverse system by defining $\alpha_i^j : A_j \to A_i$ by $\alpha_i^j a = q^{j-i}a$ for $i \le j$; we make (B_i, β_i^j) into an inverse system by defining $\beta_i^j\colon B_i \to B_j$ by $\beta_i^j b = q^{j-i}b$ for $i \le j$; and we make (C_i, γ_i^j) into an inverse system by defining $\gamma_i^j c = c$ for $i \le j$. We now have an exact sequence of inverse systems $0 \to A \xrightarrow{\theta} B \xrightarrow{\phi} C \to 0$ by defining $\theta_i\colon A_i \to B_i$ by $\theta_i a = pa$, and $\phi_i : B_i \to C_i$ to be the natural epimorphism. However $\varprojlim A_i = \varprojlim B_i = 0$, and $\varprojlim C_i \cong \mathbb{Z}/p\mathbb{Z}$, so the righthand side of the sequence for the inverse limit is not exact.

Fifth Homework Due 9:00 a.m., Monday, September 26.

(1) The purpose of this problem is to show that the Ext groups can be computed by using injective resolutions in the second variable. Let R be a ring, let M, A be R-modules, and

let $(I, \alpha_0)\colon 0 \to A \overset{\alpha_0}{\to} I_0 \overset{\alpha_1}{\to} I_1 \overset{\alpha_2}{\to} I_2 \overset{\alpha_3}{\to} \cdots$ be an injective resolution of the R-module A (i.e. the sequence is exact and all the I_i are injective R-modules).

(i) Use the fact that $\mathrm{Hom}_R(M, \_)$ is left exact to prove $\mathrm{Ext}^0_R(M, A) \cong \mathrm{H}^0(\mathrm{Hom}_R(M, I))$, where $\mathrm{Hom}_R(M, I)$ denotes the complex

$$0 \longrightarrow \mathrm{Hom}_R(M, I_0) \xrightarrow{\alpha_{1*}} \mathrm{Hom}_R(M, I_1) \xrightarrow{\alpha_{2*}} \mathrm{Hom}_R(M, I_2) \xrightarrow{\alpha_{3*}} \cdots .$$

(ii) Apply the long exact sequence for Ext in the second variable to $0 \to \ker \alpha_1 \to I_0 \to \ker \alpha_2 \to 0$ to deduce that $\mathrm{Ext}^1_R(M, A) \cong \mathrm{H}^1(\mathrm{Hom}_R(M, I))$.

(iii) Use the long exact sequence for Ext to show that $\mathrm{Ext}^n_R(M, A) \cong \mathrm{Ext}^1_R(M, \ker \alpha_n)$. Deduce that $\mathrm{Ext}^n_R(M, A) \cong \mathrm{H}^n(\mathrm{Hom}_R(M, I))$ as required.

(2) Let k be a field, let $k[X]$ and $k[X, Y]$ denote the polynomial rings in one and two variables respectively, and let k denote the $k[X, Y]$-module with X, Y acting trivially (so $aX = aY = 0$ for all $a \in k$).

(i) Prove that $\mathrm{Ext}^n_{k[X]}(k, k) \cong k$ if $n = 0$ or 1, and is 0 if $n \geq 2$ (where k denotes the $k[X]$-module with X acting trivially).

(ii) By using a change of rings theorem, prove that $\mathrm{Ext}^n_{k[X,Y]}(k[X], k) \cong \mathrm{Ext}^n_{k[X]}(k, k)$ for all $n \in \mathbb{N}$ (where $k[X]$ is the $k[X, Y]$-module with Y acting trivially).

(iii) Let $\theta\colon k[X] \to k[X]$ denote the $k[X, Y]$-map defined by $\theta 1 = X$. Prove that the induced map $\theta^*\colon \mathrm{Ext}^n_{k[X,Y]}(k[X], k) \to \mathrm{Ext}^n_{k[X,Y]}(k[X], k)$ is zero.

(iv) By considering the long exact sequence for Ext in the first variable for the exact sequence of $k[X, Y]$-modules $0 \to k[X] \xrightarrow{\theta} k[X] \to k \to 0$, prove that $\mathrm{Ext}^n_{k[X,Y]}(k, k) \cong k$ if $n = 0$ or 2, is isomorphic to $k \oplus k$ if $n = 1$, and is zero if $n > 2$.

(3) Let R be a Noetherian integral domain with field of fractions K, let M be an $R[X]$-module, and let A be a finitely generated $R[X]$-module. Suppose M is isomorphic to a direct (possibly infinite) sum of copies of K as an R-module. Prove that $\mathrm{Ext}^n_{R[X]}(A, M) = 0$ for all $n \geq 2$.

(4) Let R be a ring and let A be a right R-module. Recall that the left R-module M is finitely presented means that there exists an exact sequence $F_1 \to F_0 \to M \to 0$ with F_1 and F_0 finitely generated free left R-modules. Thus a finitely presented left R-module is finitely generated, but not conversely. (However if R is left Noetherian, then finitely generated left R-modules are finitely presented.) If $\mathrm{Tor}^R_1(A, B)$ is a torsion group for all finitely presented left R-modules B, prove that $\mathrm{Tor}^R_n(A, B)$ is a torsion group for all left R-modules B and for all $n \in \mathbb{P}$.

(5) Let R be a ring, let A be a finitely presented right R-module (see problem 4), and let (M_i) be a direct system of right R-modules. Prove that $\mathrm{Hom}_R(A, \varinjlim M_i) \cong \varinjlim \mathrm{Hom}_R(A, M_i)$ (you will have to define your isomorphism). Deduce that if R is a right Noetherian ring and B is a finitely generated right R-module, then $\mathrm{Ext}^n_R(B, \varinjlim M_i) \cong \varinjlim \mathrm{Ext}^n_R(B, M_i)$ for all $n \in \mathbb{N}$. (For the Hom part of the problem, first do the case when $A \cong R$: here you will want to use the natural isomorphism from $\mathrm{Hom}_R(R, M)$ to M defined by $f \to f(1)$. Then

prove the result when A is a finitely generated free R-module, and finally for arbitrary finitely presented A by choosing an exact sequence $F_1 \to F_0 \to A \to 0$ where F_1, F_0 are finitely generated free R-modules, and using the left exactness of $\mathrm{Hom}_R(\_, M)$.)

(6) Let R be an integral domain with field of fractions K, let A be an R-module, and let M be a K-module. Then we may also view M as an R-module. Using (5) of the chapter 6, prove that $\mathrm{Ext}^n_R(A, M) = 0$ for all $n \in \mathbb{P}$ (taking the case $n = 1$, this of course means that M is an injective R-module). Show further that if R is a PID and N is an R-submodule of M, then M/N is an injective R-module (you will need to use the fact that in a PID, every submodule of a free module is free).

Monday, September 26

Chapter 7
Limits (continued)

For the rest of this section on inverse limits, we will assume that our inverse system is indexed by the directed set $\mathbb{P}$ with $\leq$ denoting "less than or equal to" as usual. It is worth noting that in this case, the f_i^j are determined by f_i^{i+1}, because $f_i^j = f_i^{i+1} f_{i+1}^{i+2} \cdots f_{j-1}^j$. The technical condition we need to ensure that $\varprojlim$ is exact is the *Mittag-Leffler condition*. This says that if (M_i, f_i^j) is an inverse system indexed by the directed set $(\mathbb{P}, \leq)$, then for each $i \in \mathbb{P}$ there exists $j \geq i$ such that $\mathrm{im}\, f_i^j = \mathrm{im}\, f_i^k$ for all $k \geq j$. An important situation when this is trivially satisfied is when all the f_i^j are onto: this applies to examples (i), (iii) and (iv) near the end of the sixth (i.e. the previous) chapter. It can also be applied to the case of a Cartesian product of a countable number of R-modules. Indeed suppose $M_1, M_2, \ldots$ are R-modules. Then for $i \in \mathbb{P}$, we set $N_i = M_1 \oplus M_2 \oplus \cdots \oplus M_i$, and for $i \leq j$, we define $f_i^j : N_j \to N_i$ to be the natural projection with kernel $N_{i+1} \oplus \cdots \oplus N_j$. Then f_i^j is surjective for all $i \leq j$, and $\varprojlim M_i \cong \prod_{i=1}^\infty M_i$. We can now state

Theorem 1 Let $0 \to (A_i) \xrightarrow{\theta} (B_i) \xrightarrow{\phi} (C_i) \to 0$ be an exact sequence of inverse systems. If (A_i) satisfies the Mittag-Leffler condition, then the induced sequence of inverse limits $0 \to \varprojlim A_i \to \varprojlim B_i \to \varprojlim C_i \to 0$ is also exact.

Actually we will prove something stronger, and in the process define the derived functor of $\varprojlim$, namely $\varprojlim^1$.

One way to construct $\varprojlim A_i$ when the index set is $\mathbb{P}$ is to define $\alpha \colon \prod_{i=1}^\infty A_i \to \prod_{i=1}^\infty A_i$ by $\alpha(a_i) = (a_i - f_i^{i+1} a_{i+1})$. Then it is easy to verify that $\ker \alpha \cong \varprojlim A_i$ (use the construction of inverse limit at the end of the sixth chapter). We now define $\varprojlim^1 A_i = \mathrm{coker}\, \alpha$ and then we have an exact sequence of chain complexes

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \prod_{i=1}^{\infty} A_i & \xrightarrow{\theta} & \prod_{i=1}^{\infty} B_i & \xrightarrow{\phi} & \prod_{i=1}^{\infty} C_i & \longrightarrow & 0 \\
 & & \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & & \\
0 & \longrightarrow & \prod_{i=1}^{\infty} A_i & \xrightarrow{\theta} & \prod_{i=1}^{\infty} B_i & \xrightarrow{\phi} & \prod_{i=1}^{\infty} C_i & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The long exact sequence for chain complexes (Lemma 1 of chapter 5) now yields a long exact sequence

$$0 \longrightarrow \varprojlim A_i \longrightarrow \varprojlim B_i \longrightarrow \varprojlim C_i \longrightarrow \varprojlim{}^1 A_i \longrightarrow \varprojlim{}^1 B_i \longrightarrow \varprojlim{}^1 C_i \longrightarrow 0,$$

so $\varprojlim$ will be exact if (but not only if) $\varprojlim{}^1 A_i = 0$. Thus we want to prove

Theorem Let R be a ring, and let (A_i, f_i^j) be an inverse system of R-modules. If (A_i) satisfies the Mittag-Leffler condition, then $\varprojlim{}^1 A_i = 0$.

Proof First let us consider the special case when given $i \in \mathbb{P}$, there exists $j \geq i$ such that $f_i^j = 0$ (this will, of course, imply that $f_i^k = 0$ for all $k \geq j$). As above, define $\alpha\colon \prod_{i=1}^{\infty} A_i \to \prod_{i=1}^{\infty} A_i$ by $\alpha(a_i) = (a_i - f_i^{i+1} a_{i+1})$, and let $(a_i) \in \prod_{i=1}^{\infty} A_i$. Define $b_i = a_i + f_i^{i+1} a_{i+1} + \cdots + f_i^{j-1} a_{j-1}$. Then the ith component of $\alpha(b_i)$ is

$$a_i + f_i^{i+1} a_{i+1} + \cdots + f_i^{j-1} a_{j-1} - f_i^{i+1}(a_{i+1} + f_{i+1}^{i+2} a_{i+2} + \cdots + f_{i+1}^{j} a_j + f_{i+1}^{j+1} a_{j+1} + \cdots) = a_i,$$

which shows that α is onto.

Next we consider the case when all the f_i^j are onto. Given elements $a_i \in A_i$ $(i \in \mathbb{P})$, set $b_1 = 0$ and then choose $b_{i+1} \in A_{i+1}$ inductively by the condition $f_i^{i+1} b_{i+1} = b_i - a_i$. This shows that the map α of above is onto in this case as well, so we have now shown in these two special cases that $\varprojlim{}^1 A_i = 0$.

Now consider the general case, and let (B_i, g_i^j) be the inverse system with $B_i = \bigcap_{j=i}^{\infty} f_i^j A_j$ and g_i^j the restriction of f_i^j to B_j. Since (A_i) satisfies the Mittag-Leffler condition, we see that $B_i = f_i^j A_j$ for some $j \geq i$ (where j depends on i). Let $(A_i/B_i, h_i^j)$ be the inverse system with h_i^j the map induced by f_i^j. It is easy to see that (B_i) and (A_i/B_i) form inverse systems, and then we have an exact sequence of inverse systems $0 \to (B_i) \to (A_i) \to (A_i/B_i) \to 0$. By the first part, $\varprojlim{}^1(A_i/B_i) = 0$; furthermore all the maps g_i^j are onto and it follows also from the first part that $\varprojlim{}^1 B_i = 0$. From the exact sequence of inverse systems

$$0 \longrightarrow (B_i) \longrightarrow (A_i) \longrightarrow (A_i/B_i) \longrightarrow 0,$$

we obtain exact sequence $\cdots \to \varprojlim{}^1 B_i \to \varprojlim{}^1 A_i \to \varprojlim{}^1 (A_i/B_i) \to 0$, and the result follows.

This completes the proof of Theorem 1.

Exercise Let $0 \to (A_i) \to (B_i) \to (C_i) \xrightarrow{\gamma} (D_i) \to 0$ be an exact sequence of inverse systems indexed by $\mathbb{P}$. If $\ker \gamma_i : C_i \to D_i$ is finite for all $i \in \mathbb{P}$, prove that the induced map $\gamma\colon \varprojlim C_i \to \varprojlim D_i$ is onto.

In general the results involving $\varprojlim$ are not as easy as the results involving $\varinjlim$. We state without proof the following.

Theorem Let R be a ring, let $n \in \mathbb{N}$, let A, B be R-modules, and suppose $A = \bigcup_{i=1}^{\infty} A_i$ where the A_i are R-submodules of A and $A_i \subseteq A_{i+1}$ for all $i \in \mathbb{P}$. Then there is an exact sequence of abelian groups

$$0 \longrightarrow \varprojlim{}^1 \operatorname{Ext}_R^{n-1}(A_i, B) \longrightarrow \operatorname{Ext}_R^n(A, B) \longrightarrow \varprojlim \operatorname{Ext}_R^n(A_i, B) \longrightarrow 0$$

(for $n = 0$, we interpret $\operatorname{Ext}_R^{n-1}(A_i, B)$ to be 0). Though direct limits can be just considered as a generalization of union, this theorem is not true if the union is replaced by an arbitrary direct limit.

Tensor Product of Chain Complexes We will need this for the Künneth Formula. Given two chain complexes, we want to take their tensor product (in a way to be defined) so that the result is also a chain complex. When dealing with group cohomology, it will enable us to construct a $k[G \times H]$-resolution from kG and kH-resolutions. It is not immediately obvious how we should construct the tensor product of two chain complexes, so here is the definition.

Definition Let R be a ring, and let

$$A : \cdots \xrightarrow{\alpha_2} A_2 \xrightarrow{\alpha_1} A_1 \xrightarrow{\alpha_0} A_0 \longrightarrow 0$$
$$B : \cdots \xrightarrow{\beta_2} B_2 \xrightarrow{\beta_1} B_1 \xrightarrow{\beta_0} B_0 \longrightarrow 0$$

be chain complexes of R-modules. Then $A \otimes_R B$ is the chain complex of abelian groups with

$$(A \otimes_R B)_n = \bigoplus_{r+s=n} A_r \otimes B_s,$$

and boundary map ∂_n defined by

$$\partial_n(a \otimes b) = \alpha_r a \otimes b + (-1)^r a \otimes \beta_s b \quad \text{for} \quad a \in A_r, b \in B_s.$$

Similarly if

$$A:\ 0 \xrightarrow{\alpha_0} A_0 \xrightarrow{\alpha_1} A_1 \xrightarrow{\alpha_2} A_2 \longrightarrow \cdots$$
$$\text{and} \quad B:\ 0 \xrightarrow{\beta_0} B_0 \xrightarrow{\beta_1} B_1 \xrightarrow{\beta_2} B_2 \longrightarrow \cdots$$

are cochain complexes, then $A \otimes_R B$ is a cochain complex with

$$\delta_n(a \otimes b) = \alpha_{r+1}a \otimes b + (-1)^r a \otimes \beta_{s+1}b \quad \text{for} \quad a \in A_r, b \in B_s.$$

Sixth Homework Due 9:00 a.m., Monday, October 3.

(1) Let R be a ring, let M be an R-module, let $\mathcal{I}$ be a directed set, and let $\{M_i \mid i \in \mathcal{I}\}$ be a family of R-submodules of M. Assume that the partial order on $\mathcal{I}$ satisfies $i \leq j$ if and only if $M_i \supseteq M_j$. This means that we have an inverse system (M_i, f_i^j), where $f_i^j : M_j \to M_i$ denotes the natural inclusion.
   (i) Prove that $\varprojlim M_i \cong \bigcap_{i \in \mathcal{I}} M_i$.
   (ii) Assume that $R = \mathbb{Z}$. Give an example where all the M_i are uncountable, yet $\varprojlim M_i \cong \mathbb{Z}$.

(2) Let R be a ring, let M be an R-module, and let (A_i) be a direct system of R-modules (indexed by an arbitrary directed set). Prove that $\mathrm{Hom}_R(\varinjlim A_i, M) \cong \varprojlim \mathrm{Hom}_R(A_i, M)$.

(3) Let R be a ring, and let (A_i) be an inverse system of R-modules indexed by the set $\mathbb{P}$.
   (i) If $R = \mathbb{Z}$ and all the A_i are finite abelian groups, prove that $\varprojlim^1 A_i = 0$.
   (ii) If R is a field and all the A_i are finite dimensional vector spaces over R, prove that $\varprojlim^1 A_i = 0$.

(4) Let p be a prime, and let $\mathbb{Z}_p$ denote the p-adic integers.
   (i) Let $(p^i\mathbb{Z}, f_i^j)$ denote the inverse system of abelian groups indexed by $\mathbb{P}$, where for $i \leq j$, $f_i^j : p^j\mathbb{Z} \to p^i\mathbb{Z}$ denotes the inclusion map. Prove that $\varprojlim^1 p^i\mathbb{Z} \cong \mathbb{Z}_p/\mathbb{Z}$.
   (ii) Let $(p^i\mathbb{Z}_p, f_i^j)$ denote the inverse system of abelian groups (or $\mathbb{Z}_p$-modules) indexed by $\mathbb{P}$, where for $i \leq j$, $f_i^j : p^j\mathbb{Z}_p \to p^i\mathbb{Z}_p$ denotes the inclusion map. Prove that $\varprojlim^1 p^i\mathbb{Z}_p = 0$.

(5) Let p be a prime.
   (i) Prove that $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}[1/p], \mathbb{Z}) \cong \mathbb{Z}_p/\mathbb{Z}$.
   (ii) Prove that $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}) \cong \prod_q \mathbb{Z}_q$ (where the Cartesian product is over all primes q).
   (iii) Prove that $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) \cong (\prod_q \mathbb{Z}_q)/\mathbb{Z}$.

(6) Let R be an integral domain with field of fractions K.

(i) Prove that K is a direct limit of free R-modules.
(ii) Give an example of a direct system of R-modules (M_i) such that M_i is torsion free but not free for all i, yet $\varinjlim M_i$ is free.

(The two parts are not related.)

(7) Let (R_i) be a direct system of rings, let $R = \varinjlim R_i$, let A be a right R-module, let B be a left R-module, and let $n \in \mathbb{N}$.
(i) Prove that $\varinjlim A \otimes_{R_i} B \cong A \otimes_R B$.
(ii) Suppose R_j is a flat left R_i-module whenever $i \leq j$. Prove that $\varinjlim \operatorname{Tor}_n^{R_i}(A, B) \cong \operatorname{Tor}_n^R(A, B)$ for all $n \in \mathbb{N}$.

Monday, October 3

Chapter 8
The Künneth Formula

We start by stating the Künneth Formula.

Theorem (Künneth Formula) Let k be a commutative hereditary ring, let A be a chain complex of projective k-modules, let B be a chain complex of k-modules, and let $n \in \mathbb{N}$. Define

$$\pi\colon \bigoplus_{r+s=n} \mathrm{H}_r(A) \otimes_k \mathrm{H}_s(B) \longrightarrow \mathrm{H}_n(A \otimes_k B)$$

as follows. If $u \in \mathrm{H}_r(A)$ and $v \in \mathrm{H}_s(B)$ are represented by $a \in A_r$ and $b \in B_s$ respectively, then $\pi(u \otimes v)$ is represented by $a \otimes b \in (A \otimes_k B)_n$. Then there is a natural short exact sequence of k-modules

$$0 \longrightarrow \bigoplus_{r+s=n} \mathrm{H}_r(A) \otimes_k \mathrm{H}_s(B) \xrightarrow{\pi} \mathrm{H}_n(A \otimes_k B) \longrightarrow \bigoplus_{r+s=n-1} \operatorname{Tor}_1^k(\mathrm{H}_r(A), \mathrm{H}_s(B)) \longrightarrow 0$$

which splits, but not naturally.

Recall that an arbitrary ring R is right hereditary means that every right ideal of R is projective (as a right R-module). It is a well known fact (not difficult to prove) that this is equivalent to the property that every R-submodule of a projective right R-module is projective. Of course a left hereditary ring is one in which every left ideal is projective, and there exist rings which are right hereditary but not left hereditary. For commutative rings though, it is clear that the properties of being left and right hereditary are equivalent. A hereditary integral domain is called a Dedekind domain (so the Künneth formula applies if k is a Dedekind domain). Important examples of Dedekind domains to keep in mind when applying the Künneth formula are PID's and the ring of integers in an algebraic number field.

Remarks and Consequences of the Künneth formula

(i) If k is a field, then $\mathrm{Tor}_1^k(\mathrm{H}_r(A), \mathrm{H}_s(B)) = 0$ and so the Künneth formula becomes

$$\mathrm{H}_n(A \otimes_k B) \cong \bigoplus_{r+s=n} \mathrm{H}_r(A) \otimes_k \mathrm{H}_s(B)$$

for $n \in \mathbb{N}$.

(ii) Suppose (A, α_0) and (B, β_0) are projective resolutions for the projective k-modules U and V respectively. Then $\mathrm{H}_n(A) = \mathrm{H}_n(B) = 0$ for all $n \in \mathbb{P}$, $\mathrm{H}_0(A) \cong U$ and $\mathrm{H}_0(B) \cong V$. In particular $\mathrm{H}_r(A)$ and $\mathrm{H}_s(B)$ are projective k-modules for all $r, s \in \mathbb{N}$, so $\mathrm{Tor}_1^k(\mathrm{H}_r(A), \mathrm{H}_s(B)) = 0$ for all $r, s \in \mathbb{N}$. The Künneth formula now shows that $(A \otimes_k B, \alpha_0 \otimes \beta_0)$ is a projective resolution for $U \otimes_k V$. For group rings, this will show that if (P, α_0) is a projective resolution for k as kG-modules and (Q, β_0) is projective resolution for k as kH-modules, then $(P \otimes_k Q, \alpha_0 \otimes \beta_0)$ is a projective resolution for k as $k[G \times H]$-modules.

