First Homework Solutions

(1) Apply $M \otimes_R$ to the exact sequence

$$0 \to I \to R \to R/I \to 0.$$

Since $M \otimes_R -$ is right exact, it follows that

$$M \otimes_R I \xrightarrow{\alpha} M \otimes_R R \to M \otimes_R R/I \to 0$$

is exact, hence $M \otimes_R R/I \cong (M \otimes_R R)/\operatorname{im}\alpha$.

Using $m \otimes i = mi \otimes 1$, we see that $\operatorname{im}\alpha = \{m \otimes 1 \mid m \in MI\}$.

Let $\theta : M \otimes_R R \to M$ be the isomorphism defined by $\theta(m \otimes r) = mr$, and let $\pi : M \to M/MI$ be the natural epimorphism. Then $M/MI \cong M \otimes_R R / \ker(\pi\theta)$, so it remains to prove $\ker(\pi\theta) = \operatorname{im}\alpha$.

If $m \in MI$, then $\pi\theta(m \otimes 1) = \pi(m) = 0$, so $\operatorname{im}\alpha \subseteq \ker(\pi\theta)$.

If $\pi\theta(m \otimes 1) = 0$, then $\pi(m) = 0$, hence $m \in MI$ and $m \otimes 1 \in \operatorname{im}\alpha$.

This establishes $\operatorname{im}\alpha = \ker(\pi\theta)$ and it

follows that $M \otimes_R R/I \cong M/MI$ as abelian groups.

In the above, the maps were homomorphisms of abelian groups. Now if $I \lhd R$, then R/I becomes an (R,R)-bimodule, and $M \otimes_R R/I$ and M/MI become right R-modules. Then the above maps become right R-module maps and it follows that $M \otimes_R R/I \cong M/MI$ as right R-modules.

(2) First show $\hat{f}m \in \mathrm{Hom}_S(N,P)$.

(i)

$$\hat{f}m(n_1+n_2) = m\otimes(n_1+n_2) = m\otimes n_1 + m\otimes n_2$$
$$= \hat{f}(n_1) + \hat{f}(n_2)$$
$$\hat{f}m(ns) = f(m\otimes ns) = f[(m\otimes n)s] =$$
$$[f(m\otimes n)]s = (\hat{f}m\,n)s.$$

Next show $\hat{f} \in \mathrm{Hom}_R(M, \mathrm{Hom}_S(N,P))$

$$\hat{f}(m_1+m_2)n = (m_1+m_2)\otimes n = m_1\otimes n + m_2\otimes n$$
$$= (\hat{f}m_1)n + (\hat{f}m_2)n = (\hat{f}m_1 + \hat{f}m_2)n$$
$$\forall\, n \in N,$$

Hence $\hat{f}(m_1+m_2) = \hat{f}m_1 + \hat{f}m_2$.

$\left(\hat{f}(mr)\right)n = mr \otimes n = m \otimes rn = \left(\hat{f}m\right)(rn)$
$= \left(\left(\hat{f}m\right)r\right)n$ (see Exercise 7)
$\forall\, n \in N$, hence $\hat{f}(mr) = \left(\hat{f}m\right)r$.

Finally show $\hat{\ } : \mathrm{Hom}_S(M \otimes_R N, P) \longrightarrow \mathrm{Hom}_R(M, \mathrm{Hom}_S(N,P))$ is a group homomorphism.

$\left(\widehat{(f+g)}m\right)n = f+g\,(m\otimes n) = f(m\otimes n) + g(m\otimes n)$
$= \left(\hat{f}m\right)n + \left(\hat{g}m\right)n \qquad \forall\, n,$
hence $\widehat{(f+g)}m = \hat{f}m + \hat{g}m \qquad \forall\, m,$
hence $\widehat{f+g} = \hat{f} + \hat{g}$.

(ii) Now let us deal with $\sim$.
First show $\tilde{g} : M \otimes_R N \to P$ is a well defined homomorphism of right S-modules.
This is because the map $(m,n) \longmapsto (gm)n$: $M \times N \to P$ is R-balanced.

Now show $\sim : \mathrm{Hom}_R(M, \mathrm{Hom}_S(N,P)) \longrightarrow \mathrm{Hom}_S(M \otimes_R N, P)$ is a group homomorphism.

$\widetilde{(f+g)}(m\otimes n) = \left[(f+g)m\right]n = (fm+gm)n$
$= (fm)n + (gm)n$
$= \tilde{f}(m\otimes n) + \tilde{g}(m\otimes n)$.