(iii) Consider the special case $B_s = 0$ for all $s \in \mathbb{P}$. Write $M = B_0$ and let $n \in \mathbb{N}$. Since $\mathrm{H}_s(B) = 0$ for all $s \in \mathbb{P}$ and $\mathrm{H}_0(B) \cong M$, the Künneth formula now yields a natural exact sequence which splits (but not naturally)

$$0 \longrightarrow \mathrm{H}_n(A) \otimes_k M \longrightarrow \mathrm{H}_n(A \otimes_k M) \longrightarrow \mathrm{Tor}_1^k(\mathrm{H}_{n-1}(A), M) \longrightarrow 0.$$

(So for the above exact sequence M can be arbitrary, but A needs to be a complex consisting of projective k-modules). This sequence is usually referred to as the "Universal Coefficient Theorem".

(iv) There is a Künneth formula for cochain complexes. Let k be a commutative hereditary ring, let A be a cochain complex of projective k-modules, let B be a cochain complex of k-modules and let $n \in \mathbb{N}$. Then there is a natural short exact sequence of k-modules which splits (but not naturally)

$$0 \longrightarrow \bigoplus_{r+s=n} \mathrm{H}^r(A) \otimes_k \mathrm{H}^s(B) \longrightarrow \mathrm{H}^n(A \otimes_k B) \longrightarrow \bigoplus_{r+s=n+1} \mathrm{Tor}_1^k(\mathrm{H}^r(A), \mathrm{H}^s(B)) \longrightarrow 0.$$

(v) If k is not necessarily commutative but is left and right hereditary, then the exact sequence and the splitting of the exact sequence in the Künneth formula is still valid, but it is no longer a sequence of k-modules: it will be an exact sequence of abelian groups and it will split as abelian groups.

Proof of the Künneth formula Let α_r and β_s denote the boundary maps of A and B respectively. We begin by considering a special case. Suppose A is a chain complex with trivial boundary (so $\alpha_r = 0$ and $A_r \cong \mathrm{H}_r(A)$ for all $r \in \mathbb{N}$). Then $A \otimes_k B$ is the chain complex with

$$(A \otimes_k B)_n = \bigoplus_{r+s=n} A_r \otimes_k B_s$$

and boundary $\bigoplus_{r+s=n}(-1)^r \iota_r \otimes \beta_s$, where ι_r is the identity map on A_r. Thus $\mathrm{H}_n(A \otimes_k B) \cong \bigoplus_{r+s=n} \mathrm{H}_s(A_r \otimes_k B)$ and since $\mathrm{H}_s(A_r \otimes_k B) \cong A_r \otimes_k \mathrm{H}_s(B)$ by the Lemma 5 of chapter 5 (A_r is projective, so certainly flat), we deduce that

$$\pi\colon \bigoplus_{r+s=n} \mathrm{H}_r(A) \otimes_k \mathrm{H}_s(B) \longrightarrow \mathrm{H}_n(A \otimes_k B)$$

is an isomorphism. In general (i.e. when the boundary is not trivial), write $C_n = \ker \alpha_n\colon A_n \to A_{n-1}$ and $D_n = \operatorname{im} \alpha_n\colon A_n \to A_{n-1}$. Note that C_n and D_n are projective k-modules (because k is hereditary). Regard C and D as chain complexes with trivial boundary. Then $0 \to C \to A \to D \to 0$ is an exact sequence of chain complexes and hence so is $0 \to C \otimes_k B \to A \otimes_k B \to D \otimes_k B \to 0$ because D is projective. Now apply Lemma 1 of chapter 5 to obtain an exact sequence

$$\cdots \longrightarrow \mathrm{H}_{n+1}(D \otimes_k B) \xrightarrow{\partial_{n+1}} \mathrm{H}_n(C \otimes_k B) \longrightarrow \mathrm{H}_n(A \otimes_k B) \xrightarrow{\phi_n} \mathrm{H}_n(D \otimes_k B) \longrightarrow \cdots.$$

We also have an exact sequence $0 \to D_{r+1} \to C_r \to \mathrm{H}_r(A) \to 0$ for all $r \in \mathbb{N}$, and so applying the long exact sequence for Tor in the first argument yields an exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^k(\mathrm{H}_r(A), \mathrm{H}_s(B)) \longrightarrow D_{r+1} \otimes_k \mathrm{H}_s(B) \longrightarrow C_r \otimes_k \mathrm{H}_s(B) \longrightarrow \mathrm{H}_r(A) \otimes_k \mathrm{H}_s(B) \longrightarrow 0.$$

Therefore we have a commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & \displaystyle\bigoplus_{r+s=n} \mathrm{Tor}_1^k(\mathrm{H}_r(A), \mathrm{H}_s(B)) & \longrightarrow & \displaystyle\bigoplus_{r+s=n} D_{r+1} \otimes_k \mathrm{H}_s(B) & \longrightarrow & \displaystyle\bigoplus_{r+s=n} C_r \otimes_k \mathrm{H}_s(B) \\
 & & & & \delta \downarrow & & \gamma \downarrow \\
 & & \mathrm{H}_{n+1}(A \otimes_k B) & \xrightarrow{\phi_{n+1}} & \mathrm{H}_{n+1}(D \otimes_k B) & \xrightarrow{\partial_{n+1}} & \mathrm{H}_n(C \otimes_k B)
\end{array}$$

$$\begin{array}{ccccc}
\longrightarrow & \displaystyle\bigoplus_{r+s=n} \mathrm{H}_r(A) \otimes_k \mathrm{H}_s(B) & \longrightarrow & 0 & \\
 & \pi \downarrow & & & \\
\longrightarrow & \mathrm{H}_n(A \otimes_k B) & \xrightarrow{\phi_n} & \cdots, &
\end{array}$$

where δ and γ are isomorphisms by the special case when A has trivial boundary. A routine piece of diagram chasing shows that $\ker \pi = 0$, $\operatorname{im} \pi = \ker \phi_n$ and

$$\ker \partial_{n+1} \cong \bigoplus_{r+s=n} \mathrm{Tor}_1^k(\mathrm{H}_r(A), \mathrm{H}_s(B)).$$

But we have an exact sequence $0 \to \ker \phi_n \to \mathrm{H}_n(A \otimes_k B) \to \ker \partial_n \to 0$, and the required natural exact sequence follows easily.

It remains to show that the sequence splits. First consider the case when B (as well as A) is projective. Write $E_n = \ker \beta_n\colon B_n \to B_{n-1}$. Since k is hereditary, $\operatorname{im} \alpha_n$ and $\operatorname{im} \beta_n$ are projective, so we may write $A_n = C_n \oplus C'_n$ and $B_n = E_n \oplus E'_n$ for some k-submodules

C'_n and E'_n, but *not* naturally. It follows that the natural epimorphisms $C_n \to \mathrm{H}_n(A)$ and $E_n \to \mathrm{H}_n(B)$ can be extended to epimorphisms $\gamma_n \colon A_n \to \mathrm{H}_n(A)$ and $\delta_n \colon B_n \to \mathrm{H}_n(B)$ respectively, and hence to an epimorphism

$$(\gamma \otimes \delta)_n \colon (A \otimes_k B)_n \longrightarrow \bigoplus_{r+s=n} \mathrm{H}_r(A) \otimes_k \mathrm{H}_s(B).$$

If $a \in A_r$ and $b \in B_s$, then $(\gamma \otimes \delta)(\alpha_r a \otimes b + (-1)^r a \otimes \beta_s b) = 0$, because $\gamma_{r-1}\alpha_r a = 0 = \delta_{s-1}\beta_s b$. Therefore $(\gamma_{\otimes}\delta)_n$ induces a homomorphism

$$(\gamma \otimes \delta)_* \colon \mathrm{H}_n(A \otimes_k B) \longrightarrow \bigoplus_{r+s=n} \mathrm{H}_r(A) \otimes_k \mathrm{H}_s(B).$$

Now elements of $\bigoplus_{r+s=n} \mathrm{H}_r(A) \otimes_k \mathrm{H}_s(B)$ can be represented by sums of elements of the form $c \otimes e$ where $c \in C_n$ and $e \in E_n$, and then $(\gamma \otimes \delta)_* \pi$ is just induced by the map sending $c \otimes e$ to $c \otimes e$. This means that $(\gamma \otimes \delta)_* \pi$ is the identity on $\bigoplus_{r+s=n} \mathrm{H}_r(A) \otimes_k \mathrm{H}_s(B)$, i.e. the sequence splits.

Seventh Homework Due 9:00 a.m., Monday, October 10.

(1) Let p be a prime, and let $\mathbb{Z}_p$ denote the p-adic integers.
   (i) By considering $\mathbb{Z}_p$ as a subgroup of $\prod_{i=1}^{\infty} \mathbb{Z}/p^i\mathbb{Z}$, prove that every element of $\mathbb{Z}_p$ has a unique representation of the form $a_0 + pa_1 + \cdots + p^n a_n + \cdots$ where $a_i \in \mathbb{N}$ and $0 \leq a_i \leq p-1$ for all $i \in \mathbb{P}$. (We may consider $\mathbb{Z}/p^i\mathbb{Z}$ as $\{0, 1, \ldots, p^{i-1}\}$, and then for $b \in \mathbb{Z}/p^i\mathbb{Z}$, we may write $b = b_0 + b_1 p + \cdots + b_{i-1}p^{i-1}$ where $0 \leq b_j \leq p-1$.)
   (ii) Prove that $\mathbb{Z}_p$ is uncountable (you may use standard facts concerning uncountable numbers, like the fact that the set of all sequences of 0's and 1's is uncountable).
   (iii) If A is a countable free abelian group, prove that A is isomorphic to a subgroup of $\mathbb{Z}_p$.
   (iv) Let A be a countable free abelian group. Prove that there exists a descending chain of subgroups $B_1 \supset B_2 \supset B_3 \supset \cdots$ of A such that $A/B_i \cong \mathbb{Z}/p^i\mathbb{Z}$ for all $i \in \mathbb{P}$, and $\bigcap_{i=1}^{\infty} B_i = 0$.

   (You may assume that $\mathbb{Z}_p$ is an integral domain of characteristic zero and that $\bigcap_{i=1}^{\infty} p^i\mathbb{Z}_p = 0$.)

(2) For any group K, we let $\hat{K} = \varprojlim K/K_i$ where $(K/K_i, f_i^j)$ is the inverse system with indexing set $\mathcal{I}$, $\{K_i \mid i \in \mathcal{I}\}$ is the set of normal subgroups of finite index in K, and f_i^j is the natural epimorphism whenever $i \leq j$. Now let G be a group which has a finitely generated free abelian normal subgroup of finite index, and let $H \lhd G$.
   (i) Prove that there exists a family of normal subgroups of finite index in G $\{G_i \mid i \in \mathbb{P}\}$ such that $G_i \supset G_{i+1}$ for all $i \in \mathbb{P}$, and such that any subgroup of finite index in G contains one of the G_i.
   (ii) Prove that $\hat{H} \cong \varprojlim H/G_i \cap H$, $\hat{G} \cong \varprojlim G/G_i$, and $\widehat{G/H} \cong \varprojlim G/G_i H$.
   (iii) Prove that $\widehat{G/H} \cong \hat{G}/\hat{H}$.

(You may assume that if $\mathcal{J}$ is a cofinal subset of the directed set $\mathcal{K}$, and (M_i) is an inverse system indexed by $\mathcal{K}$, then $\varprojlim(M_i)$ is naturally isomorphic to the inverse limit obtained by using the directed set $\mathcal{J}$ as indexing set instead of $\mathcal{K}$.)

(3) Let p be a prime, and let K be the subfield of $\mathbb{C}$ generated by all pth power roots of unity. Prove that $\mathrm{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}_p \times \mathbb{Z}/(p-1)\mathbb{Z}$.

(4) Let k be a field, let $k[[X]]$ denote the power series ring in X, let $k[X, X^{-1}]$ denote the Laurent polynomial ring in X, and let $k[[X, X^{-1}]]$ denote the Laurent series ring in X.
(i) Prove that $\varprojlim^1 X^i k[X] \cong k[[X]]/k[X]$ and that $\varprojlim^1 X^i k[[X]] = 0$.
(ii) Prove that

$$\mathrm{Ext}^1_{k[X]}(k[X, X^{-1}], k[X]) \cong k[[X]]/k[X] \quad \text{and that} \quad \mathrm{Ext}^1_{k[[X]]}(k[[X, X^{-1}]], k[[X]]) = 0.$$

(5) Let k be an integral domain with field of fractions K, let U be a $k[X]$-module, let V be a $K[X]$-module, let (A, α_0) be a flat resolution of U with $k[X]$-modules, and let (B, β_0) be a projective resolution of V with $K[X]$-modules. Prove that $(A \otimes_{k[X]} B, \alpha_0 \otimes \beta_0)$ is a flat resolution of $U \otimes_{k[X]} V$ with $k[X]$-modules.

(6) Let p be a prime.
(i) Prove that p is a nonzero divisor in $\mathbb{Z}_p$.
(ii) If $q \in \mathbb{Z}$ and $(p, q) = 1$, prove that q is invertible in $\mathbb{Z}_p$.
(iii) Let R be a commutative ring and let S be a multiplicatively closed subset of R. Give an example to show that S^{-1} does not commute with inverse limits in general.
(iv) Let R be a ring and let A be an R-module. Give an example to show that $\mathrm{Hom}_R(A, \_)$ does not commute with $\varprojlim^1$ in general.

Monday, October 10

Chapter 9
The Künneth Formula (continued)

Exercise (cf. prob. 1 of previous homework.) Let p be a prime and let $A \neq 1$ be a free abelian group. Prove that A has a descending chain of subgroups $B_1 \supset B_2 \supset \cdots$ such that $A/B_i \cong \mathbb{Z}/p^i\mathbb{Z}$ for all i and $\bigcap_{i=1}^{\infty} B_i = 1$ if and only if $|A| \leq |\mathbb{R}|$.

Continuing the proof of the Künneth Formula, we need to show that the sequence splits in the case when B is not a projective k-module. We will need the following result.

Lemma 1 Let k be a right hereditary ring and let B be a chain complex of k-modules. Then there exists a free chain complex C and a chain map $\theta\colon C \to B$ such that the induced maps $\theta_{i*}\colon \mathrm{H}_i(C) \to \mathrm{H}_i(D)$ are isomorphisms.

Let β_i and γ_i denote the boundary maps of B and C respectively. Then θ is a chain map means that we have a sequence of k-module maps $\theta_i\colon C_i \to B_i$ which commute with the boundary maps, i.e. $\theta_i\gamma_{i+1} = \beta_{i+1}\theta_{i+1}$.

It is clear that Lemma 1 will follow from Lemmas 2 and 3 below.

Lemma 2 Let k be a ring and let B be a chain complex of k-modules. Then there exists a chain complex C of free k-modules such that $\mathrm{H}_n(C) \cong \mathrm{H}_n(B)$ for all $n \in \mathbb{N}$.

Proof We construct the complex C by induction on n. First we write $\mathrm{H}_0(B) \cong C_0/D$ where C_0 is a free k-module, and then we choose a free k-module D_1 and a k-map $\delta_1 \colon D_1 \to C_0$ such that $\operatorname{im}\gamma_1 = D$. Define γ_0 to be the zero map on C_0. Now suppose we have constructed free k-modules $C_0, \ldots, C_n$, $D_1, \ldots, D_{n+1}$, and R-maps $\gamma_i \colon C_i \to C_{i-1}$ such that $\gamma_{i-1}\gamma_i = 0$, $\ker\gamma_{i-1}/\operatorname{im}\gamma_i \cong \mathrm{H}_{i-1}(B)$ $(1 \le i \le n)$, and a map $\delta_{n+1} \colon D_{n+1} \to C_n$ such that $\gamma_n\delta_{n+1} = 0$ and $\ker\gamma_n/\operatorname{im}\delta_{n+1} \cong \mathrm{H}_n(B)$. Write $\mathrm{H}_{n+1}(B) \cong C'_{n+1}/D'_{n+1}$ and then define $C_{n+1} = C'_{n+1} \oplus D_{n+1}$ and $\gamma_{n+1} = 0 \oplus \delta_{n+1} \colon C_{n+1} \to C_n$. Finally choose a free k-module D_{n+2} and a k-epimorphism $\delta_{n+2} \colon D_{n+2} \to C_{n+1}$ such that $\operatorname{im}\delta_{n+2} = D'_{n+1} \oplus \ker\delta_{n+1}$. Then it is easy to see that $\gamma_n\gamma_{n+1} = 0$, $\ker\gamma_n/\operatorname{im}\gamma_{n+1} \cong \mathrm{H}_n(B)$, $\gamma_{n+1}\delta_{n+2} = 0$, and $\ker\gamma_{n+1}/\operatorname{im}\delta_{n+2} \cong \mathrm{H}_{n+1}(B)$, and the induction step is complete.

Lemma 3 Let k be a right hereditary ring, let B be a chain complex of k-modules, let C be a chain complex of projective k-modules, and let $\theta_n \colon \mathrm{H}_n(C) \to \mathrm{H}_n(B)$ be a k-map for each $n \in \mathbb{N}$. Then there exists a chain map $\phi : C \to B$ such that $\phi_{n*} = \theta_n$ for all $n \in \mathbb{N}$.

Proof We prove the result by induction on n. First set ϕ_{-1} to be the zero map on 0, and let β_r and γ_r denote the boundary maps of B and C respectively. For $n \in \mathbb{N}$, having constructed $\phi_r : C_r \to B_r$ such that $\phi_{r*} = \theta_r$, $\beta_r\phi_r = \phi_{r-1}\gamma_r$, $\phi_r \ker\gamma_r \subseteq \ker\beta_r$, $\phi_r \operatorname{im}\gamma_{r+1} \subseteq \beta_{r+1}$ for $0 \le r \le n-1$, we construct $\phi_n \colon C_n \to B_n$ having the same properties (thus ϕ_r will then satisfy the above properties for $0 \le r \le n$). We have a commutative diagram

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\gamma_{n+1}} & C_n & \xrightarrow{\gamma_n} & C_{n-1} & \xrightarrow{\gamma_{n-1}} & C_{n-2} & \xrightarrow{\gamma_{n-2}} & \cdots \\
 & & & & \phi_{n-1}\downarrow & & \downarrow\phi_{n-2} & & \\
\cdots & \xrightarrow{\beta_{n+1}} & B_n & \xrightarrow{\beta_n} & B_{n-1} & \xrightarrow{\beta_{n-1}} & B_{n-2} & \xrightarrow{\beta_{n-2}} & \cdots
\end{array}$$

Now θ_n is a homomorphism $\ker\gamma_n/\operatorname{im}\gamma_{n+1} \to \ker\beta_n/\operatorname{im}\beta_{n+1}$. Since $\ker\gamma_n$ is a projective k-module (k is hereditary and $\ker\gamma_n$ is a submodule of the projective k-module C_n), θ_n lifts to a homomorphism $\psi \colon \ker\gamma_n \to \ker\beta_n$ (which maps $\operatorname{im}\gamma_{n+1}$ into $\operatorname{im}\beta_{n+1}$). Also $\operatorname{im}\gamma_n$ is a projective k-module, so we may write $C_n = \ker\gamma_n \oplus D$ for some k-submodule $D \cong \operatorname{im}\gamma_n$ of C_n. Since $\phi_{n-1}\gamma_n$ maps D into $\operatorname{im}\beta_n$, there is a homomorphism $\delta \colon D \to B_n$ such that $\beta_n\delta d = \phi_{n-1}\gamma_n d$ for all $d \in D$. We may now set $\phi_n = \psi \oplus \delta \colon \ker\gamma_n \oplus D = C_n \to B_n$ and the induction step is complete.

We now show that the sequence in the Künneth formula splits when B is an arbitrary chain complex. By Lemma 1 we may choose a chain complex C of free k-modules and a chain map $\theta \colon C \to B$ such that the induced map θ_i is an isomorphism. We now have a commutative diagram with exact rows in which the top row splits and the two outside vertical maps are isomorphisms.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{r+s=n} \mathrm{H}_r(A) \otimes_k \mathrm{H}_s(C) & \longrightarrow & \mathrm{H}_n(A \otimes_k C) & \longrightarrow & \bigoplus_{r+s=n-1} \mathrm{Tor}_1^k(\mathrm{H}_r(A), \mathrm{H}_s(C)) & \longrightarrow & 0 \\
 & & \downarrow 1_* \otimes \theta_* & & \downarrow (1 \otimes \theta)_* & & \downarrow \bigoplus_{r+s=n-1} \theta_{s*} & & \\
0 & \longrightarrow & \bigoplus_{r+s=n} \mathrm{H}_r(A) \otimes_k \mathrm{H}_s(B) & \longrightarrow & \mathrm{H}_n(A \otimes_k B) & \longrightarrow & \bigoplus_{r+s=n-1} \mathrm{Tor}_1^k(\mathrm{H}_r(A), \mathrm{H}_s(B)) & \longrightarrow & 0
\end{array}$$

The Five Lemma shows that the middle vertical map is an isomorphism and a routine diagram chase now shows that the bottom row splits, as required.

Group Rings To define the cohomology of a group G, we need to know what the group ring of G is.

Definition Let G be a group and let k be a commutative ring. Then the group ring of G over k is the associative k-algebra which is a free k-module with basis $\{g \mid g \in G\}$ and multiplication defined distributively using the group multiplication in G.

Thus we can consider kG as consisting of all formal sums $\sum_{g \in G} a_g g$ where $a_g \in k$ and $a_g = 0$ for all but finitely many $g \in G$. If $\alpha = \sum_{g \in G} a_g g$ and $\beta = \sum_{g \in G} b_g g$ $(a_g, b_g \in k)$, then the multiplication is defined according to the formula

$$\alpha\beta = \sum_{g,h \in G} a_g b_h \, gh = \sum_{g \in G} \big(\sum_{h \in H} a_{gh^{-1}} b_h \big) g.$$

There is no reason why we cannot make the same definition with k noncommutative, and there are further generalizations of the concept of a group ring such as a crossed product; however for group cohomology it seems best to restrict to the case k is commutative. We will be especially interested in the cases when k is a field or the integers $\mathbb{Z}$.

Remarks

(i) We identify G with the subset $\{1g \mid g \in G\}$ of kG: thus G can be thought of as a subgroup of the group of units of kG.

(ii) We identify k with the subset $\{a_1 1 \mid a_1 \in k\}$ of kG: thus k can be thought of as a subring (or k-subalgebra) of kG. Then we have $kG = k$ if and only if $G = 1$.

(iii) We have an isomorphism of rings ($= \mathbb{Z}$-algebras) $kG \otimes_{\mathbb{Z}} k \cong k$, and this means that often problems involving kG can be reduced to problems involving $\mathbb{Z}G$.

(iv) kG is commutative if and only if G is abelian.

(v) Suppose G is a finitely generated free abelian group with free generators $\{x_1, x_2, \ldots, x_n\}$. Then kG is isomorphic (as a k-algebra) to the Laurent polynomial ring

$$k[X_1, X_1^{-1}, X_2, X_2^{-1}, \ldots, X_n, X_n^{-1}]$$

in n indeterminants.

(vi) kG is finitely generated as a k-module if and only if G is finite.

A useful property of kG is that it can be defined using the following universal property which we shall quote without proof (this can be especially useful when constructing homomorphisms from kG to another ring).

Proposition Let G be a group, let k be a commutative ring, let R be a k-algebra, let U denote the units of R, and let $\theta\colon G \to U$ be a group homomorphism. Then there exists a unique k-algebra homomorphism $\phi\colon kG \to R$ such that $\phi g = \theta g$ for all $g \in G$. Moreover if S is a k-algebra and H is a subgroup of the group of units of S isomorphic to G with the above universal property (i.e. if $\theta\colon H \to U$ is a group homomorphism, then there exists a unique k-algebra homomorphism $\phi\colon S \to R$ such that $\phi h = \theta h$ for all $h \in H$), then there exists a unique k-algebra isomorphism $\psi\colon kG \to S$ such that $\psi g = \alpha g$ for all $g \in G$, where $\alpha\colon G \to H$ is a group isomorphism.

Exercise Let k be a commutative ring and let $n \in \mathbb{P}$. Prove that $kG \cong k[X]/(X^n - 1)$ (as k-algebras).

Using the above proposition we can define a k-algebra homomorphism $\varepsilon\colon kG \to k$ be $\varepsilon g = 1$ for all $g \in G$. The map ε is called the augmentation map and $\ker \varepsilon$ is called the augmentation ideal of kG, or the augmentation ideal of G over k.

Exercise Let kG be the group ring of the group G over the commutative ring k, and let I denote the augmentation ideal of kG. Prove that I is a free k-module with k-basis $\{g - 1 \in G \mid g \in G\backslash 1\}$.

The augmentation ideal of $\mathbb{Z}G$ will be denoted by the small German letter corresponding to the capital Latin letter used to name the group. Thus by the above exercise, $\mathfrak{g}$ is a free $\mathbb{Z}$-module with $\mathbb{Z}$-basis $\{g - 1 \mid g \in G\backslash\}$.

Theorem Let G be a group, let k be a commutative ring, and let I denote the augmentation ideal of kG.

(i) $I \cong \mathfrak{g} \otimes_{\mathbb{Z}} k$ as kG-modules.
(ii) $\mathfrak{g}/\mathfrak{g}^2 \cong G/G'$ as $\mathbb{Z}$-modules. (G' denotes the commutator subgroup of G.)
(iii) $I/I^2 \cong G/G' \otimes_{\mathbb{Z}} k$ as k-modules.

Proof (i) For $\alpha \in \mathfrak{g}$ and $x \in k$, define $\theta\colon \mathfrak{g} \otimes_{\mathbb{Z}} k \to I$ by $\theta\alpha \otimes x = x\alpha$. Then θ is a k-map which commutes with the action of G (here G acts by right multiplication on $\mathfrak{g}$ and trivially on k). Therefore θ is a kG-module homomorphism. Since $\mathfrak{g}$ is free as a k-module on $\{g - 1 \mid g \in G\backslash 1\}$ (see previous exercise) we may well define a k-module map $\phi\colon I \to \mathfrak{g} \otimes k$ by $\phi(g - 1) = g \otimes 1$. Then it is easily checked that θ and ϕ are inverse to each other, hence θ (and ϕ) is a kG-isomorphism and the result is proven.

(ii) Since $\mathfrak{g}$ is free as a $\mathbb{Z}$-module on $\{g - 1 \mid g \in G\backslash 1\}$, we can well define a group homomorphism $\phi\colon \mathfrak{g} \to G/G'$ by $\phi(g - 1) = G'g$ ($g \in G\backslash 1$). Now $\mathfrak{g}^2$ is generated as a $\mathbb{Z}$-module by $\{(g - 1)(h - 1) \mid g, h \in G\}$, and

$$\phi(g-1)(h-1) = \phi\big((gh-1)-(g-1)-(h-1)\big) = G'ghg^{-1}h^{-1} = 1,$$

so ϕ induces a group homomorphism $\bar{\phi}\colon \mathfrak{g}/\mathfrak{g}^2 \to G/G'$. We now define a $\psi\colon G \to \mathfrak{g}/\mathfrak{g}^2$ by $\psi g = \mathfrak{g}^2 + (g-1)$. Then it is easily checked that ψ is a group homomorphism and so it induces a group homomorphism $\bar{\psi}\colon G/G' \to \mathfrak{g}/\mathfrak{g}^2$ such that $\bar{\psi}G'g = \mathfrak{g}^2 + (g-1)$. Moreover $\bar{\phi}$ and $\bar{\psi}$ are inverses to each other, and the result follows.

(iii) From (ii) we have an exact sequence $\mathfrak{g}^2 \to \mathfrak{g} \to G/G' \to 0$. Applying $\otimes_{\mathbb{Z}} k$ to this sequence, we obtain an exact sequence

$$\mathfrak{g}^2 \otimes_{\mathbb{Z}} k \xrightarrow{\alpha} \mathfrak{g} \otimes_{\mathbb{Z}} k \longrightarrow G/G' \otimes_{\mathbb{Z}} k \to 0.$$

If we identify $\mathfrak{g} \otimes_{\mathbb{Z}} k$ with I via the isomorphism θ of (i), then $\operatorname{im}\alpha$ becomes identified with I^2. Thus $I/I^2 \cong \mathfrak{g}/\operatorname{im}\alpha$ and the result follows.

Eighth Homework Due 9:00 a.m., Monday, October 17.

(1) Let $2 \le p, q \in \mathbb{Z}$ with $(p,q) = 1$. Define an inverse system of abelian groups (= $\mathbb{Z}$-modules) by $M_i = \mathbb{Z}$ for all $i \in \mathbb{P}$ and $f_i^{i+1}\colon M_{i+1} \to M_i$ to be multiplication by q.
   (i) Prove that $\operatorname{Ext}^n_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, M_i)$ is finite for all $i \in \mathbb{P}$.
   (ii) Prove that $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, \varprojlim M_i) = 0$.
   (iii) Prove that $\varprojlim \operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, M_i) \cong \mathbb{Z}/q\mathbb{Z}$.

(2) Let R be a ring, let F be a free R-module, and let (M_i) be an inverse system of R-modules indexed by $\mathbb{P}$. Suppose $0 \to K \to F \to A \to 0$ is an exact sequence of R-modules such that $\operatorname{Hom}_R(K, M_i)$ is finite for all $i \in \mathbb{P}$. Prove that $\operatorname{Ext}^n_R(A, \varprojlim M_i) \cong \varprojlim \operatorname{Ext}^n_R(A, M_i)$ for $n = 0, 1$.

(3) Let R be a commutative hereditary ring, and let A, B, C be R-modules. Prove that $\operatorname{Tor}^1_R(A, \operatorname{Tor}^1_R(B, C)) \cong \operatorname{Tor}^1_R(\operatorname{Tor}^1_R(A, B), C)$. (Consider $\mathrm{H}_2(P \otimes_R Q \otimes_R T)$.)

(4) Let R be a ring and let A be a cochain complex. Prove that there exists an injective cochain complex of R-modules with the same cohomology as A.

(5) Let k be a right hereditary ring, let A be a projective chain complex of right k-modules, and let B, C be chain complexes of left k-modules. Suppose $\theta\colon B \to C$ is a chain map such that the induced map $\theta_*\colon \mathrm{H}_n(B) \to \mathrm{H}_n(C)$ is an isomorphism for all $n \in \mathbb{N}$. Prove that $(1 \otimes \theta)_*\colon \mathrm{H}_n(A \otimes_k B) \to \mathrm{H}_n(A \otimes_k C)$ is an isomorphism for all $n \in \mathbb{N}$.

(6) Let k be a commutative ring and let G, H be groups. Prove that $k[G \times H] \cong kG \otimes_k kH$ as k-algebras. (Recall that if k is a commutative ring and A, B are k-algebras, then we can make $A \otimes_k B$ into a k-algebra by defining multiplication according to the formula $(\alpha_1 \otimes \beta_1)(\alpha_2 \otimes \beta_2) = \alpha_1\alpha_2 \otimes \beta_1\beta_2$ ($\alpha_1, \alpha_2 \in A$, $\beta_1, \beta_2 \in B$). Since elements of the form $\alpha \otimes \beta$ generate $A \otimes_k B$ as a k-module, this uniquely defines the multiplication on $A \otimes_k B$.)

(7) Let A be an abelian group such that for each $n \in \mathbb{P}$, A has only finitely many elements of order n.

(i) Let p be a prime and let $B_i = \mathbb{Z}/p^i\mathbb{Z}$. If (B_i, f_i^j) is the direct system indexed by $\mathbb{P}$, where the f_i^j are the natural inclusions for $i \leq j$, prove that $\varprojlim^1 \operatorname{Ext}^0_{\mathbb{Z}}(B_i, A) = 0$.

(ii) Use the quoted theorem at the end of chapter 7 to prove that

$$\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, A) \cong \prod_q \varprojlim A/(Aq^i),$$

where the Cartesian product is over all primes q.

Monday, October 17

Chapter 10
Group Cohomology

Let G be a group, let k be a commutative ring, and let M be a kG-module. Then to say that G acts trivially on M means that $mg = m$ for all $g \in G$. We can always regard k as a kG-module with G acting trivially, and then if I is the augmentation ideal of kG, we have $k \cong kG/I$ as kG-modules. If N is any kG-module, then N can be considered as a $\mathbb{Z}G$-module and $N \otimes_{kG} k \cong N/N\mathfrak{g}$ as kG-modules (use HW1 prob. 1).

We can now define the homology and cohomology groups of G with coefficients in a $\mathbb{Z}G$-module M.

Definition Let G be a group and let $n \in \mathbb{N}$.

(i) If M is a left $\mathbb{Z}G$-module, then the nth homology group of G with coefficients in M is $\mathrm{H}_n(G, M) = \operatorname{Tor}_n^{\mathbb{Z}G}(\mathbb{Z}, M)$.

(ii) If N is a right $\mathbb{Z}G$-module, then the nth cohomology group of G with coefficients in N is $\mathrm{H}^n(G, N) = \operatorname{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, N)$.

In both (i) and (ii), $\mathbb{Z}$ is regarded as a right $\mathbb{Z}G$-module with G acting trivially, and $\mathrm{H}_n(G, \mathbb{Z})$ is often termed the nth homology group of G, and $\mathrm{H}^n(G, \mathbb{Z})$ is often termed the nth cohomology group of G. The reason for using right coefficient modules for cohomology but left coefficient modules for homology, is that then we can use one and the same projective resolution of right $\mathbb{Z}G$-modules for $\mathbb{Z}$ to calculate the relevant Ext and Tor groups.

Remarks and Examples Let us calculate $\mathrm{H}_n(G, M)$ and $\mathrm{H}^n(G, N)$ in some special cases.

(1) If $G = 1$, then $\mathrm{H}_0(G, M) \cong M$ as $\mathbb{Z}$-modules, and $\mathrm{H}_n(G, M) = \operatorname{Tor}_n^{\mathbb{Z}G}(\mathbb{Z}, M) = 0$ if $n \neq 0$.

(2) If $G = 1$, then $\mathrm{H}^0(G, N) \cong N$ as $\mathbb{Z}$-modules, and $\mathrm{H}^n(G, N) = \operatorname{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, N) = 0$ if $n \neq 0$.

(3) $\mathrm{H}_0(G, M) \cong M/\mathfrak{g}M$.

(4) $\mathrm{H}^0(G, N) \cong N^G$. (Here $N^G = \{x \in N \mid xg = x \text{ for all } g \in G\}$.)

(5) $\mathrm{H}_n(G, M) = 0$ for all $n \in \mathbb{P}$ if M is a flat $\mathbb{Z}G$-module.

(6) $\mathrm{H}^n(G, N) = 0$ for all $n \in \mathbb{P}$ if N is an injective $\mathbb{Z}G$-module.

(7) Let k be a commutative ring. We could consider homology and cohomology groups "over k", but we would not get anything new. For example we could define $\mathrm{H}_n(G,M) = \mathrm{Tor}_n^{kG}(k,M)$ for any left kG-module M. However $\mathrm{Tor}_n^{kG}(k,M) \cong \mathrm{Tor}_n^{\mathbb{Z}G}(\mathbb{Z},M)$ because if (P,α_0) is a projective resolution of $\mathbb{Z}$ with $\mathbb{Z}G$-modules, then $(P\otimes_{\mathbb{Z}} k, \alpha_0\otimes 1)$ is a projective resolution of k with kG-modules, and we have

$$P\otimes_{\mathbb{Z}} k\otimes_{kG} M \cong P\otimes_{\mathbb{Z}}\mathbb{Z}\otimes_{\mathbb{Z}G} M \cong P\otimes_{\mathbb{Z}G} M.$$

A similar comment applies to cohomology using $\mathrm{Ext}_{kG}(k,N)\cong \mathrm{Ext}_{\mathbb{Z}G}(\mathbb{Z},N)$ for any right kG-module N (remark 5 of chapter 6).

(8) We have $\mathrm{H}_1(G,\mathbb{Z}) = \mathrm{Tor}_1^{\mathbb{Z}G}(\mathbb{Z},\mathbb{Z}) \cong \mathrm{Tor}_1^{\mathbb{Z}G}(\mathbb{Z}G/\mathfrak{g},\mathbb{Z}G/\mathfrak{g})$. Applying HW3 prob. 7(iii), we see that

$$\mathrm{H}_1(G,\mathbb{Z}) \cong \mathfrak{g}/\mathfrak{g}^2 \cong G/G'$$

by the Theorem of chapter 9.

(9) We have $\mathrm{H}^1(G,\mathbb{Z}) = \mathrm{Ext}^1_{\mathbb{Z}G}(\mathbb{Z},\mathbb{Z})$, so application of the long exact sequence for Ext in the first variable to the exact sequence $0\to\mathfrak{g}\xrightarrow{\alpha}\mathbb{Z}G\to\mathbb{Z}\to 0$ (where α is the natural inclusion) yields an exact sequence

$$0\to \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z},\mathbb{Z})\to \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G,\mathbb{Z})\xrightarrow{\alpha^*}\mathrm{Hom}_{\mathbb{Z}G}(\mathfrak{g},\mathbb{Z})\to \mathrm{H}^1(G,\mathbb{Z})\to 0,$$

because $\mathrm{Ext}^1_{\mathbb{Z}G}(\mathbb{Z}G,\mathbb{Z})=0$. Now if $f\in\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G,\mathbb{Z})$, then $f(g-1)=f(1)(g-1)=0$ for all $g\in G$, so $f(\mathfrak{g})=0$. Therefore $\alpha^*f=0$ and it follows that $\mathrm{H}_1(G,\mathbb{Z})\cong\mathrm{Hom}_{\mathbb{Z}G}(\mathfrak{g},\mathbb{Z})$. If $f\in\mathrm{Hom}_{\mathbb{Z}G}(\mathfrak{g},\mathbb{Z})$, then $f\big((g-1)(h-1)\big)=\big(f(g-1)\big)(h-1)=0$ for all $g,h\in G$, hence $f(\mathfrak{g}^2)=0$ and we deduce that $\mathrm{H}_1(G,\mathbb{Z})\cong\mathrm{Hom}_{\mathbb{Z}G}(\mathfrak{g}/\mathfrak{g}^2,\mathbb{Z})$. The Theorem of chapter 9 shows that $\mathfrak{g}/\mathfrak{g}^2\cong G/G'$ and we conclude that $\mathrm{H}^1(G,\mathbb{Z})\cong\mathrm{Hom}_{\mathbb{Z}}(G/G',\mathbb{Z})$.

Long Exact Sequences Let G be a group, and let $0\to A\to B\to C\to 0$ be an exact sequence of left kG-modules. Then the long exact sequence for $\mathrm{Tor}^{\mathbb{Z}G}(\mathbb{Z},\_)$ yields a long exact sequence

$$\begin{aligned}\cdots\longrightarrow \mathrm{H}_n(G,A)\longrightarrow \mathrm{H}_n(G,B)\longrightarrow \mathrm{H}_n(G,C)\longrightarrow \mathrm{H}_{n-1}(G,A)\longrightarrow\cdots\\ \cdots\longrightarrow \mathrm{H}_0(G,A)\longrightarrow \mathrm{H}_0(G,B)\longrightarrow \mathrm{H}_0(G,C)\longrightarrow 0.\end{aligned}$$

This sequence is often termed the long exact homology sequence. Similarly an exact sequence of right kG-modules yields a long exact sequence

$$\begin{aligned}0\longrightarrow \mathrm{H}^0(G,A)\longrightarrow \mathrm{H}^0(G,B)\longrightarrow \mathrm{H}^0(G,C)\longrightarrow\cdots\\ \cdots\longrightarrow \mathrm{H}^n(G,A)\longrightarrow \mathrm{H}^n(G,B)\longrightarrow \mathrm{H}^n(G,C)\longrightarrow \mathrm{H}^{n+1}(G,A)\longrightarrow\cdots.\end{aligned}$$

Inflation and Restriction Let G and H be groups, let M be a $\mathbb{Z}H$-module, and let $\theta\colon G\to H$ be a group homomorphism. Then θ extends to a ring homomorphism $\theta\colon\mathbb{Z}G\to\mathbb{Z}H$,

so by HW3 prob. 6, θ induces well defined group homomorphisms $\theta_n^* : \mathrm{Ext}^n_{\mathbb{Z}H}(\mathbb{Z}, M) \to \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, M)$ for all $n \in \mathbb{N}$. In other words, θ induces well defined group homomorphisms $\theta_n^* \colon \mathrm{H}^n(H, M) \to \mathrm{H}^n(G, M)$ for all $n \in \mathbb{N}$. On identifying $\mathrm{H}^1(H, \mathbb{Z})$ with $\mathrm{Hom}_{\mathbb{Z}}(H/H', \mathbb{Z})$, this is just the homomorphism you would expect; namely given $f \in \mathrm{Hom}_{\mathbb{Z}}(H/H', \mathbb{Z})$ and $g \in G$, then $\theta_n^* f(G'g) = f(H'\theta g)$. In the special case when $H = G/K$ for some $K \lhd G$ and $\theta \colon G \to H$ is the natural epimorphism, then θ_n^* is called the inflation map; and in the special case when $G \leqslant H$ and θ is the inclusion, then θ_n^* is called the restriction map.

Shapiro's Lemma This is the cohomological version of Frobenius reciprocity in character theory of finite groups: in fact some mathematicians think that Shapiro's lemma should be called Frobenius reciprocity. In any case it is an important tool when performing calculations. First we need the following easy but fundamental result.

Proposition Let $H \leqslant G$ be groups, and let R be a commutative ring. Then RG is free both as a left RH-module and as a right RH-module.

Proof Let T be a right transversal for H in G. Then it is easy to see that T is a free basis for RG as a left RH-module. Similarly by taking a left transversal S for H in G, we see that RG is free as a right RH-module, with basis S as required.

The above proposition shows in particular that if P is a projective RG-module, then P is also a projective RH-module. Thus we can now apply Lemma 7 of chapter 5 to deduce that

Theorem (Shapiro's lemma) Let $H \leqslant G$ be groups, let R be a commutative ring, let M be a left RH-module, and let $n \in \mathbb{N}$. Then $\mathrm{H}_n(G, \mathbb{Z}G \otimes_{\mathbb{Z}H} M) \cong \mathrm{H}_n(H, M)$.

There is a similar result for cohomology: we just state it, leaving the proof to a future homework exercise (homework 11, prob. 1).

Theorem (Shapiro's lemma) Let $H \leqslant G$ be groups, let R be a commutative ring, let N be an RH-module, and let $n \in \mathbb{N}$. Then $\mathrm{H}^n(G, \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, N)) \cong \mathrm{H}^n(H, N)$.

Diagonal Action Let G, H be groups, let k be a commutative ring, let M be a kG-module and let N be a kH-module. Then we can make $M \otimes_k N$ into a $k[G \times H]$-module by defining $m \otimes n\,(g, h) = mg \otimes nh$. If M and N are free, then it is easy to see that $M \otimes_k N$ is also free. It follows that if M and N are projective, then $M \otimes_k N$ is also projective.

Now suppose M and N are both kG-modules. Then we can make $M \otimes_k N$ into a kG-module by defining $m \otimes n\, g = mg \otimes ng$. Similarly we can make $\mathrm{Hom}_k(M, N)$ into a kG-module by defining $(fg)m = f(mg^{-1})g$ for all $f \in \mathrm{Hom}_k(M, N)$, $g \in G$, $m \in M$. These actions are often called the diagonal actions of G on $M \otimes_k N$ and $\mathrm{Hom}_k(M, N)$. They are of great importance when dealing with group cohomology, partly because of the following fundamental result.

Theorem Let G be a group, let k be a commutative ring, let P be a projective kG-module, and let M be a kG-module which is projective as a k-module. Then $P \otimes_k M$ is a projective kG-module.

Proof Let N be any kG-module. By HW1 prob. 2, there is an isomorphism of k-modules between $\mathrm{Hom}_k(P \otimes_k M, N)$ and $\mathrm{Hom}_k(P, \mathrm{Hom}_k(M, N))$ which is manifestly natural, hence it induces an isomorphism

$$\mathrm{Hom}_{kG}(P \otimes_k M, N) \cong \mathrm{Hom}_{kG}(P, \mathrm{Hom}_k(M, N)).$$

Since M is projective as a k-module, $\mathrm{Hom}_k(M, \_)$ is exact, and since P is a projective kG-module, $\mathrm{Hom}_{kG}(P, \_)$ is also exact. It follows that $\mathrm{Hom}_{kG}(P \otimes_k M, \_)$ is exact, which means that $P \otimes_k M$ is a projective kG-module.

Ninth Homework Due 9:00 a.m., Monday, October 24.