Since this is true $\forall\ m, n \in M, N$ and the $m \otimes n$ generate $M \otimes_R N$ as an abelian group, it follows that

$$\widetilde{f+g} = \tilde{f} + \tilde{g}.$$

(iii) $\sim\wedge$ = identity : $\tilde{\hat{f}}(m \otimes n) = (\hat{f}\, m)n = f(m \otimes n)$

Since this is true $\forall\ m, n \in M, N$ and the $m \otimes n$ generate $M \otimes_R N$ as an abelian group, it follows that $\tilde{\hat{f}} = f \quad \forall\ f \in \mathrm{Hom}_S(M \otimes_R N, P)$. Therefore $\sim\wedge$ = identity.

$\wedge\sim$ = identity : $\left(\hat{\tilde{f}}\, m\right)n = \tilde{f}(m \otimes n) = (fm)n.$

Since this is true $\forall\ n \in N$, we deduce that $\hat{\tilde{f}}\, m = fm \quad \forall\ m \in M$ and hence $\hat{\tilde{f}} = f$ $\forall\ f \in \mathrm{Hom}_R(M, \mathrm{Hom}_S(N, P))$. Therefore $\wedge\sim =$ identity as required. It follows that $\wedge$ and $\sim$ are inverse to each other and hence

$$\mathrm{Hom}_S(M \otimes_R N, P) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_S(N, P))$$

as abelian groups.

(3) First show $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$.

Define $\beta : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ by $\beta(a,b) = ab$. Then β is a balanced map because

$\beta(a_1 + a_2, b) = (a_1 + a_2)b = a_1 b + a_2 b = \beta(a_1, b) + \beta(a_2, b)$

$\beta(a, b_1 + b_2) = a(b_1 + b_2) = ab_1 + ab_2 = \beta(a, b_1) + \beta(a, b_2)$

$\beta(an, b) = anb = \beta(a, nb)$ for $n \in \mathbb{Z}$.

It follows that the rule

$$a \otimes b \longmapsto ab$$

defines a group homomorphism

$$\theta : \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$$

which is obviously onto. Now define

$\phi : \mathbb{Q} \to \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ by $\phi(a) = a \otimes 1$.

Then

$\theta\phi(a) = \theta(a \otimes 1) = a \quad \forall\, a \in \mathbb{Q}$,

so $\theta\phi = 1_{\mathbb{Q}}$.

Consider $\phi\theta(a \otimes b)$ where $a, b \in \mathbb{Q}$. Write $b = \frac{r}{s}$ where $r \in \mathbb{Z}$, $s \in \mathbb{Z} \setminus 0$. Then

$$\phi\theta(a \otimes b) = \phi(ab) = ab \otimes 1 =$$

$$\frac{ar}{s} \otimes 1 = \frac{a}{s} \otimes r = \frac{a}{s} \otimes \frac{sr}{s} = \frac{as}{s} \otimes \frac{r}{s}$$

$= a \otimes b$. Since the elements $a \otimes b$ generate $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ as an abelian group,

it follows that $\phi\theta = 1_{\mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Q}}$ and hence

$\mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Q} \cong \mathbb{Q}$ as $\mathbb{Z}$-modules.

Now show $\mathbb{Q}\otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} = 0$.

$$q\otimes a = \tfrac{q}{2}\otimes 2a = \tfrac{q}{2}\otimes 0 = 0$$

$\forall\, q\in\mathbb{Q}$, $a\in\mathbb{Z}/2\mathbb{Z}$. Since the elements $q\otimes a$ generate $\mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Z}/2\mathbb{Z}$, it follows that $\mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Z}/2\mathbb{Z} = 0$.

Thus $\mathbb{Q}\otimes_{\mathbb{Z}}(\mathbb{Q}\otimes\mathbb{Z}/2\mathbb{Z}) \cong$

$\mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Q} \oplus \mathbb{Q}\otimes\mathbb{Z}/2\mathbb{Z} \cong$

$\mathbb{Q}\oplus 0 = \mathbb{Q}$.

(4) For $H_r(A)$ we have three cases to consider.

(i) $n=0$ $A_0 = \mathbb{Z}/8\mathbb{Z}$, $\ker \alpha_0 = \mathbb{Z}/8\mathbb{Z}$, $\operatorname{im} \alpha_1 = 4\mathbb{Z}/8\mathbb{Z}$. So $H_0(A) \cong \mathbb{Z}/4\mathbb{Z}$.

(ii) n even, $\neq 0$. $A_n = \mathbb{Z}/8\mathbb{Z}$,

$\ker \alpha_n = 2\mathbb{Z}/8\mathbb{Z}$, $\operatorname{im} \alpha_{n+1} = 4\mathbb{Z}/8\mathbb{Z}$.

So $H_n(A) \cong \mathbb{Z}/2\mathbb{Z}$.

(iii) n odd $A_n = \mathbb{Z}/4\mathbb{Z}$, $\ker \alpha_n = 2\mathbb{Z}/4\mathbb{Z}$,

$\operatorname{im} \alpha_{n+1} = 2\mathbb{Z}/4\mathbb{Z}$. So $H_n(A) = 0$.