(1) Let $R = \mathbb{Z}[X_1, X_2, \dots]$ be a polynomial ring in any number of variables, let A be a right R-module, let B be left R-module, and let $n, p \in \mathbb{N}$. Suppose B is free as a $\mathbb{Z}$-module.
   (i) If P is a projective R-module, prove that $P \otimes_R B$ is a projective $\mathbb{Z}$-module.
   (ii) Prove that $\mathrm{Tor}_n^R(A, B/pB) \cong \mathrm{Tor}_n^R(A, B) \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \oplus \mathrm{Tor}_1^{\mathbb{Z}}(\mathrm{Tor}_{n-1}^R(A, B), \mathbb{Z}/p\mathbb{Z})$ as abelian groups.

   (If $n = 0$ in the above, then we interpret Tor_{n-1} to be 0.)

(2) Let k be a field, let U be a $k[X]$-module, and let V be a $k[Y]$-module.
   (i) Prove that $U \otimes_k V$ can be made into a $k[X, Y]$-module according to the rule $u \otimes v\, X = uX \otimes v$, $u \otimes v\, Y = u \otimes (vY)$ for $u \in U$, $v \in V$. Prove further that if U and V are projective, then $U \otimes_k V$ is a projective $k[X, Y]$-module.
   (ii) Let $n \in \mathbb{N}$. Use the Künneth Theorem to prove that

$$\mathrm{Tor}_n^{k[X,Y]}(U \otimes_k V, k) \cong \bigoplus_{r+s=n} \mathrm{Tor}_r^{k[X]}(U, k) \otimes_k \mathrm{Tor}_s^{k[Y]}(V, k)$$

   as $k[X, Y]$-modules. (Of course k denotes the module with X and Y acting trivially; that is $aX = aY = 0$ for all $a \in k$).

(3) Let R be a ring, let A be a chain complex of right R-modules, let B be a chain complex of left R-modules, and let $\alpha_n \colon A_n \to A_{n-1}$ denote the boundary maps of A. Suppose A_n and $\mathrm{im}\, \alpha_{n+1}$ are flat R-modules for all $n \in \mathbb{N}$.
   (i) Prove that $\ker \alpha_n$ is a flat R-module for all $n \in \mathbb{N}$.
   (ii) Prove that there is a natural short exact sequence of abelian groups

$$0 \longrightarrow \bigoplus_{r+s=n} \mathrm{H}_r(A) \otimes_R \mathrm{H}_s(B) \xrightarrow{\pi} \mathrm{H}_n(A \otimes_R B) \longrightarrow \bigoplus_{r+s=n-1} \mathrm{Tor}_1^R(\mathrm{H}_r(A), \mathrm{H}_s(B)) \longrightarrow 0.$$

(4) Let $A \lhd G$ be groups with G/A finite.
   (i) If $\mathbb{Z}A$ is a Noetherian ring, prove that $\mathbb{Z}G$ is a Noetherian ring.
   (ii) Suppose A is a finitely generated free abelian group. Prove that $\mathbb{Z}A$ is a Noetherian ring. Deduce that $\mathbb{Z}G$ is also a Noetherian ring.

(By a Noetherian ring, we mean a ring which is both left and right Noetherian.)

(5) Let $A \lhd G$ be groups such that A is finitely generated abelian and G/A is finite, let (M_i) be a direct system of $\mathbb{Z}G$-modules, and let $n \in \mathbb{N}$. Prove that $\mathrm{H}^n(G, \varinjlim M_i) \cong \varinjlim \mathrm{H}^n(G, M_i)$.

(6) Let (G_i) be a direct system of groups, and let $n \in \mathbb{N}$.
   (i) Prove that $\mathbb{Z}[\varinjlim G_i] \cong \varinjlim \mathbb{Z}G_i$.
   (ii) In the case that the direct limit is a union (i.e. all the corresponding maps f_i^j are inclusions), prove that $\mathrm{H}_n(\varinjlim G_i, \mathbb{Z}) \cong \varinjlim \mathrm{H}_n(G_i, \mathbb{Z})$. (Use HW6, prob. 7 and the Proposition in Shapiro's lemma.)
   (iii) Prove that $\mathrm{H}^1(\varinjlim G_i, \mathbb{Z}) \cong \varprojlim \mathrm{H}^1(G_i, \mathbb{Z})$.

Monday, October 24

Chapter 11
Group Cohomology, basic results

To get some examples, we shall calculate the cohomology of a finite cyclic group. Let k be a commutative ring, let $n \in \mathbb{P}$, and let $G = \langle g \rangle$ be a finite cyclic group of order n. We are going to determine $\mathrm{H}^r(G, k)$ (where G is acting trivially on k.) The cases $r = 0$ (answer k) and $r = 1$ (answer $\mathrm{Hom}(G, k)$) were covered at the beginning of chapter 10. Let us write $k_n = \{a \in k \mid na = 0\}$. Then $\mathrm{H}^1(G, k) \cong k_n$. We can obtain the other cohomology groups by dimension shifting. We have an exact sequence

$$0 \longrightarrow k\mathfrak{g} \longrightarrow kG \xrightarrow{\varepsilon} k \longrightarrow 0, \tag{1}$$

where ε is the augmentation map and $k\mathfrak{g}$ denotes the augmentation ideal of kG. We can also define a kG-epimorphism $\alpha\colon kG \to k\mathfrak{g}$ by $\alpha 1 = g - 1$. If $\nu = \sum_{i=0}^{n-1} g^i$, then it is not difficult to see that $\ker \alpha = k\nu$ and that $k\nu \cong k$ as kG-modules. Thus we also have an exact sequence

$$0 \longrightarrow k \longrightarrow kG \xrightarrow{\alpha} k\mathfrak{g} \longrightarrow 0. \tag{2}$$

Applying the long exact sequence for Ext_{kG} in the first variable to the exact sequences (1) and (2), we see that $\mathrm{H}^r(G, k) \cong \mathrm{H}^{r+2}(G, k)$ for all $r \geq 1$, $\mathrm{H}^2(G, k) \cong \mathrm{Ext}^1_{kG}(\mathfrak{g}, k)$, and there is an exact sequence

$$0 \longrightarrow \mathrm{Ext}^0_{kG}(\mathfrak{g}, k) \xrightarrow{\alpha_0^*} \mathrm{Ext}^0_{kG}(kG, k) \xrightarrow{\rho} \mathrm{Ext}^0_{kG}(k, k) \longrightarrow \mathrm{Ext}^1_{kG}(\mathfrak{g}, k) \xrightarrow{\alpha_1^*} \mathrm{Ext}^0_{kG}(kG, k),$$

where ρ is the restriction map from $\mathrm{Hom}_{kG}(kG, k) \to \mathrm{Hom}_{kG}(k\nu, k)$. Therefore $\mathrm{Ext}^1_{kG}(\mathfrak{g}, k) \cong \mathrm{Hom}_{kG}(k, k)/\operatorname{im}\rho$. If $f \in \mathrm{Hom}_{kG}(kG, k)$, then $f(x) = f(1)$ for all $x \in G$, and we see that $f(\nu) = nf(1)$. It follows that $\operatorname{im}\rho = nk$ and hence $\mathrm{Ext}^1_{kG}(\mathfrak{g}, k) \cong k/nk$. Summing up,

$$\mathrm{H}^r(G, k) \cong \begin{cases} k & \text{if } r = 0, \\ k_n & \text{if } r \text{ is odd}, \\ k/nk & \text{if } 0 < r \text{ is even}. \end{cases}$$

Let us apply the diagonal action to the direct product of two groups. Let k be a commutative hereditary ring, let G, H be groups, let (P, α_0) be a projective resolution of k as kG-modules, and let (Q, β_0) be a projective resolution of k as kH-modules. Then the Künneth formula shows that $(P \otimes_k Q, \alpha_0 \otimes \beta_0)$ is a projective resolution of $k \otimes_k k$ as $k[G \times H]$-modules. Obviously $G \times H$ acts trivially on $k \otimes_k k$, so $k \otimes_k k \cong k$. Thus we can calculate the groups $\mathrm{H}^n(G \times H, k)$ using the projective resolution $(P \otimes_k Q, \alpha_0 \otimes \beta_0)$. We require the following lemma.

Lemma Let G, H be groups, let k be a commutative ring, let P and Q be chain complexes of kG and kH-modules respectively, and let U and V be kG and kH-modules respectively. Then there is a natural homomorphism of chain complexes

$$\theta\colon \mathrm{Hom}_{kG}(P, U) \otimes_k \mathrm{Hom}_{kH}(Q, V) \longrightarrow \mathrm{Hom}_{k[G\times H]}(P \otimes_k Q, U \otimes_k V)$$

defined by $(\theta f \otimes g)(u \otimes v) = fu \otimes vg$ for $f \in \mathrm{Hom}_{kG}(P_r, U)$, $g \in \mathrm{Hom}_{kH}(Q_s, k)$, $u \in P_r$, $v \in Q_s$.

The proof is routine with the necessary maps set up. Even though there is an awkward sign $(-1)^r$ involved in the definition of the tensor product of chain complexes, no sign is needed in the definition of θ.

Exercise Show that θ need not always be an isomorphism.

There are several hypotheses which will make θ an isomorphism. The most convenient seems to be the following.

Theorem In the above Lemma, assume that P is a finitely generated projective kG-module and that $U = k$ (where as usual, G is acting trivially on k). Then θ is an isomorphism.

Proof Because $\mathrm{Hom}_{kG}(\_, k)$ commutes with *finite* direct sums, it follows easily that we need only consider the case $P = kG$. But then we have an inverse map defined by $\phi f = \varepsilon \otimes h$ for $f \in \mathrm{Hom}_{k[G\times H]}(P \otimes_k Q, k \otimes_k V)$, where $\varepsilon\colon kG \to k$ is the augmentation map (so $\varepsilon g = 1$ for all $g \in G$), and $h\colon Q \to V$ is defined by $hq = f(1 \otimes q)$ for all $q \in Q$.

We now want to apply the Künneth formula to the cochain complexes $\mathrm{Hom}_{kG}(P, k)$ and $\mathrm{Hom}_{kH}(Q, k)$, but some words of caution are necessary here. To apply the Künneth formula, we need the $\mathrm{Hom}_{kG}(P_r, k)$ to be projective as k-modules for all $r \in \mathbb{N}$. If $P_r \cong kG$, then this is certainly so because $\mathrm{Hom}_{kG}(kG, k) \cong k$ as k-modules. It follows that this is also the case if P_r is a finitely generated projective kG-module. However this is not so if P_r is not finitely generated: if $P_r \cong \bigoplus_{i=1}^{\infty} kG$, then $\mathrm{Hom}_{kG}(P_r, k) \cong \prod_{i=1}^{\infty} k$, (see exercise 6(i) on the first chapter), which is not projective in general (though it certainly will be if k is a field).

Exercise Show that $\prod_{i=1}^{\infty} \mathbb{Z}$ is *not* a free abelian group. (Write $A = \prod_{i=1}^{\infty} \mathbb{Z}$. We think of A as all infinite sequences (a_i) with $a_i \in \mathbb{Z}$. Let $B = \{(a_i) \mid \text{for each } n \in \mathbb{P},\ a_i \in 2^n\mathbb{Z} \text{ for all but finitely many } i\}$. Show that B is an uncountable subgroup of A, and $B/2B$ is countable. Now if A is free, then so is B (we use the theorem that subgroups of free abelian groups are free) and now we have a contradiction.)

We need the following definition.

Definition Let G be a group and let k be a commutative ring. Then we say that G is of type FP_∞ over k if the trivial kG-module k admits a projective resolution with finitely generated kG-modules. We say that G is of type FP_∞ to mean that G is of type FP_∞ over $\mathbb{Z}$.

Exercise If kG is Noetherian, prove that G is of type FP_∞ over k. Deduce that if G has a finitely generated abelian subgroup of finite index, then G is of type FP_∞.

Thus the above exercise shows that there are plenty of groups of type FP_∞, in particular all finite groups. If G is of type FP_∞ over $\mathbb{Z}$, then $\mathbb{Z}$ has a projective resolution (P, α_0) in which all the modules are finitely generated. Since $(P \otimes_{\mathbb{Z}} k, \alpha_0 \otimes 1)$ is a projective resolution of k with finitely generated kG-modules, it follows that G is of type FP_∞ over k. It is an open problem at the moment to as whether the reverse is true: namely that if G is of type FP_∞ over k, then G is of type FP_∞ over $\mathbb{Z}$.

We now assume that G is of type FP_∞ over k, so we assume that the chain complex P consists of finitely generated projective kG-modules, and the chain complex Q consists of projective kH-modules (not necessarily finitely generated) and apply the Künneth formula for cochain complexes to obtain a natural exact sequence of kG-modules which splits:

$$0 \longrightarrow \bigoplus_{r+s=n} \mathrm{H}^r(\mathrm{Hom}_{kG}(P,k)) \otimes_k \mathrm{H}^s(\mathrm{Hom}_{kH}(Q,k)) \longrightarrow \mathrm{H}^n(\mathrm{Hom}_{kG}(P,k) \otimes_k \mathrm{Hom}_{kH}(Q,k))$$
$$\longrightarrow \bigoplus_{r+s=n+1} \mathrm{Tor}_1^k(\mathrm{H}^r(\mathrm{Hom}_{kG}(P,k)), \mathrm{H}^s(\mathrm{Hom}_{kH}(Q,k))) \longrightarrow 0.$$

Combining the above Theorem with $\mathrm{H}^r(\mathrm{Hom}_{kG}(P,k)) \cong \mathrm{H}^r(G,k)$ and $\mathrm{H}^s(\mathrm{Hom}_{kH}(Q,k)) \cong \mathrm{H}^s(H,k)$, we obtain an exact sequence of k-modules which splits (remember that the two hypotheses we need are that G is of type FP_∞ and that k is commutative hereditary):

$$0 \longrightarrow \bigoplus_{r+s=n} \mathrm{H}^r(G,k) \otimes_k \mathrm{H}^s(H,k) \longrightarrow \mathrm{H}^n(G \times H, k) \longrightarrow \bigoplus_{r+s=n+1} \mathrm{Tor}_1^k(\mathrm{H}^r(G,k), \mathrm{H}^s(H,k)) \longrightarrow 0.$$

Thus once $\mathrm{H}^n(G,k)$ has been calculated for G cyclic, it can be calculated when G is any finitely generated abelian group (since $\mathbb{Z}G$ is Noetherian when G is finitely generated abelian).

Remark In view of this stronger result (as compared with the original version which required that H be of type FP_∞), the hypotheses for problem 9 of the eleventh homework can be weakened, but I will let the original problem stand.

Example $\mathrm{H}^4(\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}, \mathbb{Z})$. Since $\mathbb{Z}$ is a commutative hereditary ring, we can apply the sequence of above to obtain a split exact sequence

$$0 \longrightarrow \bigoplus_{r+s=4} \mathrm{H}^r(\mathbb{Z}/6\mathbb{Z},\mathbb{Z}) \otimes_{\mathbb{Z}} \mathrm{H}^s(\mathbb{Z}/3\mathbb{Z},\mathbb{Z}) \longrightarrow \mathrm{H}^4(\mathbb{Z}/6\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z},\mathbb{Z})$$
$$\longrightarrow \bigoplus_{r+s=5} \mathrm{Tor}_1^{\mathbb{Z}}(\mathrm{H}^r(\mathbb{Z}/6\mathbb{Z},\mathbb{Z}),\mathrm{H}^s(\mathbb{Z}/3\mathbb{Z},\mathbb{Z})) \longrightarrow 0.$$

Using the calculation for the cohomology of a finite cyclic group, we have $\mathrm{H}^0(\mathbb{Z}/6\mathbb{Z},\mathbb{Z}) = \mathbb{Z} = \mathrm{H}^0(\mathbb{Z}/3\mathbb{Z},\mathbb{Z})$, $\mathrm{H}^r(\mathbb{Z}/6\mathbb{Z},\mathbb{Z}) = 0 = \mathrm{H}^r(\mathbb{Z}/3\mathbb{Z},\mathbb{Z})$ if r is odd, $\mathrm{H}^r(\mathbb{Z}/6\mathbb{Z},\mathbb{Z}) \cong \mathbb{Z}/6\mathbb{Z}$ and $\mathrm{H}^r(\mathbb{Z}/3\mathbb{Z},\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}$ if $0 < r$ is even. It follows easily that $\mathrm{H}^4(\mathbb{Z}/6\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z},\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$.

Universal Coefficient Theorem We can also apply the Universal Coefficient Theorem to group cohomology. Let G be a group of type FP_∞ and let (P,α_0) be a projective resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ with finitely generated $\mathbb{Z}G$-modules. Let $n \in \mathbb{N}$ and let M be a $\mathbb{Z}G$-module which is free as a $\mathbb{Z}$-module. Since $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, M) \cong M$, we see that $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, M)$ is free as a $\mathbb{Z}$-module and it follows that $\mathrm{Hom}_{\mathbb{Z}G}(Q, M)$ is free as a $\mathbb{Z}$-module for any finitely generated projective $\mathbb{Z}G$-module Q. Therefore $\mathrm{Hom}_{\mathbb{Z}G}(P, M)$ is a chain complex of projective $\mathbb{Z}$-modules, so we can apply the Universal Coefficient Theorem to obtain a split exact sequence (of k-modules)

$$0 \longrightarrow \mathrm{H}^n(\mathrm{Hom}_{\mathbb{Z}G}(P,M)) \otimes_{\mathbb{Z}} k \longrightarrow \mathrm{H}^n(\mathrm{Hom}_{\mathbb{Z}G}(P,M) \otimes_{\mathbb{Z}} k) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(\mathrm{H}^{n+1}(G,M),k) \longrightarrow 0.$$

Now $\mathrm{Hom}_{\mathbb{Z}G}(P,M)\otimes_{\mathbb{Z}}k$ is naturally isomorphic to $\mathrm{Hom}_{kG}(P\otimes_k k, M\otimes_k k)$ (map $f\otimes x$ for $x \in k$ to the homomorphism $f\otimes\hat{x}$, where $\hat{x}$ denotes that map multiplication by x on k; that this map is an isomorphism depends on P consisting of finitely generated projective $\mathbb{Z}G$-modules: it is easy to see that it is an isomorphism in the case $P = \mathbb{Z}G$, hence it is an isomorphism for any finitely generated projective $\mathbb{Z}G$-module) and $(P \otimes_k k, \alpha_0 \otimes 1)$ is a projective resolution of k by the Theorem at the end of the last chapter. By remark (7) of chapter 10, $\mathrm{H}^n(\mathrm{Hom}_{kG}(P\otimes_k k, M \otimes_k k)) \cong \mathrm{H}^n(G, M \otimes_k k)$ and we deduce that $\mathrm{H}^n(\mathrm{Hom}_{\mathbb{Z}G}(P,M) \otimes_{\mathbb{Z}} k) \cong \mathrm{H}^n(G,M)$. Therefore

$$\mathrm{H}^n(G, M \otimes_{\mathbb{Z}} k) \cong \mathrm{H}^n(G,M) \otimes_{\mathbb{Z}} k \oplus \mathrm{Tor}_1^{\mathbb{Z}}(\mathrm{H}^{n+1}(G,M),k).$$

Dual Universal Coefficient Theorem for group homology Using similar techniques as for the Künneth formula and Universal Coefficient Theorems, one can obtain formulae which involve both H^n and H_n. Here is one typical formula.

Theorem Let G be a group and let k be a commutative ring. Then there is a natural exact sequence of k-modules which splits

$$0 \longrightarrow \mathrm{Ext}_{\mathbb{Z}}^1(\mathrm{H}_{n-1}(G,\mathbb{Z}),k) \longrightarrow \mathrm{H}^n(G,k) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_n(G,\mathbb{Z}),k) \longrightarrow 0,$$

for all $n \in \mathbb{N}$ (where we interpret $\mathrm{H}_{n-1}(G,\mathbb{Z}) = 0$ if $n = 0$).

It may be asked why one considers the groups $\mathrm{H}^n(G,M)$ instead of the apparently more general $\mathrm{Ext}_{kG}^n(A,B)$: surely we are losing something. The next result shows that we lose

very little. Furthermore when we come to the Lyndon-Hochschild-Serre spectral sequence of a group extension, we will want to work with H^n rather than Ext^n.

Theorem Let k be a commutative ring, let $n \in \mathbb{N}$, let G be a group, and let A, B be kG-modules. If A is projective as a k-module, then $\mathrm{Ext}^n_{kG}(A, B) \cong \mathrm{H}^n(G, \mathrm{Hom}_k(A, B))$.

Of course, $\mathrm{Hom}_k(A, B)$ has the diagonal action as described at the end of the last chapter.

Proof Let (P, α_0) be a resolution of the kG-module k with projective kG-modules. Then $(P \otimes_k A, \alpha_0 \otimes 1)$ is a resolution (because A is projective and hence flat as a k-module) of the kG-module $k \otimes_k A \cong A$ and by the Theorem at the end of the last chapter, this resolution is projective as kG-modules. Therefore

$$\begin{aligned}\mathrm{Ext}^n_{kG}(A, B) &\cong \mathrm{H}^n(\mathrm{Hom}_{kG}(P \otimes_k A, B)) \cong \mathrm{H}^n(\mathrm{Hom}_{kG}(P, \mathrm{Hom}_k(A, B))) \\ &\cong \mathrm{Ext}^n_{kG}(k, \mathrm{Hom}_k(A, B)) \cong \mathrm{H}^n(G, \mathrm{Hom}_k(A, B))\end{aligned}$$

by Remark (7) of chapter 10.

Thus the only thing we have lost is that we need to assume that A is projective as a k-module. However one has long exact sequences and the Universal Coefficient Theorem available to deduce the more general results. In any case the hypothesis that A is projective as a k-module is vacuous if k is a field (and this is an important case).

There is a similar result for Tor_n and H_n which we now state; the proof is left to a future homework exercise (homework 12, prob. 7).

Theorem Let k be a commutative ring, let A be a right kG-module, let B be a left kG-module, and let $n \in \mathbb{N}$. If A is flat as a k-module, then $\mathrm{Tor}^{kG}_n(A, B) \cong \mathrm{H}_n(G, A \otimes_k B)$.

Transfer map If $H \leqslant G$ are groups, then the restriction map gives us a homomorphism $\mathrm{H}^r(G, M) \to \mathrm{H}^r(H, M)$; we would like to have a map going the other way, but this is not always available. However in the case $[G : H] < \infty$, we do have a map $\mathrm{H}^r(H, M) \to \mathrm{H}^r(G, M)$ which is called the transfer: it is often also called the corestriction. We define it as follows. Let $\{x_1, \ldots, x_n\}$ be a right transversal for H in G, and let (P, α_0) be a projective resolution of $\mathbb{Z}$ with $\mathbb{Z}G$-modules. Then for each $r \in \mathbb{N}$, we can define a map $\theta_r \colon \mathrm{Hom}_{\mathbb{Z}H}(P_r, M) \to \mathrm{Hom}_{\mathbb{Z}G}(P_r, M)$ by $\theta_r f = \sum_{i=1}^n x_i f x_i^{-1}$, or more explicitly

$$(\theta_r f)u = \sum_{i=1}^n f(ux_i^{-1})x_i \quad \text{for } u \in P_r.$$

Then the θ_r commute with the coboundary maps α_{r*} and hence they induce homomorphisms $\theta_{r*} \colon \mathrm{H}^r(H, M) \to \mathrm{H}^r(G, M)$, usually denoted $\mathrm{tr}_{H,G}$ and called the transfer homomorphism. The following Lemma is extremely important ($\mathrm{res}_{G,H}$ denotes the restriction map from G to H).

Lemma Let $H \leqslant G$ be groups with $n = [G : H] < \infty$, let $r \in \mathbb{N}$, let M be a $\mathbb{Z}H$-module, and let $\sigma \in \mathrm{H}^r(G, M)$. Then $\mathrm{tr}_{H,G} \mathrm{res}_{G,H} \sigma = n\sigma$.

Proof Let $\{x_1, \dots, x_n\}$ be a right transversal for H in G, and let (P, α_0) be a projective resolution of $\mathbb{Z}$. If $f \in \mathrm{Hom}_{\mathbb{Z}G}(P_r, M)$ is represented by σ, then $\sum_{i=1}^n x_i f x_i^{-1}$ is represented by $\mathrm{tr}_{H,G} \mathrm{res}_{G,H} \sigma$. Since $f \in \mathrm{Hom}_{\mathbb{Z}G}(P_r, M)$, $x_i f x_i^{-1} = f$ and hence $\sum_{i=1}^n x_i f x_i^{-1} = nf$. Therefore $\mathrm{tr}_{H,G} \mathrm{res}_{G,H} \sigma$ is represented by nf and the result follows.

Tenth Homework Due 9:00 a.m., Monday, October 31.

(1) Let (G_i) be a direct system of groups with $\varinjlim G_i = G$, let U be a $\mathbb{Z}G$-module, and let $n \in \mathbb{P}$. Prove that $\varinjlim \mathrm{Tor}_n^{\mathbb{Z}G_i}(U, \mathbb{Z}G) = 0$. (Roughly speaking the argument goes as follows: an element in $\varinjlim \mathrm{Tor}_n^{\mathbb{Z}G_i}(U, \mathbb{Z}G)$ is the image of an element $x \in \mathrm{Tor}_n^{\mathbb{Z}G_i}(U, \mathbb{Z}G_j)$ for some $j \geq i$, and then the image of $x \in \mathrm{Tor}_n^{\mathbb{Z}G_j}(U, \mathbb{Z}G_j)$ is zero.)

(2) Let H, G be groups, let $n \in \mathbb{N}$, let A be a right $\mathbb{Z}G$-module, and let B be a left $\mathbb{Z}G$-module. Sketch a proof that a group homomorphism $\theta\colon H \to G$ induces a natural map of abelian groups $\theta_{n*}\colon \mathrm{Tor}_n^{\mathbb{Z}H}(A, B) \to \mathrm{Tor}_n^{\mathbb{Z}G}(A, B)$. What does the map $\theta_{1*}\colon \mathrm{H}_1(H, \mathbb{Z}) \to \mathrm{H}_1(G, \mathbb{Z})$ correspond to group theoretically (i.e. in terms of the groups G, H not using H_1)?

(3) Let (G_i) be a direct system of groups with $\varinjlim G_i = G$ and let U be a right $\mathbb{Z}G$-module.
   (i) Prove that if P is a projective left $\mathbb{Z}G$-module and $n \in \mathbb{P}$, then $\varinjlim \mathrm{Tor}_n^{\mathbb{Z}G_i}(U, P) = 0$.
   (ii) Prove that if M is any left $\mathbb{Z}G$-module, then $\varinjlim \mathrm{Tor}_1^{\mathbb{Z}G_i}(U, M) \cong \mathrm{Tor}_1^{\mathbb{Z}G}(U, M)$.
   (iii) Prove that if M is any left $\mathbb{Z}G$-module, then $\varinjlim \mathrm{Tor}_n^{\mathbb{Z}G_i}(U, M) \cong \mathrm{Tor}_n^{\mathbb{Z}G}(U, M)$ for all $n \in \mathbb{N}$.

(4) Let G be a finite group. Prove that $\mathrm{H}^1(G, \mathbb{Z}) = 0$. Deduce that $\mathrm{H}^2(G, \mathfrak{g}) = 0$.

(5) Let $H \leqslant G$ be groups, and let T be a right transversal for H in G.
   (i) If $I \lhd_{\mathrm{r}} \mathbb{Z}H$, prove that $I\mathbb{Z}G = \bigoplus_{t \in T} It$.
   (ii) If $\mathfrak{g}$ is a finitely generated right ideal of $\mathbb{Z}G$, prove that G is a finitely generated group.
   (iii) If $\mathbb{Z}G$ is right Noetherian, prove that G has the maximum condition on subgroups (i.e. there does not exist a strictly ascending chain of subgroups $H_1 < H_2 < H_3 < \cdots \leqslant G$).

(6) Let G be a group. Prove that G is a finitely generated group if and only if $\mathfrak{g}$ is a finitely generated $\mathbb{Z}G$-module. (If $G = \langle g_1, g_2, \dots \rangle$, prove that $\mathfrak{g} = (g_1 - 1)\mathbb{Z}G + (g_2 - 1)\mathbb{Z}G + \cdots$.)

(7) Let $G = A \rtimes H$ be the split extension (i.e. semi-direct product) of the group A with the group H. Thus $A \lhd G$, $G = AH$ and $A \cap H = 1$. Let M be a $\mathbb{Z}G$-module with G acting trivially and let $n \in \mathbb{N}$. Prove that $\mathrm{H}^n(H, M)$ is naturally isomorphic to a subgroup of $\mathrm{H}^n(G, M)$, but show by example that $\mathrm{H}^n(A, M)$ need not be isomorphic to a subgroup of $\mathrm{H}^n(G, M)$.

(8) Let G be a group. Prove that $\mathrm{H}^0(G, \mathbb{Z}G) \cong \mathbb{Z}$ if G is finite, and that $\mathrm{H}^0(G, \mathbb{Z}G) = 0$ if G is infinite.

(9) Let $G \cong \mathbb{Z}$ be the infinite cyclic group and let M be a $\mathbb{Z}G$-module.

(i) Prove that $\mathbb{Z}G \cong \mathfrak{g}$ as $\mathbb{Z}G$-modules. (Let $G = \langle g \rangle$ and map 1 to $g - 1$.)
(ii) Prove that $\mathrm{H}^n(G, M) = 0$ for all $n \geq 2$.
(iii) Prove that G acting by right multiplication on $\mathbb{Z}G$ induces the identity action on $\mathrm{H}^1(G, \mathbb{Z}G)$.
(iv) Prove that $\mathrm{H}^1(G, \mathbb{Z}G)$ is a quotient of $\mathbb{Z}$.
(v) Prove that $\mathrm{H}^1(G, \mathbb{Z}G) \cong \mathbb{Z}$.

Monday, October 31

Chapter 12
Group Cohomology, further results

As a quick application of the transfer map at the end of the last chapter, we have the following.

Proposition Let G be a finite group, let M be a $\mathbb{Z}G$-module, let $n = |G|$, and let $r \in \mathbb{P}$. Then $n\,\mathrm{H}^r(G, M) = 0$ (thus if G is a finite group, all the cohomology groups $\mathrm{H}^r(G, M)$ are torsion groups for all $r > 0$).

Proof Let $\sigma \in \mathrm{H}^r(G, M)$ and let 1 denote the identity group. Then $\mathrm{res}_{G,1}\,\sigma = 0$ because $\mathrm{H}^r(1, M) = 0$ for all $r \in \mathbb{P}$. Therefore by the Lemma at the end of the last chapter, $n\sigma = 0$.

There is also a transfer map $\mathrm{H}_r(G, M) \to \mathrm{H}_r(H, M)$ when $H \leqslant G$: this will be covered in a future homework exercise (homework 13, prob. 4). In the case H is the Sylow p-subgroup of the finite group G and $r = 1$, this is an important tool in the theory of finite groups, especially in the theory of finite simple groups.

The Trace Map Let $H \leqslant G$ be groups. Since $\mathbb{Z}G$ is a free $\mathbb{Z}$-module with basis $\{g \mid g \in G\}$, we can define a map $\mathrm{tr}_H \colon \mathbb{Z}G \to \mathbb{Z}H$ by $\mathrm{tr}_H\, g = 0$ if $g \in G \backslash H$, and $\mathrm{tr}_H\, g = g$ if $g \in H$ (tr stands for "trace", and hopefully it will not be confused with the transfer map defined at the end of the last chapter); it is easy to verify that tr_H is both a right and left $\mathbb{Z}H$-map (though it will not be a $\mathbb{Z}G$-map). This map has many applications, one of which is the following result.

Theorem Let $H \leqslant G$ be groups and let M be a right $\mathbb{Z}H$-module. If $[G : H]$ is finite, then $\mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M) \cong M \otimes_{\mathbb{Z}H} \mathbb{Z}G$ as right $\mathbb{Z}G$-modules.

Proof Define $\theta \colon M \otimes_{\mathbb{Z}H} \mathbb{Z}G \to \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M)$ by $(\theta m \otimes \alpha)\beta = m\,\mathrm{tr}_H\,\alpha\beta$ for $m \in M$ and $\alpha, \beta \in \mathbb{Z}G$. Then it is easy to see that θ is a well defined $\mathbb{Z}G$-map. This map does not depend on $[G : H]$ being finite. To define a map the other way, first let T be a right transversal for H in G. Then for $f \in \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M)$, set $\phi f = \sum_{t \in T} f(t^{-1}) \otimes t \in M \otimes_{\mathbb{Z}H} \mathbb{Z}G$ (this does depend on $[G : H] < \infty$); we note that this does not depend on the choice of transversal T because $f((ht)^{-1}) \otimes ht = f(t^{-1}) \otimes t$. This is obviously a $\mathbb{Z}$-map: in fact it is a $\mathbb{Z}G$-map because for $f \in \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M)$ and $g \in G$,

$$\begin{aligned}(\phi f)g &= \sum_{t\in T}(f(t^{-1})\otimes t)g\\ &= \sum_{t\in T} f(gt^{-1})\otimes t\end{aligned}$$

by replacing t with tg in the summation and noting that $\{tg \mid t \in T\}$ is still a right transversal for H in G,

$$\begin{aligned}&= \sum_{t\in T}(fg)t^{-1}\otimes t\\ &= \phi(fg).\end{aligned}$$

Then for $\alpha = \sum_{t\in T}\alpha_t t \in \mathbb{Z}G$ ($\alpha_t \in \mathbb{Z}H$) and $m \in M$, $\phi\theta(m\otimes\alpha) = \sum_{t\in T} m(\mathrm{tr}_H\, \alpha t^{-1})\otimes t = \sum_{t\in T} m\alpha_t \otimes t = \sum_{t\in T} m\otimes \alpha_t t = m\otimes\alpha$. Therefore $\phi\theta$ is the identity map. Also for $f \in \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M)$, $(\theta\phi f)1 = \theta(\sum_{t\in T} f(t^{-1})\otimes t)1 = \sum_{t\in T} f(t^{-1})\,\mathrm{tr}_H\, t = f(1)$, hence $\theta\phi$ is also the identity. This establishes the required isomorphism.

Combining this Theorem with the first homework problem in the case $H = 1$, we obtain the following result which was mentioned earlier.

Theorem Let G be a finite group and let $n \in \mathbb{P}$. Then $\mathrm{H}^n(G, \mathbb{Z}G) = 0$.

Eleventh Homework Due 9:00 a.m., Monday, November 7.

(1) Let $H \leqslant G$ be groups, let M be a $\mathbb{Z}H$-module, let $n \in \mathbb{N}$, and let (P, α_0) be a projective resolution of $\mathbb{Z}$ with $\mathbb{Z}G$-modules.
   (i) Prove that $\mathrm{H}^n(H, M) \cong \mathrm{H}^n(\mathrm{Hom}_{\mathbb{Z}G}(P, \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M)))$ (use HW1 prob. 2(iii)).
   (ii) Prove that $\mathrm{H}^n(G, \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M)) \cong \mathrm{H}^n(H, M)$.

(2) This and the next problem are for those students who know a little about Lie algebras. Let k be a field, let $\mathfrak{g}$ be a Lie algebra over k, let $U\mathfrak{g}$ denote the universal enveloping algebra of $\mathfrak{g}$, and let $I\mathfrak{g}$ denote the augmentation ideal of $U\mathfrak{g}$. Thus if we view $\mathfrak{g}$ as a subspace of $U\mathfrak{g}$, then $I\mathfrak{g}$ is the ideal of $U\mathfrak{g}$ generated by $\mathfrak{g}$.
   (i) Prove that the natural injection $\iota\colon \mathfrak{g} \to U\mathfrak{g}$ maps $[\mathfrak{g},\mathfrak{g}]$ into $(I\mathfrak{g})^2$. Deduce that ι induces a map $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}] \to I\mathfrak{g}/(I\mathfrak{g})^2$.
   (ii) Prove that there is a k-module map $\tau\colon I\mathfrak{g} \to \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ which maps $(I\mathfrak{g})^2$ to 0. Deduce that τ induces a k-module map $I\mathfrak{g}/(I\mathfrak{g})^2 \to \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$.
   (iii) Prove that $I\mathfrak{g}/(I\mathfrak{g})^2 \cong \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$. (Same proof as for augmentation ideals of group rings).

(3) Let $\mathfrak{g}$ be a Lie algebra over a commutative ring k, and let M be a $\mathfrak{g}$-module. Then we may view M as a $U\mathfrak{g}$-module, and we define $\mathrm{H}^n(\mathfrak{g}, M) = \mathrm{Ext}^n_{U\mathfrak{g}}(k, M)$, where $I\mathfrak{g}$ is

acting trivially on k. If M is a trivial $\mathfrak{g}$-module (i.e. $M\,I\mathfrak{g} = 0$), prove that $\mathrm{H}^1(\mathfrak{g}, M) \cong \mathrm{Hom}_k(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}], M)$.

(4) Prove that $\mathrm{H}^4(\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}) \cong (\mathbb{Z}/3\mathbb{Z})^6 \oplus \mathbb{Z}/6\mathbb{Z}$.

(5) Let G, H be any groups, and let k be a commutative hereditary ring. Prove that there is a short exact sequence of k-modules

$$0 \longrightarrow \bigoplus_{r+s=n} \mathrm{H}_r(G,k) \otimes_k \mathrm{H}_s(H,k) \longrightarrow \mathrm{H}_n(G \times H, k) \longrightarrow \bigoplus_{r+s=n-1} \mathrm{Tor}_1^k(\mathrm{H}_r(G,k), \mathrm{H}_s(H,k)) \longrightarrow 0$$

which splits. (Of course this sequence is natural, but don't verify this.)

(6) Let k be a commutative ring, let G be a group, and let M be a $\mathbb{Z}G$-module which is free as a $\mathbb{Z}$-module. Prove that

$$\mathrm{H}_n(G, M \otimes_{\mathbb{Z}} k) \cong \mathrm{H}_n(G, M) \otimes_{\mathbb{Z}} k \oplus \mathrm{Tor}_1^{\mathbb{Z}}(\mathrm{H}_{n-1}(G, M), k)$$

as k-modules (if $n = 0$, interpret $\mathrm{H}_{n-1}(G, M) = 0$).

(7) Let $n, r \in \mathbb{N}$, and let $G = \mathbb{Z}^n$. Prove that $\mathrm{H}^r(G, \mathbb{Z}) \cong \mathbb{Z}^{\binom{n}{r}}$ where $\binom{n}{r}$ denotes the binomial coefficient $\dfrac{n!}{r!\,(n-r)!}$ (interpret $\mathbb{Z}^{\binom{n}{r}} = 0$ for $n < r$. First do the case $n = 1$ using HW10 prob. 9(i)). Deduce that $\mathrm{H}^r(G, k) \cong k^{\binom{n}{r}}$ as k-modules for any commutative ring k.

(8) Let G be a finitely generated free abelian group, let H be a finite group, and let $n \in \mathbb{N}$. Prove that $\mathrm{H}^n(G \times H, \mathbb{Z}) \cong \bigoplus_{r+s=n} \mathrm{H}^r(G, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathrm{H}^s(H, \mathbb{Z})$.

(9) Let G be a group, let k be a commutative ring, let A be a flat right kG-module, and let B be a left kG-module.
   (i) If P is a right kG-module, prove that $(P \otimes_k A) \otimes_{kG} B$ is naturally isomorphic to $P \otimes_{kG} (A \otimes_k B)$. (Just establish the isomorphism: it is not necessary to verify that it is natural.) Of course G is acting diagonally on $A \otimes_k B$ and $P \otimes_k A$.
   (ii) If P is a flat right kG-module, prove that $P \otimes_k A$ is also a flat right kG-module.
   (iii) Prove that $\mathrm{Tor}_n^{kG}(A, B) \cong \mathrm{H}_n(G, A \otimes_k B)$ for all $n \in \mathbb{N}$.

Monday, November 7

Chapter 13
Graded Algebras

Notation Let G be a group, let k be a commutative ring, and let M be a kG-module. Then it is standard to define $\mathrm{H}^*(G, M) = \bigoplus_{r \in \mathbb{N}} \mathrm{H}^r(G, M)$. Similarly if N is a left kG-module, then one defines $\mathrm{H}_*(G, N) = \bigoplus_{r \in \mathbb{N}} \mathrm{H}_r(G, N)$.

Group cohomology ($\mathrm{H}^*(G, M)$) is studied much more than group homology ($\mathrm{H}_*(G, N)$), and the reason seems to be that if k is a commutative ring, then $\mathrm{H}^*(G, k)$ can be made into

a ring, in fact a k-algebra, whereas there does not appear to be a corresponding construction for group homology. In general one cannot make $\mathrm{Ext}_R(M,N)$ into a ring. The fact that one can make $\mathrm{H}^*(G,k)$ into a ring depends crucially on the diagonal action of G on the tensor product of two kG-modules. Actually one can make this construction using bialgebras (Hopf algebras without an antipode), rather than just for group rings as we are going to do here.

So let G be a group, let k be a commutative ring, and let M be a kG-module. We are going to make $\mathrm{H}^*(G,k)$ into an anticommutative graded k-algebra, and $\mathrm{H}^*(G,M)$ into a graded $\mathrm{H}^*(G,k)$-module. Let us recall what is meant by an anticommutative graded algebra, and what is meant by a graded module.

Definition Let k be a commutative ring, and let A be a k-algebra (not necessarily commutative). Suppose we can write $A = \bigoplus_{r\in\mathbb{N}} A_r$ (as a k-module) such that $A_rA_s \subseteq A_{r+s}$ for all $r,s \in \mathbb{N}$. Then we say that A is a graded k-algebra. The elements A_r are called the homogeneous elements of degree r. To say that $a \in A$ has degree r will mean that a is homogeneous of degree r; i.e. $a \in A_r$.

Examples

(i) Any algebra A can be trivially graded by setting $A_0 = A$ and $A_r = 0$ for all $r \in \mathbb{P}$.

(ii) Any polynomial ring $k[X_1, X_2, \dots]$ is a graded algebra, by letting A_r be the homogeneous polynomials of total degree r. Thus for $k[X,Y]$, we let $A_0 = k$, $A_1 = kX + kY$, $A_2 = kX^2 + kXY + kY^2$, $A_3 = kX^3 + kX^2Y + kXY^2 + kY^3$, and in general $A_r = kX^r + kX^{r-1}Y + \cdots + kY^r$.

Exercise Let k be a commutative ring, and let $A = \bigoplus_{r\in\mathbb{N}} A_r$ be a graded k-algebra. Prove that A_0 is a subring of A, and that $\bigoplus_{r\geq n} A_r$ is a two-sided ideal of A for all $n \in \mathbb{N}$.

If A is a graded k-algebra, then is should be obvious what the definition of a graded A-module is.

Definition Let k be a commutative ring, let A be a graded k-algebra, and let M be an A-module. Then M is a graded A-module means that we can write $M = \bigoplus_{r\in\mathbb{N}} M_r$ (as k-modules) such that $M_rA_s \subseteq M_{r+s}$ for all $r,s \in \mathbb{N}$. The elements of M_r are called the homogeneous elements of degree r of M

Examples

(i) If A is a graded k-algebra, then A is itself a graded A-module.

(ii) Let $A = k[X_1, X_2, \dots]$ be a polynomial ring, graded by total degree as above, and let k denote the A-module with all the X_i acting trivially (so $bX_i = 0$ for all $b \in k$ and for all i). Then $M = k$ is a graded A-module with $M_0 = k$ and $M_r = 0$ for all $r \in \mathbb{P}$.

Exercise Let k be a commutative ring and let A be a graded k-algebra. Prove that $\bigoplus_{r=n}^{\infty} A_r$ is a graded A-module for all $n \in \mathbb{N}$.

It is obvious what is meant by a commutative graded k-algebra: it is a graded k-algebra which is commutative. However the algebra $\mathrm{H}^*(G,k)$ turns out not to be commutative, but *anticommutative*. This means that if $A = \bigoplus_{r\in\mathbb{N}} A_r$ is an anticommutative graded k-algebra,

then $a_r a_s = (-1)^{rs} a_s a_r$ for all $a_r \in A_r$ and for all $a_s \in A_s$ $(r, s \in \mathbb{N})$. The reason for $\mathrm{H}^*(G, k)$ being anticommutative rather than commutative is the sign involved when defining the tensor product of two chain complexes, as we shall see later.

Examples

(i) If A is a commutative graded k-algebra, then $\bigoplus_{r\in\mathbb{N}} A_{2r}$ is an anticommutative graded subalgebra of A. Also if A has no elements of odd degree, then the concepts of commutative and anticommutative coincide.

(ii) If k is a field of characteristic 2, then the concepts of commutative and anticommutative coincide.

(iii) Let k be a commutative ring, let A be the free k-module of rank 4 with basis $\{1, a, b, c\}$, and we make A into a k-algebra by defining a multiplication as follows. 1 will be the identity, thus $11 = 1$, $1a = a = a1$, $1b = b = b1$ and $1c = c = c1$. Also $a^2 = b^2 = c^2 = ac = ca = bc = cb = 0$, and $ab = -ba = c$. We grade A so that $k1$ are the homogeneous elements of degree 0, $ka + kb$ are the homogeneous elements of degree 1, and kab are the homogeneous elements of degree 2. Then A is an anticommutative graded k-algebra.

Exercise Verify the above statement, that indeed A is an anticommutative graded k-algebra (you need to check the associative law).

In fact there is an easy construction of A as follows. Let B be the k-algebra with basis $\{1, x\}$, where the multiplication is defined by $11 = 1$, $1x = x = x1$, and $x^2 = 0$. It is easy to check that this yields a k-algebra structure on B (associative law is easy to check in this case). In fact $B \cong k[X]/(X^2)$ as k-algebras. We can make B into an anticommutative graded k-algebra by letting $k1$ be the homogeneous elements of degree 0, and kx be the homogeneous elements of degree 1. Then $A \cong B \otimes_k B$ as anticommutative k-algebras, where the tensor product of anticommutative graded k-algebras is as defined in the next paragraph.

If A, B are anticommutative graded k-algebras, then their tensor product becomes an anticommutative graded k-algebra with product

$$(a \otimes b)(a' \otimes b') = (-1)^{\deg b \deg a'} aa' \otimes bb',$$

where b, a' are homogeneous elements of B and A respectively. It is then routine to check that $A \otimes_k B$ is indeed an anticommutative graded k-algebra. Perhaps the only thing which is not immediately obvious is the associative law. First we verify it for homogeneous elements: suppose $a, a', a'' \in A$ and $b, b', b'' \in B$ are homogeneous of degrees r, r', r'' and s, s', s'' respectively. Then a computation shows that

$$\big((a \otimes b)(a' \otimes b')\big)(a'' \otimes b'') = (-1)^{r's + r''s + r''s'} aa'a'' \otimes bb'b'' = (a \otimes b)\big((a' \otimes b')(a'' \otimes b'')\big).$$

The case for general (not necessarily homogeneous elements) now follows from the k-bilinearity of the tensor product. It is a general technique that when one is trying to prove something about graded algebras, often one needs only check that the required equalities hold for the homogeneous elements.

Exercise Let k be a commutative ring, and let A, B be graded anticommutative k-algebras. Prove that the map $\tau\colon A \otimes_k B \to B \otimes_k A$ defined by $\tau a \otimes b = (-1)^{\deg a \deg b} b \otimes a$ (a, b homogeneous) is an isomorphism of graded k-algebras.

Cup Products After the digression into graded algebras, we can now return to cup products. Let

$$(P, \varepsilon)\colon \cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

be a projective resolution of $\mathbb{Z}$ with $\mathbb{Z}G$-modules. Then by the Künneth formula and the Theorem at the end of the tenth chapter, $(P \otimes_{\mathbb{Z}} P, \varepsilon \otimes \varepsilon)$ is a projective resolution of $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}$ with projective $\mathbb{Z}G$-modules, where G is acting diagonally. We also have a natural isomorphism of $\mathbb{Z}G$-modules $\mu\colon \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \to \mathbb{Z}$ defined by $\mu a \otimes b = ab$ for $a, b \in \mathbb{Z}$. Thus if $\pi = \mu(\varepsilon \otimes \varepsilon)$, then $(P \otimes_{\mathbb{Z}} P, \pi)$ is a projective resolution of $\mathbb{Z}$ with $\mathbb{Z}G$-modules. By Lemma 1 of chapter 2, there exists a chain map

$$\theta\colon P \to P \otimes_{\mathbb{Z}} P \tag{1}$$

extending the identity map on $\mathbb{Z}$.

Twelfth Homework Due 9:00 a.m., Monday, November 14.

(1) Let G be a finite cyclic group, and suppose $0 \to A \xrightarrow{\alpha} B \to C \to 0$ is an exact sequence of $\mathbb{Z}G$-modules.
   (i) Prove that there is an isomorphism $\theta\colon \mathrm{H}^1(G, A) \to \mathrm{H}^3(G, A)$ such that $\theta(\ker \alpha_{1*}) = \ker \alpha_{3*}$. ($\alpha_{i*}$ is the map $\mathrm{H}^i(G, A) \to \mathrm{H}^i(G, B)$.)
   (ii) Prove that there is an exact hexagon of groups

$$\begin{array}{ccccc} & & \mathrm{H}^1(G,A) & \longrightarrow & \mathrm{H}^1(G,B) \\ & \nearrow & & & \searrow \\ \mathrm{H}^2(G,C) & & & & \mathrm{H}^1(G,C) \\ & \nwarrow & & & \swarrow \\ & & \mathrm{H}^2(G,B) & \longleftarrow & \mathrm{H}^2(G,A) \end{array}$$

(2) Let G be a finite cyclic group. Then an important tool in number theory is the *Herbrand quotient* $h(A)$, defined as follows. Let A be a $\mathbb{Z}G$-module such that the cohomology groups $\mathrm{H}^1(G, A)$ and $\mathrm{H}^2(G, A)$ are both finite. Then we set $h(A) = |\mathrm{H}^2(G, A)|/|\mathrm{H}^1(G, A)|$ (so $h(A)$ is some rational number).
   (i) Prove that if $0 \to A \to B \to C \to 0$ is an exact sequence of abelian groups, then $|B| = |A||C|$ (assume for simplicity that A, B, C are all finite, though the result is still true without this hypothesis if correctly interpreted).
   (ii) Let $0 \to A \to B \to C \to 0$ be an exact sequence of $\mathbb{Z}G$-modules, and suppose that $\mathrm{H}^i(G, A)$, $\mathrm{H}^i(G, B)$ and $\mathrm{H}^i(G, C)$ are finite for $i = 1, 2$. By splitting the exact hexagon of the previous problem up into short exact sequences, prove that $h(B) = h(A)h(C)$.

(3) Let G be a finite group.
   (i) By considering the exact sequence of $\mathbb{Z}G$-modules $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ (where G acts trivially on each term), prove that $\mathrm{H}^2(G, \mathbb{Z}) \cong \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$.
   (ii) If $n \in \mathbb{P}$, prove that $\mathrm{H}^2(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$ (using (i), but not the formula I gave in class for the cohomology of a cyclic group).
   (iii) Using the Universal Coefficient Theorem for group cohomology, deduce from (ii) that $\mathrm{H}^2(\mathbb{Z}/n\mathbb{Z}, k) \cong k/nk$ as k-modules. (Again, don't use the formula I gave in class for the cohomology of a cyclic group, but you may assume that $\mathrm{H}^3(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = 0$.)

(4) Let G be a finite group. Using the techniques and results of the previous problem, prove
   (i) $\mathrm{H}^2(G, \mathbb{Z}) \cong G/G'$.
   (ii) Prove that $\mathrm{H}^r(G, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{H}^{r+1}(G, \mathbb{Z})$ for all $r \in \mathbb{P}$.
   (iii) Prove that $\mathrm{H}^{r+1}(G, \mathbb{Z}) \cong \varinjlim \mathrm{H}^r(G, \mathbb{Z}/n\mathbb{Z})$ for all $r \in \mathbb{P}$. Here $(\mathbb{Z}/m\mathbb{Z}, f_m^n)$ is the direct system of cyclic groups where there is a map $f_m^n \colon \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ if and only if $m|n$, and all the f_m^n are monomorphisms.

(5) Let k be a commutative ring and let M be a right kG-module.
   (i) Prove that $M \otimes_k kG \cong M \otimes_k kG$ as kG-modules, where on the left G is acting diagonally, and on the right M is viewed as a k-module. On both sides, kG is a right kG-module via right multiplication by kG. (Map $m \otimes g$ to $mg^{-1} \otimes g$.)
   (ii) Prove that if M is free as a k-module and F is a free kG-module, then $M \otimes_k F$ (where G is acting diagonally) is a free kG-module.
   (iii) Prove that if M is projective as a k-module, and P is a projective kG-module, then $M \otimes_k P$ is a projective kG-module (where G is acting diagonally).

(6) Let p be a prime, let k be a field of characteristic p, and let $G = H \rtimes P$ be the split extension of the normal subgroup H with the subgroup P. If P is the Sylow p-subgroup of G, prove that $\mathrm{H}^n(G, k) \cong \mathrm{H}^n(P, k)$ as k-modules for all $n \in \mathbb{N}$.

(7) Let p be a prime, let k be a field of characteristic p, let G be a finite p-group, let M be a $\mathbb{Z}G$-module which is projective as a $\mathbb{Z}$-module, and let $n \in \mathbb{N}$. If $\mathrm{H}^n(G, M \otimes_{\mathbb{Z}} k) = 0$, prove that $\mathrm{H}^{n+1}(G, M) = 0$.

(8) Let $1 \leq p \in \mathbb{R}$. One defines L^p-cohomology groups $\bar{\mathrm{H}}^n(G, M)$ in the same way as the ordinary cohomology groups except that one requires M to be a Banach space such that the action of G on M is continuous, that maps are continuous with respect to the L^p-norm, and that one replaces the image with the closure of the image.

   It turns out that $\bar{\mathrm{H}}^0(G, M)$ is M^G (just as in the ordinary cohomology case, and the proof is the same). Let $\mathrm{L}^p(G)$ denote the Banach space $\{\sum_{g \in G} a_g g \mid a_g \in \mathbb{C} \text{ and } \sum_{g \in G} |a_g|^p < \infty\}$. Then G acts (continuously) by right multiplication on $\mathrm{L}^p(G)$: $\big(\sum_{g \in G} a_g g\big)x = \sum_{g \in G} a_g gx$ for $x \in G$, and in this way $\mathbb{C}G$ is a $\mathbb{C}G$-submodule of $\mathrm{L}^p(G)$ (in fact, $\mathrm{L}^p(G)$ is the closure of $\mathbb{C}G$ in the L^p-norm).

   Prove that $\bar{\mathrm{H}}^0(G, \mathrm{L}^p(G)) = 0$ if and only if G is infinite.

(9) Let $H \leqslant G$ be groups such that $H \cong \mathbb{Z}$, and let M be a $\mathbb{Z}G$-module. Suppose $[G : H] = n < \infty$.

(i) Prove that $\mathrm{H}^r(H, M) = 0$ for all $r > 1$ (use HW10 prob. 9(i)).

(ii) Prove that $n\,\mathrm{H}^r(G, M) = 0$ for all $r > 1$.

Monday, November 14

Chapter 14
Cup Products

Let M be a $\mathbb{Z}G$-module and let k be a commutative ring (one possibility that we shall be examining presently is the case $M = k$; when considering k as a $\mathbb{Z}G$-module or kG-module, unless otherwise stated, we shall always assume that the action of G on k is trivial: i.e. $ag = a$ for all $a \in A$ and $g \in G$). We shall calculate $\mathrm{H}^*(G, M)$ and $\mathrm{H}^*(G, k)$ using the projective resolution P of $\mathbb{Z}$ at the end of the last chapter. Suppose $u \in \mathrm{H}^r(G, M)$ and $x \in \mathrm{H}^s(G, k)$. We want to define the product ux: this will make $\mathrm{H}^*(G, k)$ into a ring in the case $M = k$, and $\mathrm{H}^*(G, M)$ into a right $\mathrm{H}^*(G, k)$-module. Choose $f \in \mathrm{Hom}_{\mathbb{Z}G}(P_r, M)$ and $g \in \mathrm{Hom}_{\mathbb{Z}G}(P_s, k)$ representing u and x respectively (thus we assume that $\alpha^*_{r+1} f = 0 = \beta^*_{s+1} g$). Then $f \otimes g \in \mathrm{Hom}_{\mathbb{Z}G}(P_r \otimes_{\mathbb{Z}} P_s, M)$, where $(f \otimes g)(a \otimes b) = f(a)\, g(b)$ for $a \in P_r$ and $b \in P_s$. Let us calculate the boundary of $f \otimes g$: it is

$$\alpha^*_{r+1} f \otimes g + (-1)^r f \otimes \beta^*_{s+1} g = 0.$$

Therefore $f \otimes g$ represents an element $u'x'$ of $\mathrm{H}^{r+s}(\mathrm{Hom}_{\mathbb{Z}G}(P \otimes_{\mathbb{Z}} P, M))$. It is not difficult to see that this element depends only on u and x, and not on the choice of f and g. Using Lemma 2 of chapter 2, it follows that $\theta^*(u'x')$ is a well defined element of $\mathrm{H}^{r+s}(\mathrm{Hom}_{\mathbb{Z}G}(P, M))$, which does not depend on the choice of θ. Therefore $(f \otimes g)\theta$ represents a well defined element of $\mathrm{H}^{r+s}(G, M)$, which we shall denote by ux.

We make the following two notes.

(i) ux does not depend on the choice of f, g, and θ: it only depends on u and x.

(ii) It is important to compute ux using the same resolution of $\mathbb{Z}$ as used for u and x.

If v is an arbitrary element of $\mathrm{H}^*(G, M)$, then we shall use the notation v_i to denote the ith component of v: thus $v_i \in \mathrm{H}^i(G, M)$ and $v = \sum_{i \in \mathbb{N}} v_i$. We can now define the cup product vy of arbitrary elements in $\mathrm{H}^*(G, M)$ and $\mathrm{H}^*(G, k)$ respectively by

$$(vy)_r = \sum_{i+j=r} v_i y_j.$$

There is an obvious way to give $\mathrm{H}^*(G, M)$ a grading: namely we let the homogeneous elements of degree n be the elements of $\mathrm{H}^n(G, M)$. We now have the following theorem.

Theorem Let G be a group, let k be a commutative ring, let M be a right kG-module, and let $\pi\colon \mathbb{Z} \to k$ be the unique ring homomorphism (so $\pi a = a1$ for all $a \in \mathbb{Z}$). Then $\mathrm{H}^*(G, k)$ is a graded anticommutative k-algebra with a 1, and $\mathrm{H}^*(G, M)$ is a graded right $\mathrm{H}^*(G, k)$-module. If (P, α_0) is a projective resolution for $\mathbb{Z}$ with $\mathbb{Z}G$-modules and $e \in \mathrm{H}^0(G, k)$

represents $\pi\alpha_0 \in \mathrm{Hom}_{\mathbb{Z}G}(P_0, k)$, then $ue = u$ for all $u \in \mathrm{H}^*(G, M)$. (Thus $\mathrm{H}^*(G, k)$ has a 1 and $\mathrm{H}^*(G, M)$ is a unital $\mathrm{H}^*(G, k)$-module.)

Proof We shall leave the proof of the properties of e as an exercise. Everything else is now clear except the anticommutativity: we must prove that if $x \in \mathrm{H}^r(G, k)$ and $y \in \mathrm{H}^s(G, k)$, then $xy = (-1)^{rs}yx$.

Let (P, α_0) be a projective resolution for $\mathbb{Z}$ with $\mathbb{Z}G$-modules, let $f \in \mathrm{Hom}_{\mathbb{Z}G}(P_r, k)$ represent x, let $g \in \mathrm{Hom}_{\mathbb{Z}G}(P_s, k)$ represent y, and let $\mu\colon \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \to \mathbb{Z}$ denote the $\mathbb{Z}G$-module map defined by $\mu(a \otimes b) = ab$. Then $(P \otimes_{\mathbb{Z}} P, \mu(\alpha_0 \otimes \alpha_0))$ is also a projective resolution of $\mathbb{Z}$ with $\mathbb{Z}G$-modules, so there exists a chain map $\theta\colon P \to P \otimes_{\mathbb{Z}} P$ extending the identity map on $\mathbb{Z}$. By definition, $(f \otimes g)\theta$ and $(g \otimes f)\theta \in \mathrm{Hom}_{\mathbb{Z}G}(P_{r+s}, k)$ represent xy and $yx \in \mathrm{H}^{r+s}(G, k)$ respectively, and application of Lemma 2 of chapter 2 shows that

$$\theta^*\colon \mathrm{H}^{r+s}(\mathrm{Hom}_{\mathbb{Z}G}(P \otimes_{\mathbb{Z}} P, k)) \longrightarrow \mathrm{H}^{r+s}(\mathrm{Hom}_{\mathbb{Z}G}(P, k))$$

is an isomorphism. Therefore we want to show that $f \otimes g$ and $(-1)^{rs}g \otimes f$ represent the same element in $\mathrm{H}^{r+s}(\mathrm{Hom}_{\mathbb{Z}G}(P \otimes_{\mathbb{Z}} P, , k))$.

Define a chain map $\tau\colon P \otimes_{\mathbb{Z}} P \to P \otimes_{\mathbb{Z}} P$ by $\tau(a \otimes b) = (-1)^{rs}(b \otimes a)$ for $a \in P_r$ and $b \in P_s$ (we need the $(-1)^{rs}$ to ensure that τ commutes with the boundary maps). Then $(f \otimes g)\tau = (-1)^{rs}(g \otimes f)$, and the induced map

$$\tau^*\colon \mathrm{H}^{r+s}(\mathrm{Hom}_{\mathbb{Z}G}(P \otimes_{\mathbb{Z}} P)) \longrightarrow \mathrm{H}^{r+s}(\mathrm{Hom}_{\mathbb{Z}G}(P \otimes_{\mathbb{Z}} P))$$

is the identity by Lemma 2 of chapter 2. This proves the result.

Exercise Prove that $ue = u$ for all $u \in \mathrm{H}^*(G, M)$ in the above Theorem. Here is an outline of what to do.

(i) By dimension shifting, we may assume that $u \in \mathrm{H}^0(G, M)$ (you will need to use the fact that every module can be embedded in an injective module).
(ii) Prove that $ue = u$ for all $u \in \mathrm{H}^0(G, M)$ in the case $P_0 = \mathbb{Z}G$ and $\theta_0 g = g \otimes g$ for all $g \in G$.
(iii) Prove that $ue = u$ for all $u \in \mathrm{H}^0(G, M)$ for arbitrary P_0.

Various group maps between the various $\mathrm{H}^n(G, M)$ are in fact ring and/or module maps, and a large number of results can now be read off. For example

Theorem (Künneth theorem) Let G, H be groups, and let k be a commutative hereditary ring. Suppose G is of type FP_∞ over k. Then there is a natural monomorphism of anticommutative graded k-algebras $\pi\colon \mathrm{H}^*(G, k) \otimes_k \mathrm{H}^*(H, k) \to \mathrm{H}^*(G \times H, k)$. If k is a field, then π is an epimorphism.

Remarks

(i) A homomorphism of graded k-algebras will normally mean an algebra homomorphism which respects the grading. In this case it is obvious that π respects the grading.

(ii) When k is a field, it follows from the above Theorem that once we have calculated the ring structure of $\mathrm{H}^*(G, k)$ for G cyclic, then we have calculated the ring structure of $\mathrm{H}^*(G, k)$

for any finitely generated abelian group (since any finitely generated abelian group is a direct product of cyclic groups).

(iii) Unfortunately there does not appear to be any useful ring structure on the Tor term in the Künneth formula, so if k is a commutative hereditary ring which is not a field (and the important case here is when $k = \mathbb{Z}$), it is not clear how to calculate the ring structure of $\mathrm{H}^*(G \times H, k)$ from that of $\mathrm{H}^*(G, k)$ and $\mathrm{H}^*(H, k)$. In fact when k is a field, in general the cohomology rings with coefficients in $\mathbb{Z}$ are far more complicated than the cohomology rings with coefficients in k.

Proof of Theorem This is just the result at the bottom of the first lemma of chapter 11; all that we need to do is to verify that the map π respects multiplication (i.e. the cup product). Since π respects addition (because it is a group homomorphism), it will be sufficient to check this on homogeneous elements.

Let (P, α_0) be a projective resolution of $\mathbb{Z}$ with $\mathbb{Z}G$-modules, and let (Q, β_0) be a projective resolution of $\mathbb{Z}$ with $\mathbb{Z}H$-modules. Define $\mu\colon \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \to \mathbb{Z}$ by $\mu(a \otimes b) = ab$ for $a, b \in \mathbb{Z}$. then $(P \otimes_{\mathbb{Z}} P, \mu(\alpha_0 \otimes \alpha_0))$ is a projective resolution of $\mathbb{Z}$ with $\mathbb{Z}G$-modules, and $(Q \otimes_{\mathbb{Z}} Q, \mu(\beta_0 \otimes_{\mathbb{Z}} \beta_0))$ is a projective resolution of $\mathbb{Z}$ with $\mathbb{Z}H$-modules. Let $\theta\colon P \to P \otimes_{\mathbb{Z}} P$ and $\phi\colon Q \to Q \otimes_{\mathbb{Z}} Q$ be chain maps extending the identity on $\mathbb{Z}$.

Let $r, s \in \mathbb{N}$, let $u \in \mathrm{H}^r(G, k)$ be represented by $f \in \mathrm{Hom}_{\mathbb{Z}G}(P_r, k)$, let $u' \in \mathrm{H}^{r'}(G, k)$ be represented by $f' \in \mathrm{Hom}_{\mathbb{Z}G}(P_{r'}, k)$, let $v \in \mathrm{H}^s(G, k)$ be represented by $g \in \mathrm{Hom}_{\mathbb{Z}G}(P_s, k)$, and let $v' \in \mathrm{H}^{s'}(G, k)$ be represented by $g' \in \mathrm{Hom}_{\mathbb{Z}G}(P_{s'}, k)$. We need to show that $\pi\big((u \otimes v)(u' \otimes v')\big) = \pi(u \otimes v)\pi(u' \otimes v')$. By definition, $\pi(u \otimes v)$ is represented by $f \otimes g \in \mathrm{Hom}_{\mathbb{Z}G}((P_r \otimes_{\mathbb{Z}} Q_s), k)$ and $\pi(u' \otimes v')$ is represented by $f' \otimes g' \in \mathrm{Hom}_{\mathbb{Z}G}((P_{r'} \otimes_{\mathbb{Z}} Q_{s'}), k)$. Define a chain map

$$\tau\colon P \otimes_{\mathbb{Z}} P \otimes_{\mathbb{Z}} Q \otimes_{\mathbb{Z}} Q \longrightarrow P \otimes_{\mathbb{Z}} Q \otimes_{\mathbb{Z}} P \otimes_{\mathbb{Z}} Q$$

by $\tau(p \otimes p' \otimes q \otimes q') = (-1)^{\deg p \deg q'} p \otimes q \otimes p' \otimes q'$, where p, p' and q, q' are homogeneous elements of P and Q respectively (we need the $(-1)^{\deg p \deg q}$ to ensure that τ commutes with the boundary maps). Then by definition of the cup product, $(u \otimes v)(u' \otimes v')$ is represented by

$$(f \otimes g \otimes f' \otimes g')\tau(\theta \otimes \phi) \in \mathrm{Hom}_{\mathbb{Z}[G \times H]}(P_{r+r'} \otimes_{\mathbb{Z}} Q_{s+s'}, k).$$

Also uu' is represented by $(f \otimes f')\theta \in \mathrm{Hom}_{\mathbb{Z}G}(P_{r+r'}, k)$, and vv' is represented by $(g \otimes g')\phi \in \mathrm{Hom}_{\mathbb{Z}H}(Q_{s+s'}, k)$. Therefore $\pi(uu' \otimes vv')$ is represented by

$$(f \otimes f' \otimes g \otimes g')(\theta \otimes \phi) \in \mathrm{Hom}_{\mathbb{Z}[G \times H]}(P_{r+r'} \otimes_{\mathbb{Z}} Q_{s+s'}, k).$$

But $(f \otimes g \otimes f' \otimes g')\tau = (-1)^{r's}(f \otimes f' \otimes g \otimes g')$ and $(u \otimes v)(u' \otimes v') = (-1)^{r's} uu' \otimes vv'$, and the result follows.

We list some further properties of the cup product.

Theorem Let G, H be a groups, let k be a commutative ring, let L, M, N be kG-modules, let $u \in \mathrm{H}^*(G, M)$, and let $x \in \mathrm{H}^*(G, k)$.

(i) If $\theta\colon H \to G$ is a homomorphism, then $\theta^*(ux) = \theta^*(u)\theta^*(x)$.

(ii) If $\phi\colon M \to N$ is a kG-map, then $\phi_*(ux) = \phi_*(u)x$.
(iii) If $0 \to L \to M \to N \to 0$ is exact and $\delta\colon \mathrm{H}^*(G,N) \to \mathrm{H}^*(G,L)$ is the map induced by this sequence, then $\delta(vx) = (\delta v)x$ for $v \in \mathrm{H}^*(G,N)$.
(iv) If $H \leqslant G$ and $[G:H] < \infty$, then $\mathrm{tr}_{H,G}((\mathrm{res}_{G,H}\, u)y) = u\,\mathrm{tr}_{H,G}\, y$ for $y \in \mathrm{H}^*(H,k)$.

The Bockstein Map This is an important map in group cohomology. Among other things, it can facilitate the calculation of the cohomology of a cyclic group, especially its ring structure. There are several versions of the Bockstein, the most useful seems to be the following.

Definition of the Bockstein Map Let p be a prime, let k be the field $\mathbb{Z}/p\mathbb{Z}$, and let G be a group. Then we have a short exact sequence of $\mathbb{Z}G$-modules $0 \to k \to \mathbb{Z}/p^2\mathbb{Z} \to k \to 0$ (where G acts trivially on each term), hence the long exact sequence for cohomology yields an exact sequence

$$\cdots \longrightarrow \mathrm{H}^n(G,k) \xrightarrow{\beta_n} \mathrm{H}^{n+1}(G,k) \longrightarrow \cdots,$$

and we call β_n the Bockstein map. We can extend β_n to be defined on $\mathrm{H}^n(G,K)$ for an arbitrary field K of characteristic p by using a variant of the Universal Coefficient Theorem for group cohomology of chapter 11.

Exercise Let $k \subseteq K$ be fields, let G be a group of type FP_∞ over k, and let M be a kG-module. Prove that $\mathrm{H}^r(G, M \otimes_k K) \cong \mathrm{H}^r(G,M) \otimes_k K$ for all $r \in \mathbb{N}$.

Thus for an arbitrary field K of characteristic p, we define the Bockstein (still denoted β_n) from $\mathrm{H}^n(G,K)$ to $\mathrm{H}^{n+1}(G,K)$ to be $\beta_n \otimes 1$, though we need in this case to assume that G is of type FP_∞ over k.

How is β_n defined at the cochain level? Let

$$(P,\alpha_0)\colon \cdots \xrightarrow{\alpha_2} P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} \mathbb{Z} \longrightarrow 0$$

be a projective resolution of $\mathbb{Z}$ with $\mathbb{Z}G$-modules, and let $u \in \mathrm{H}^n(G,k)$ be represented by $f \in \mathrm{Hom}_{\mathbb{Z}G}(P_n,k)$ (so we assume that $f\alpha_{n+1} = 0$). Since P is projective, we may lift f to a map $\hat{f} \in \mathrm{Hom}_{\mathbb{Z}G}(P_n, \mathbb{Z}/p^2\mathbb{Z})$. Then (see chapter 5) $\hat{f}\alpha_{n+1}\colon P_{n+1} \to \mathbb{Z}/p^2\mathbb{Z}$ has image contained in $p\mathbb{Z}/p^2\mathbb{Z} = k$, because $f\alpha_{n+1} = 0$. Then $\hat{f}\alpha_{n+1} \in \mathrm{Hom}_{\mathbb{Z}G}(P_{n+1},k)$ represents $\beta_n u$. We now have the following properties of the Bockstein map.

Theorem Let G be a group, let p be a prime, and let K be a field of characteristic p. Assume that G is of type of FP_∞ in the case $K \neq \mathbb{Z}/p\mathbb{Z}$.

(i) $\beta_{n+1}\beta_n = 0$ (because $\alpha_{n+1}\alpha_{n+2} = 0$).
(ii) $\beta_0 = 0$.
(iii) Suppose $r \in \mathbb{N}$, $x \in \mathrm{H}^r(G,K)$ and $y \in \mathrm{H}^*(G,K)$. Then $\beta(xy) = (\beta x)y + (-1)^r x\beta y$.

Proof We shall leave the proofs of (i) and (ii) as exercises, and just prove (iii).

We may assume that $K = k = \mathbb{Z}/p\mathbb{Z}$ and that y is homogeneous of degree s for some $s \in \mathbb{N}$. Let

$$(P,\alpha_0)\colon \cdots \xrightarrow{\alpha_2} P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} \mathbb{Z} \longrightarrow 0$$

be a projective resolution of $\mathbb{Z}$ with $\mathbb{Z}G$-modules, let $f \in \mathrm{Hom}_{\mathbb{Z}G}(P_r, k)$ and $g \in \mathrm{Hom}_{\mathbb{Z}G}(P_s, k)$ represent x and y respectively, and let $\mu\colon \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \to \mathbb{Z}$ be the map defined by $\mu(a \otimes b) = ab$. Then $(P \otimes_{\mathbb{Z}} P, \mu(\alpha_0 \otimes \alpha_0))$ is also a projective resolution of $\mathbb{Z}$, so there exists a chain map $\theta\colon P \to P \otimes_{\mathbb{Z}} P$ extending the identity map on $\mathbb{Z}$. Then xy is represented by $(f \otimes g)\theta \in \mathrm{Hom}_{\mathbb{Z}G}(P_{r+s}, k)$.

Lift f and g to elements $\hat{f}$ and $\hat{g}$ of $\mathrm{Hom}_{\mathbb{Z}G}(P_r, \mathbb{Z}/p^2\mathbb{Z})$ and $\mathrm{Hom}_{\mathbb{Z}G}(P_s, \mathbb{Z}/p^2\mathbb{Z})$ respectively. Then $(\hat{f} \otimes \hat{g})\theta \in \mathrm{Hom}_{\mathbb{Z}G}(P_{r+s}, \mathbb{Z}/p^2\mathbb{Z})$ lifts $(f \otimes g)\theta$ and so $\beta_{r+s}(xy)$ is represented by $(\hat{f} \otimes \hat{g})\theta_{r+s}\alpha_{r+s+1} \in \mathrm{Hom}_{\mathbb{Z}G}(P_{r+s+1}, k)$. Let ∂ denote the boundary map on $P \otimes_{\mathbb{Z}} P$; thus $\partial_{r+s}(u \otimes v) = \alpha_r u \otimes v + (-1)^r u \otimes \alpha_s v$ for $u \in P_r$ and $v \in P_s$. Then $\beta_{r+s}(xy)$ is represented by

$$(\hat{f} \otimes \hat{g})\partial_{r+s+1}\theta = (\hat{f}\alpha_{r+1} \otimes \hat{g} + (-1)^r \hat{f} \otimes \hat{g}\alpha_{s+1})\theta.$$

I claim that (under the appropriate identifications), $\hat{f}\alpha_{r+1} \otimes \hat{g} = \hat{f}\alpha_{r+1} \otimes g$ and $\hat{f} \otimes \hat{g}\alpha_{s+1} = f \otimes \hat{g}\alpha_{s+1}$. Once this is established, then the result follows because $\hat{f}\alpha_{r+1}$ represents βx, g represents y, f represents x, and $\hat{g}\alpha_{s+1}$ represents βy.

We prove the claim; without loss of generality, we need only prove $\hat{f}\alpha_{r+1} \otimes \hat{g} = \hat{f}\alpha_{r+1} \otimes g$. If $u \in P_{r+1}$, $v \in Q_s$, then $\hat{f}\alpha_{r+1} \otimes \hat{g}\,(u \otimes v) = (\hat{f}\alpha_{r+1}u)(\hat{g}v)$. Let $\pi\colon \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ be the natural epimorphism, and let $\psi\colon \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z}$ be the natural injection. Now $\hat{f}\alpha_{r+1}u \in p\mathbb{Z}/p^2\mathbb{Z}$ (by construction of the Bockstein map), so we may write $\hat{f}\alpha_{r+1}u = \psi x_0$ for some $x_0 \in \mathbb{Z}/p\mathbb{Z}$. Using the Lemma below $(\hat{f}\alpha_{r+1}u)(\hat{g}v) = \psi(x_0\pi\hat{g}v)$. By definition, $\pi\hat{g} = g$, so $(\hat{f}\alpha_{r+1}u)(\hat{g}v) = \psi(x_0gv) = (\psi x_0)gv$. Therefore

$$(\hat{f}\alpha_{r+1}u)(\hat{g}v) = (\hat{f}\alpha_{r+1}u)(gv)$$

for all u, v, hence $\hat{f}\alpha_{r+1} \otimes \hat{g} = \hat{f}\alpha_{r+1} \otimes g$ and the claim is established.

Lemma Let p be a prime, let $x, y \in \mathbb{Z}/p^2\mathbb{Z}$, let $\pi\colon \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ be the natural epimorphism, and let $\psi\colon \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z}$ be the natural injection. Suppose $x \in p\mathbb{Z}/p^2\mathbb{Z}$ and $x = \psi x_0$ where $x_0 \in \mathbb{Z}/p\mathbb{Z}$. Then $xy = \psi(x_0\pi y)$.

Thirteenth Homework Due 9:00 a.m., Monday, November 28.

(1) Let $G = \langle g \rangle$ be a cyclic group, and let M be a $\mathbb{Z}G$-module.
   (i) Prove that multiplication by $1 - g$ on M (i.e. the map $m \mapsto m(1 - g)$ for all $m \in M$) induces the zero map on $\mathrm{H}^r(G, M)$ for all $r \in \mathbb{N}$.
   (ii) Prove that if M is finite and $M^G = 0$, then multiplication by $1 - g$ on M is a $\mathbb{Z}G$-automorphism.
   (iii) Prove that if M is finite and $M^G = 0$, then $\mathrm{H}^r(G, M) = 0$ for all $r \in \mathbb{N}$.

(2) Let G be a finite group and let M be a finitely generated $\mathbb{Z}G$-module. Prove that $\mathrm{H}^r(G, M)$ is a finitely generated abelian group for all $r \in \mathbb{N}$. Deduce that $\mathrm{H}^r(G, M)$ is a finite group for all $r \in \mathbb{P}$.

(3) Let $H \leqslant G$ be groups and let M be a $\mathbb{Z}H$-module. Prove that $M \otimes_{\mathbb{Z}H} \mathbb{Z}G$ is isomorphic to $\mathbb{Z}G$-submodule of $\mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M)$.

(4) Let $H \leqslant G$ be groups such that $[G : H] = n$ is finite, let $\{x_1, \ldots, x_n\}$ be a right transversal for H in G, and let M be a left $\mathbb{Z}H$-module. Let (P, α_0) be a projective resolution of $\mathbb{Z}$ with $\mathbb{Z}G$-modules. For $r \in \mathbb{N}$, define $\theta\colon P_r \otimes_{\mathbb{Z}} M \to P_r \otimes_{\mathbb{Z}H} M$ by $\theta(p \otimes m) = \sum_{i=1}^{n} p x_i^{-1} \otimes x_i m$.
   (i) Prove that θ is a well defined map that does not depend on the choice of transversal $\{x_1, \ldots, x_n\}$.
   (ii) Prove that θ induces a well defined map $\hat{\theta}\colon P_r \otimes_{\mathbb{Z}G} M \to P_r \otimes_{\mathbb{Z}H} M$.
   (iii) Prove that $\hat{\theta}$ induces a well defined map $\theta_r^* : \mathrm{H}_r(G, M) \to \mathrm{H}_r(H, M)$ for all $r \in \mathbb{N}$.

(5) Let k be a field, and let A be a finitely generated anticommutative graded k-algebra: this means that there exists a finite subset X of A such that every element of A can be written as a (finite) k-linear sum of products of elements of X. Let B denote the subalgebra of A consisting of sums of elements of even degree.
   (i) Prove that there exists a finite subset Y of A consisting of homogeneous elements of A such that every element of A can be written as a k-linear sum of products of elements of Y.
   (ii) Prove that there exists a finite subset Z of B consisting of homogeneous elements of B such that every element of B can be written as a k-linear sum of products of elements of Z. Deduce that B is a Noetherian ring.

(6) (Continuation of the previous problem.) Let k be a field, let A be a finitely generated anticommutative graded k-algebra, let B be the subalgebra of A consisting of sums of elements of even degree, and let M be the subset of A consisting of sums of elements of odd degree.
   (i) Prove that M is a finitely generated B-submodule of A. Deduce that M is a Noetherian B-module.
   (ii) Prove that A is a (right and left) Noetherian ring.

(7) Let k be a field with characteristic *not* equal to two, let A be an anticommutative graded k-algebra, let B be the subalgebra of A consisting of sums of elements of even degree, and let M be the subset of A consisting of sums of elements of odd degree.
   (i) Prove that the B is a central subalgebra of A.
   (ii) Prove that if $x \in A$ is homogeneous of odd degree, then $x^2 = 0$.
   (iii) Let X be a finite subset of order n consisting of homogeneous elements of M. Prove that any product $a_1 a_2 \ldots$ of elements of A, with at least $n + 1$ of the $a_i \in X$, is 0.
   (iv) If N is an ideal of A which is generated by a finite number of elements of M (so N is finitely generated as a right A-module), prove that N is a nilpotent ideal (i.e. there exists $t \in \mathbb{P}$ such that $N^t = 0$).

(8) Let G be a group, let $1 \leq p \in \mathbb{R}$, let $\alpha \in \mathrm{L}^p(G)$ (see HW 12, prob. 8), and let $\theta\colon \mathbb{C}G \to \mathrm{L}^p(G)$ denote the inclusion map.
   (i) Suppose $g \in G$ has infinite order. If $\alpha g - \alpha \in \mathbb{C}G$, prove that $\alpha \in \mathbb{C}G$. Deduce that $\theta_{1*}\colon \mathrm{H}^1(G, \mathbb{C}G) \to \mathrm{H}^1(G, \mathrm{L}^p(G))$ is injective.
   (ii) Suppose G is infinite and we can write $G = \bigcup_{i=1}^{\infty} G_i$, where the G_i are finite subgroups of G and $G_1 \subseteq G_2 \subseteq \cdots$. Prove that there exists $\alpha \in \mathrm{L}^p(G) \backslash \mathbb{C}G$ such that $\alpha g - \alpha \in \mathbb{C}G$ for all $g \in G$. Deduce that $\theta_{1*}\colon \mathrm{H}^1(G, \mathbb{C}G) \to \mathrm{H}^1(G, \mathrm{L}^p(G))$ is *not* injective.

Monday, November 28

Chapter 15
Cohomology Rings

We have already introduced the notation for $\bigoplus_{i=0}^{\infty} \mathrm{H}^i(G, M)$: this is standard. We shall also use the less standard though often seen

$$\mathrm{H}^{\mathrm{ev}}(G, M) = \bigoplus_{i=0}^{\infty} \mathrm{H}^{2i}(G, M) \quad \text{and} \quad \mathrm{H}^{\mathrm{odd}}(G, M) = \bigoplus_{i=0}^{\infty} \mathrm{H}^{2i+1}(G, M).$$

Exterior Algebra The anticommutative graded k-algebra B of Example (iii) of chapter 13 is often called the *exterior algebra* on the one generator x. The exterior algebra on the d-generators $\{x_1, \ldots, x_d\}$ is the algebra $B \otimes_k \cdots \otimes_k B$ (where B appears d-times in the foregoing tensor product), and is denoted $\mathrm{E}_k[x_1, \ldots, x_d]$ (of course when we are taking tensor products, we are doing it as anticommutative k-algebras, so that what results is an anticommutative k-algebra). Thus $\mathrm{E}_k[x_1, \ldots, x_d] \cong \mathrm{E}_k[x_1] \otimes_k \cdots \otimes_k \mathrm{E}_k[x_d]$, as a k-module is free of rank 2^d, and the x_i satisfy $x_i^2 = 0$ and $x_i x_j = -x_j x_i$ for $i \neq j$.

The Cohomology Ring of a Cyclic Group Let k be a commutative ring and let G be a finite cyclic group. We shall calculate the ring structure on $\mathrm{H}^*(G, k)$; we have already calculated the additive structure (see chapter 11). Let us recall some of the proof. We have an exact sequence

$$0 \longrightarrow k\mathfrak{g} \longrightarrow kG \xrightarrow{\varepsilon} k \longrightarrow 0, \tag{1}$$

where ε is the augmentation map and $k\mathfrak{g}$ denotes the augmentation ideal of kG. We also have a kG-epimorphism $\alpha\colon kG \to k\mathfrak{g}$ defined by $\alpha 1 = g - 1$, and then we have an exact sequence

$$0 \longrightarrow k \longrightarrow kG \xrightarrow{\alpha} k\mathfrak{g} \longrightarrow 0. \tag{2}$$

Applying the long exact sequence for Ext_{kG} in the second variable to (1) and (2) respectively and using $\mathrm{H}^r(G, kG) = 0$ for all $r \in \mathbb{P}$ (because G is finite, see chapter 12), we obtain isomorphisms $\gamma_r\colon \mathrm{H}^r(G, k) \to \mathrm{H}^{r+1}(G, k\mathfrak{g})$ and $\delta_r\colon \mathrm{H}^r(G, k\mathfrak{g}) \to \mathrm{H}^{r+1}(G, k)$ for all $r \in \mathbb{P}$. Thus if we set $\theta_r = \delta_{r+1}\gamma_r\colon \mathrm{H}^r(G, k) \to \mathrm{H}^{r+2}(G, k)$, then θ_r is an isomorphism for all $r \in \mathbb{P}$. Let us consider two important cases.

Case 1 $k = \mathbb{Z}/p\mathbb{Z}$. Let p be a prime and let $k = \mathbb{Z}/p\mathbb{Z}$. We shall assume that $p||G|$ (if $p \nmid |G|$, then as we have seen earlier $\mathrm{H}^r(G, k) = 0$ for all $r \in \mathbb{P}$, and then we have $\mathrm{H}^*(G, k) \cong k$). The long exact cohomology sequence in the second variable applied to (1) yields an exact sequence

$$\cdots \longrightarrow \mathrm{H}^0(G, k) \xrightarrow{\gamma_0} \mathrm{H}^1(G, k\mathfrak{g}) \longrightarrow 0,$$

because $\mathrm{H}^1(G, kG) = 0$ (G is finite). We claim that γ_0 is an isomorphism: this is not hard to do by direct calculation, but we shall show this using previous results. Because δ_1 is an isomorphism, we see that $\mathrm{H}^1(G, k\mathfrak{g}) \cong \mathrm{H}^2(G, k)$. From the results on the cohomology of a cyclic group from chapter 11, we see that $\mathrm{H}^2(G, k) \cong k$ and we deduce that $\mathrm{H}^1(G, k\mathfrak{g}) \cong k$.

Since $\mathrm{H}^0(G,k) \cong k$, it follows that γ_0 is an isomorphism as claimed. We now see that α_0 is an isomorphism and hence α_r is an isomorphism for all $r \in \mathbb{N}$.

Set $x = \theta_0 1 \in \mathrm{H}^2(G,k)$. Using part (iii) of the third theorem of the previous chapter twice, we see that $\theta_{r+s}(uv) = (\theta_r u)v$ for all $u \in \mathrm{H}^r(G,k)$ and $v \in \mathrm{H}^s(G,k)$, so applying this with $u = 1$ and $v = y$, we deduce that $\theta_r y = xy$ for all $y \in \mathrm{H}^r(G,k)$. Thus

$$\text{multiplication by } x \text{ is an isomorphism } \mathrm{H}^r(G,k) \to \mathrm{H}^{r+2}(G,k) \text{ for all } r \in \mathbb{N}. \qquad (3)$$

Define a ring homomorphism ϕ from the polynomial ring $k[X] \to \mathrm{H}^{\mathrm{ev}}(G,k)$ by $\phi X = x$. Then (1) above shows that ϕ is an isomorphism, hence $\mathrm{H}^{\mathrm{ev}}(G,k) \cong k[X]$. We now have a further subdivision of cases, namely p is odd and p is even.

Case 1a p is odd. Let y be any nonzero element of $\mathrm{H}^1(G,k)$. Because k is a field of characteristic not equal to two, HW 13, prob. 7(ii) shows that $y^2 = 0$. Using (3), it now follows that $\mathrm{H}^*(G,k)$ is the k-algebra with k-basis $\{x^i y^j \mid i \in \mathbb{Z} \text{ and } j = 0 \text{ or } 1\}$, and multiplication is determined by the rules $y^2 = 0$ and $xy = -yx$. Another way of saying this is that $\mathrm{H}^r(G,k) \cong k[X] \otimes_k \mathrm{E}_k[y]$. Alternatively we can write $\mathrm{H}^*(G,k) \cong k[x,y]/(x^2, xy - yx)$, the free k-algebra on generators $\{x,y\}$ factored out by the two-sided ideal generated by y^2 and $xy - yx$.

Case 1b $p = 2$. Here we need a further subdivision of cases, depending on whether $4||G|$. We will just do the case $|G| = 2$ and leave the others as exercises, so assume now that $|G| = 2$. Let $g \in G\backslash 1$ (so $G = \{1, g\}$). Since $(g-1)g = g^2 - g = 1 - g = g - 1$ in kG (where we have used the fact that $-1 = 1$ in characteristic 2) and $k\mathfrak{g} = (g-1)kG$, it follows that $\sigma g = \sigma$ for all $\sigma \in \mathfrak{g}$ and hence $\mathfrak{g} \cong k$ as kG-modules. Since it was proved earlier that γ_r is an isomorphism for all $r \in \mathbb{N}$, we have now established that $\gamma_r \colon \mathrm{H}^r(G,k) \to \mathrm{H}^{r+1}(G,k)$ is an isomorphism for all $r \in \mathbb{N}$. Set $w = \gamma_0 1 \in \mathrm{H}^1(G,k)$. Using part (iii) of the third theorem of the previous chapter, we see that $\gamma_{r+s}(uv) = (\gamma_r u)v$ for all $u \in \mathrm{H}^r(G,k)$ and $v \in \mathrm{H}^s(G,k)$, so applying this with $u = 1$ and $v = y$, we deduce that $\gamma_r y = wy$ for all $y \in \mathrm{H}^r(G,k)$. Thus

$$\text{multiplication by } w \text{ is an isomorphism } \mathrm{H}^r(G,k) \to \mathrm{H}^{r+1}(G,k) \text{ for all } r \in \mathbb{N}. \qquad (4)$$

Define a ring homomorphism from the polynomial ring $k[X] \to \mathrm{H}^*(G,k)$ by $\psi X = w$. Then (4) above shows that ψ is an isomorphism and hence $\mathrm{H}^*(G,k) \cong k[X]$.

Exercise Compute the cohomology ring for the finite cyclic group G with coefficients $\mathbb{Z}/2\mathbb{Z}$ when $|G| \neq 2$.

Summing up, we have the following theorem.

Theorem Let p be a prime, and let $G = \mathbb{Z}/n\mathbb{Z}$ where $n \in \mathbb{P}$ and $p|n$.

(i) If p is odd or $4|n$, then $\mathrm{H}^*(G,k) \cong k[x,y]/(x^2, xy-yx)$, where $\deg x = 2$ and $\deg y = 1$.
(ii) If $p = 2$ and $4 \nmid n$, then $\mathrm{H}^*(G,k) \cong k[X]$ where $\deg X = 1$.

Exercise Using the Universal Coefficient theorem, show that the above result remains true if k is replaced by any field of characteristic p.

Case 2 $k = \mathbb{Z}$. Let $n = |G|$. The computation of the cohomology of a finite cyclic group from chapter 11 shows that $\mathrm{H}^0(G,\mathbb{Z}) \cong \mathbb{Z}$, $\mathrm{H}^r(G,\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$ if $r \in \mathbb{P}$ is even, and $\mathrm{H}^r(G,\mathbb{Z}) = 0$

if $r \in \mathbb{P}$ is odd. Thus we need only worry about $\mathrm{H}^{\mathrm{ev}}(G,k)$. As before we have θ_r is an isomorphism for all $r \in \mathbb{P}$; however we can only assert that γ_0 is onto and hence we only have that θ_0 is onto. Let $x = \theta_0 1 \in \mathrm{H}^2(G,\mathbb{Z})$. Using the third theorem of the previous chapter again, we see that $\theta_r y = xy$ for all $y \in \mathrm{H}^r(G,\mathbb{Z})$. Therefore multiplication by x is an isomorphism $\mathrm{H}^r(G,\mathbb{Z}) \to \mathrm{H}^{r+2}(G,\mathbb{Z})$ for all $r \in \mathbb{P}$, and is an epimorphism for $r = 0$. Define a ring homomorphism ϕ from the polynomial ring $\mathbb{Z}[X] \to \mathrm{H}^*(G,\mathbb{Z})$ by $\phi X = x$. Then ϕ is an epimorphism with kernel (nX). We conclude that $\mathrm{H}^*(G,\mathbb{Z}) \cong \mathbb{Z}[X]/(nX)$, where $\deg X = 2$.

The Bockstein Let us compute the Bockstein map in the case $G = \mathbb{Z}/p\mathbb{Z}$ where p is a prime. The exact sequence $0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$ yields an exact sequence

$$0 \longrightarrow \mathrm{H}^0(G,\mathbb{Z}/p\mathbb{Z}) \longrightarrow \mathrm{H}^0(G,\mathbb{Z}/p\mathbb{Z}^2) \longrightarrow \mathrm{H}^0(G,\mathbb{Z}/p\mathbb{Z}) \xrightarrow{\beta_0} \mathrm{H}^1(G,\mathbb{Z}/p\mathbb{Z}) \longrightarrow \cdots$$
$$\cdots \longrightarrow \mathrm{H}^r(G,\mathbb{Z}/p\mathbb{Z}) \longrightarrow \mathrm{H}^r(G,\mathbb{Z}/p\mathbb{Z}^2) \longrightarrow \mathrm{H}^r(G,\mathbb{Z}/p\mathbb{Z}) \xrightarrow{\beta_r} \mathrm{H}^{r+1}(G,\mathbb{Z}/p\mathbb{Z}) \longrightarrow \cdots$$

Now $\mathrm{H}^r(G,\mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$ for all $r \in \mathbb{N}$, $\mathrm{H}^r(G,\mathbb{Z}/p^2\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$ for all $r \in \mathbb{P}$, (use the results on the cohomology of a cyclic group from chapter 11) and $\mathrm{H}^0(G,\mathbb{Z}/p^2\mathbb{Z}) \cong \mathbb{Z}/p^2\mathbb{Z}$. Therefore $\beta_0 = 0$ and maps in the above sequence after that are alternately an isomorphism and zero. It follows that $\beta_{2r} = 0$ and β_{2r+1} is an isomorphism for all $r \in \mathbb{N}$; in particular β_1 is onto. Using this and the theorem for the cohomology ring of a finite cyclic group, when p is an odd prime and $k = \mathbb{Z}/p\mathbb{Z}$, we may now write

$$\mathrm{H}^*(G,k) \cong k[u] \otimes_k \mathrm{E}_k[u]$$

where u is any nonzero element of $\mathrm{H}^1(G,k)$. There is a natural isomorphism between $\mathrm{H}^1(G,k)$ and $\mathrm{Hom}(G,k) = \mathrm{Hom}(\mathbb{Z}/p\mathbb{Z},\mathbb{Z}/p\mathbb{Z})$, and $\mathrm{Hom}(\mathbb{Z}/p\mathbb{Z},\mathbb{Z}/p\mathbb{Z})$ has a canonical nonzero element, namely the identity map ι, so it is conventional to choose u to correspond to ι. Doing this can be helpful in making certain diagrams commutative.

Elementary Abelian p-groups Let p be a prime, let k be the field $\mathbb{Z}/p\mathbb{Z}$, and let G be an elementary abelian p-group. Then we may write $G = G_1 \times \cdots \times G_d$ where $d \in \mathbb{P}$ and the G_i are groups of order p. Using the Künneth theorem (see previous chapter) we have $\mathrm{H}^*(G,k) \cong \mathrm{H}^*(G_1,k) \otimes_k \cdots \otimes_k \mathrm{H}^*(G_d,k)$. For $i = 1,\ldots,d$, choose $u_i \in \mathrm{H}^1(G_i,k)\backslash 0$. We can now state

Theorem Let $\beta\colon \mathrm{H}^1(G_i,k) \to \mathrm{H}^2(G_i,k)$ denote the relevant Bockstein map.

(i) If p is odd, then $\mathrm{H}^*(G,k) \cong k[\beta u_1,\ldots,\beta u_d] \otimes_k \mathrm{E}[u_1,\ldots,u_d]$.
(ii) If $p = 2$, then $\mathrm{H}^*(G,k) \cong k[u_1,\ldots,u_d]$.

Fourteenth Homework Due 9:00 a.m., Monday, December 5.

(1) Let G be a finite cyclic group and let M be a finite (i.e. $|M| < \infty$) $\mathbb{Z}G$-module.
(i) If $M = M^G$, prove that the Herbrand quotient $h(M)$ (see HW 12, prob. 2) is 1.
(ii) If $M^G = 0$, prove that $h(M) = 1$ (use HW 13, prob. 1).

(2) (Continuation of the previous problem.) Let G be a finite cyclic group and let M be a finite $\mathbb{Z}G$-module.
   (i) Prove that $h(M) = 1$ when M is finite.
   (ii) Suppose B is a $\mathbb{Z}G$-submodule of the $\mathbb{Z}G$-module A. Assume that $\mathrm{H}^i(G, A)$, $\mathrm{H}^i(G, B)$ and $\mathrm{H}^i(G, A/B)$ are finite for $i = 1, 2$. If A/B is finite, prove that $h(A) = h(B)$.

(3) Let k be a commutative hereditary ring. Determine the ring structure of $\mathrm{H}^*(\mathbb{Z}, k)$. Hence determine the ring structure of $\mathrm{H}^*(\mathbb{Z}\times\mathbb{Z}, k)$. (The same result is still true if the hypothesis that k is hereditary is dropped.)

(4) Let $G = \mathbb{Z} \times \mathbb{Z}$ let p be a prime, and let $k = \mathbb{Z}/p\mathbb{Z}$. Prove that $\beta_r \colon \mathrm{H}^r(G, \mathbb{Z}/p\mathbb{Z}) \to \mathrm{H}^{r+1}(G, \mathbb{Z}/p\mathbb{Z})$ (where β_r is the Bockstein map) is the zero map for all $r \in \mathbb{N}$.

(5) Let $H \leqslant G$ be groups such that $[G : H] < \infty$, let k be a field of characteristic p, and assume that $p \nmid [G : H]$. Let $R = \mathrm{res}_{G,H} \mathrm{H}^*(G, k)$ and let $T = \ker \mathrm{tr}_{H,G}$, where $\mathrm{tr}_{H,G} \colon \mathrm{H}^*(H, k) \to \mathrm{H}^*(G, k)$ denotes the transfer map.
   (i) Prove that $\mathrm{H}^*(H, k) = R \oplus T$ (as k-modules).
   (ii) Prove that $RR \subseteq R$.
   (iii) Prove that $RT = TR \subseteq T$.

(6) (Continuation of the previous problem.) Let $H \leqslant G$ be groups such that $[G : H] < \infty$, let k be a field of characteristic p, and assume that $p \nmid [G : H]$. If $I, J \lhd_{\mathrm{r}} \mathrm{H}^*(G, k)$ and $\big(\mathrm{res}_{G,H} I\big)\, \mathrm{H}^*(H, k) = \big(\mathrm{res}_{G,H} J\big)\, \mathrm{H}^*(H, k)$, prove that $I = J$. Deduce that if $\mathrm{H}^*(H, k)$ is right Noetherian, then so is $\mathrm{H}^*(G, k)$.

(7) Let $G = \mathbb{Z}/2\mathbb{Z} = \langle g \rangle$ (the cyclic group of order two generated by the element g), and let T denote the $\mathbb{Z}G$-module $\mathbb{Z}$ with G-action defined by $tg = -t$ for all $t \in T$.
   (i) Prove that $T \cong \mathfrak{g}$ as $\mathbb{Z}G$-modules. Deduce that there is an exact sequence $0 \to T \to \mathbb{Z}G \to \mathbb{Z} \to 0$.
   (ii) Prove that $(g+1)g = g+1$ in $\mathbb{Z}G$. Deduce that $(g+1)\mathbb{Z}G \cong \mathbb{Z}$ as $\mathbb{Z}G$-modules (where as usual, we assume that the action on $\mathbb{Z}$ is given by $ag = a$ for all $a \in \mathbb{Z}$).
   (iii) Define a $\mathbb{Z}G$-map $\theta \colon \mathbb{Z}G \to T$ by $\theta 1 = 1$. Prove that $\ker \theta = (1 + g)\mathbb{Z}G$. Deduce that there is an exact sequence of $\mathbb{Z}G$-modules $0 \to \mathbb{Z} \to \mathbb{Z}G \to T \to 0$.

(8) (Continuation of the previous problem.) Let $G = \mathbb{Z}/2\mathbb{Z} = \langle g \rangle$, and let T denote the $\mathbb{Z}G$-module $\mathbb{Z}$ with G-action defined by $tg = -t$ for all $t \in T$.
   (i) Let $r \in \mathbb{N}$. Using the exact sequences of the previous problem and the results for the cohomology of a cyclic group from chapter 11, prove that $\mathrm{H}^r(G, T) = 0$ if r is even, and $\mathrm{H}^r(G, T) \cong \mathbb{Z}/2\mathbb{Z}$ if r is odd.
   (ii) Let $K = \mathbb{Z}/4\mathbb{Z}$ be the $\mathbb{Z}G$-module with G-action defined by $ag = -a$ for all $a \in K$. Prove that $\mathrm{H}^r(G, K) \cong \mathbb{Z}/2\mathbb{Z}$ for all $r \in \mathbb{N}$.

(9) (Continuation of the previous problem.) Let $G = \mathbb{Z}/2\mathbb{Z} = \langle g \rangle$, let $K = \mathbb{Z}/4\mathbb{Z}$ be the $\mathbb{Z}G$-module with G-action defined by $ag = -a$ for all $a \in K$, and let $r \in \mathbb{N}$. Then

(prove this) we have an exact sequence of $\mathbb{Z}G$-modules $0 \to \mathbb{Z}/2\mathbb{Z} \to K \to \mathbb{Z}/2\mathbb{Z} \to 0$, where G acts trivially on $\mathbb{Z}/2\mathbb{Z}$. Therefore we can define a "twisted Bockstein" map $\beta'_r \colon \mathrm{H}^r(G, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{H}^{r+1}(G, \mathbb{Z}/2\mathbb{Z})$ to be the map associated to the long exact cohomology sequence for $0 \to \mathbb{Z}/2\mathbb{Z} \to K \to \mathbb{Z}/2\mathbb{Z} \to 0$. Prove that β'_r is an isomorphism if r is even, and β'_r is zero if r is odd.

Monday, December 5

Chapter 16
Cohomology Rings (continued)

The Bockstein again We give another way to describe the Bockstein map, which will be of use when calculating the cohomology ring of an abelian group with coefficients in $\mathbb{Z}$. Let p be a prime. Then we have an exact sequence of $\mathbb{Z}$-modules $0 \to \mathbb{Z} \xrightarrow{\mu} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/p\mathbb{Z} \to 0$, where μ is multiplication by p and π is the natural epimorphism. We can make this into an exact sequence of $\mathbb{Z}G$-modules by letting G act trivially on each term (i.e. $mg = m$ for all m and for all $g \in G$). The long exact cohomology sequence applied to this yields an exact sequence

$$0 \longrightarrow \mathrm{H}^0(G, \mathbb{Z}) \xrightarrow{\mu_{0*}} \mathrm{H}^0(G, \mathbb{Z}) \xrightarrow{\pi_{0*}} \mathrm{H}^0(G, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\partial_0} \mathrm{H}^1(G, \mathbb{Z}) \xrightarrow{\mu_{1*}} \cdots$$
$$\cdots \xrightarrow{\pi_{n*}} \mathrm{H}^n(G, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\partial_n} \mathrm{H}^{n+1}(G, \mathbb{Z}) \xrightarrow{\mu_{n+1*}} \mathrm{H}^{n+1}(G, \mathbb{Z}) \xrightarrow{\pi_{n+1*}} \cdots \quad (1)$$

I claim that $\pi_{(n+1)*}\,\partial_n \colon \mathrm{H}^n(G, \mathbb{Z}/p\mathbb{Z}) \to \mathrm{H}^{n+1}(G, \mathbb{Z}/p\mathbb{Z})$ is in fact the Bockstein map. In the construction of the Bockstein map from chapter 12, we have an exact sequence of $\mathbb{Z}G$-modules $0 \to \mathbb{Z}/p\mathbb{Z} \xrightarrow{\mu'} \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{\pi'} \mathbb{Z}/p\mathbb{Z} \to 0$, also with G acting trivially on each term, where μ' is multiplication by p and π' is the natural epimorphism. Putting this together with the exact sequence from above, we obtain the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\mu} & \mathbb{Z} & \xrightarrow{\pi} & \mathbb{Z}/p\mathbb{Z} & \longrightarrow & 0 \\
 & & \pi \downarrow & & \psi \downarrow & & \iota \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}/p\mathbb{Z} & \xrightarrow{\mu'} & \mathbb{Z}/p^2\mathbb{Z} & \xrightarrow{\pi'} & \mathbb{Z}/p\mathbb{Z} & \longrightarrow & 0
\end{array}$$

where ψ is the natural epimorphisms and ι is the identity map. Applying the long exact cohomology sequence and using the fact that it is natural, we obtain a commutative diagram

$$\begin{array}{ccc}
\mathrm{H}^n(G, \mathbb{Z}/p\mathbb{Z}) & \xrightarrow{\partial_n} & \mathrm{H}^{n+1}(G, \mathbb{Z}) \\
\iota_{n*} \downarrow & & \downarrow \pi_{(n+1)*} \\
\mathrm{H}^n(G, \mathbb{Z}/p\mathbb{Z}) & \xrightarrow{\beta_n} & \mathrm{H}^{n+1}(G, \mathbb{Z}/p\mathbb{Z})
\end{array}$$

where β_n is the Bockstein map. Since ι_{n*} is the identity map, it follows that $\beta_n = \pi_{(n+1)*}\,\partial_n$ as claimed.

Cohomology rings with coefficients in $\mathbb{Z}$ We first need the following lemma, which does not depend on the cup product structure of the cohomology ring.

Theorem Let G be an elementary abelian p-group and let $n \in \mathbb{P}$. Then $p\,\mathrm{H}^n(G, \mathbb{Z}) = 0$.

Proof By the results on the cohomology of a cyclic group from chapter 11, the result is certainly true if $|G| = p$. In general if $|G| > p$, write $G = H \times J$ where $|H|, |J| < |G|$. By the Künneth formula (applicable since G has type FP_∞ and $\mathbb{Z}$ is a commutative hereditary ring), we have a split exact sequence

$$0 \longrightarrow \bigoplus_{r+s=n} \mathrm{H}^r(G,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathrm{H}^s(H,\mathbb{Z}) \longrightarrow \mathrm{H}^n(G \times H,\mathbb{Z}) \longrightarrow \bigoplus_{r+s=n+1} \mathrm{Tor}_1^{\mathbb{Z}}(\mathrm{H}^r(G,\mathbb{Z}), \mathrm{H}^s(G,\mathbb{Z})) \longrightarrow 0.$$

If A and B are $\mathbb{Z}$-modules and $pA = pB = 0$, then $p(A \otimes_{\mathbb{Z}} B) = 0 = p\,\mathrm{Tor}_{\mathbb{Z}}(A,B)$. Thus the result follows by induction on $|G|$.

Let G be an elementary abelian p-group, let $k = \mathbb{Z}/p\mathbb{Z}$, and let us return to the exact sequence (1). The above result shows that in this case $\mu_{n*} = 0$ for all $n \in \mathbb{P}$, hence we obtain an exact sequence

$$0 \longrightarrow \mathrm{H}^n(G,\mathbb{Z}) \xrightarrow{\pi_{n*}} \mathrm{H}^n(G,k) \xrightarrow{\partial_n} \mathrm{H}^{n+1}(G,\mathbb{Z}) \longrightarrow 0$$

for all $n \in \mathbb{P}$. Thus for $n \in \mathbb{N}$,

$$\ker \partial_n = \ker \pi_{n+1*} \partial_n = \ker \beta_n.$$

Define

$$\begin{cases} \tilde{\mathrm{H}}^n(G,\mathbb{Z}) = \mathrm{H}^n(G,\mathbb{Z}) & \text{if } n > 0, \\ \tilde{\mathrm{H}}^0(G,\mathbb{Z}) = k, \end{cases}$$

so $\tilde{\mathrm{H}}^*(G,\mathbb{Z}) \cong \mathrm{H}^*(G,\mathbb{Z})/(p)$ as anticommutative graded k-algebras. Now π_{0*} induces an isomorphism $\tilde{\mathrm{H}}^0(G,\mathbb{Z}) \to \mathrm{H}^0(G,k)$ because $p\,\mathrm{H}^0(G,k) = 0$, hence π_* induces a k-algebra monomorphism $\tilde{\mathrm{H}}^*(G,\mathbb{Z}) \to \mathrm{H}^*(G,k)$ with image $\operatorname{im} \pi_* = \ker \partial_*$. It follows that $\tilde{\mathrm{H}}^*(G,\mathbb{Z}) \cong \ker \beta_*$ (this *is* a ring isomorphism, even though β_* is not ring homomorphism). Thus to calculate $\tilde{\mathrm{H}}^*(G,\mathbb{Z})$ (and hence also $\mathrm{H}^*(G,\mathbb{Z})$), it will be sufficient to determine the kernel of the Bockstein map.

Example Let G be an odd prime, let $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, and let $\{u, v\}$ be a k-basis for $\mathrm{H}^1(G,k)$. Then $\tilde{\mathrm{H}}^*(G,\mathbb{Z}) \cong k[\beta u, \beta v] \otimes_k E_k[u\beta v - v\beta u]$.

Remarks Since $\mathrm{H}^1(G,k) \cong \mathrm{Hom}(G,k)$, it follows that $\mathrm{H}^1(G,k) \cong k \oplus k$, hence $\mathrm{H}^1(G,k)$ has a k-basis consisting of two elements. Also when evaluating the tensor product above, it should be born in mind that $\beta u, \beta v$ have degree two and that $u\beta v - v\beta u$ has degree three; thus $E_k[u\beta v - v\beta u]$ has no elements in degrees one and two.

Proof Set $x = \beta u$ and $y = \beta v$. From the theorem at the end of the previous chapter, $\mathrm{H}^*(G,k) \cong k[x,y] \otimes_k E_k[u,v]$. Thus we may write the general element of $\mathrm{H}^*(G,k)$ uniquely in the form $f_1 + f_2 u + f_3 v + f_4 uv$ where $f_i \in k[x,y]$ for $i = 1,2,3,4$. We want to calculate $\ker \beta$. Note that $\beta f_i = 0$ for all i; this is because $\beta(x^i y^j) = (\beta x) x^{i-1} y^j + x \beta(x^{i-1} y^j)$ and $\beta x = \beta y = 0$, so we can prove this by induction on $i + j$.

$$\beta(f_1 + f_2u + f_3v + f_4uv) = 0 \iff f_2x + f_3y = 0 = f_4(xv - yu)$$
$$\iff f_4 = 0 \text{ and } f_2 = yf,\ f_3 = -xf$$

for some $f \in k[x, y]$. Thus if $w \in \tilde{\mathrm{H}}^*(G, k)$, then $\beta w = 0$ if and only if w can be written in the form $f_1 + f(yu - xv)$ for some $f, f_1 \in k[x, y]$, and the result follows.

Exercise Let $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and let $\{x, y\}$ be a k-basis for $\mathrm{H}^1(G, k)$. Prove that $\tilde{\mathrm{H}}^*(G, \mathbb{Z}) \cong k[x^2, y^2, x^2y + xy^2]$ (i.e. the subring of the polynomial ring over k in the variables x, y generated by $x^2, y^2, x^2y + xy^2$.)

Copies of handwritten solutions to all the homework problems are available on request.

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