$H_r(A \otimes_{\mathbb{Z}} \mathbb{Z}/4\mathbb{Z})$ Write $B_n = (A \otimes_{\mathbb{Z}} \mathbb{Z}/4\mathbb{Z})_n$.

Then if n is even, $B_n = \mathbb{Z}/8\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}/4\mathbb{Z}$, and for n odd $B_n = \mathbb{Z}/4\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}/4\mathbb{Z}$. Also for $n \neq 0$,

$\alpha_{2n} \otimes 1$ is multiplication by 2,

$\alpha_{2n+1} \otimes 1$ is multiplication by 4, i.e. the zero map.

Thus we can write the chain complex $A \otimes_{\mathbb{Z}} \mathbb{Z}/4\mathbb{Z}$ as

$$\cdots \to B_2 \xrightarrow{\beta_2} B_1 \xrightarrow{\beta_1} B_0 \xrightarrow{\beta_0} 0$$

where $B_n \cong \mathbb{Z}/4\mathbb{Z} \ \forall n$

$\beta_0 = 0 = \beta_{2n+1}$ for $n \in \mathbb{N}$

β_{2n} is multiplication by 2 for $n \in \mathbb{P}$.

It follows that $H_0(B) = \ker \beta_0 / \operatorname{im} \beta_1 = \frac{\mathbb{Z}/4\mathbb{Z}}{0} \cong \mathbb{Z}/4\mathbb{Z}$,

$H_{2n+1}(B) = \frac{\mathbb{Z}/4\mathbb{Z}}{2\mathbb{Z}/4\mathbb{Z}} \cong \mathbb{Z}/2\mathbb{Z}$,

$H_{2n}(B) = \frac{2\mathbb{Z}/4\mathbb{Z}}{0} \cong \mathbb{Z}/2\mathbb{Z}$

as required.

(1) Another method.

Define $\tau : M \times R/I \longrightarrow M/MI$ by

$$\tau(m, r+I) = mr + MI$$

Then τ is well defined and R-balanced,

[If $r_1 + I = r_2 + I$, then $r_1 - r_2 \in I$ and

$\tau(m, r_1 + I) = mr_1 + MI =$

$mr_1 + m(r_2 - r_1) + MI = mr_2 + MI =$

$\tau(m, r_2 + I)$

$\tau(m_1 + m_2, r + I) = m_1 r + m_2 r + I$

$= m_1 r + I + m_2 r + I =$

$\tau(m_1, r+I) + \tau(m_2, r+I)$.

$\tau(ms, r+I) = msr + I = \tau(msr, I)$]

Thus τ induces a group homo

$$\theta : M \otimes_R R/I \longrightarrow M/MI$$

satisfying $\theta(m \otimes (r+I)) = mr + MI$.

Define $\psi : M \to M \otimes_R R/I$ by

$$\psi m = m \otimes 1$$

Then ψ is a group homo and for $m \in M$, $i \in I$,

$$\psi(mi) = mi \otimes 1 = m \otimes i = 0.$$

Therefore $\psi(I) = 0$ and so ψ induces a group homo $\phi : M/MI \to M \otimes_R R/I$

satisfying $\phi(m + MI) = m \otimes (1 + I)$.

Then $\theta\phi(m + MI) = \theta(m \otimes (1 + I))$

$= m + MI$, so $\theta\phi$ is the identity,

and $\phi\theta(m \otimes (r + I)) = \phi(mr + I)$

$= mr \otimes (1 + I) = m \otimes (r + I)$, so

$\phi\theta$ is the identity. Therefore θ is an isomorphism of abelian groups as required.

If $I \triangleleft R$, then

$$\theta((m \otimes (r + I)s)) = \theta(m \otimes (rs + I)) = mrs + MI,$$

$$(\theta(m \otimes (r + I)))s = (mr + MI)s = mrs + MI,$$

so θ is an isomorphism of R-modules.

Second Homework Solutions

(1) $\ker \alpha_0 = R$ and $\operatorname{im} \alpha_0 = (x^2 + (x^4))$.

(i) Therefore $H_0(A) = \dfrac{R}{(x^2+(x^4))} \cong \dfrac{k[x]}{(x^2)}$,

so $\underline{H_0(A) \cong k \oplus k}$ as k-modules

Spse. $n \in \mathbb{P}$. Then $\ker \alpha_n = (x^2 + (x^4))$ and $\operatorname{im} \alpha_{n+1} = (x^2+(x^4))$. Therefore

$$\underline{H_n(A) = 0} \quad \forall\, n \in \mathbb{P}.$$

(ii) $\left[A \otimes_R \dfrac{k[x]}{(x^3)}\right]_n = A_n \otimes_R \dfrac{k[x]}{(x^3)} \cong \dfrac{k[x]}{(x^3)}$; we shall use this to identify $\left[A \otimes_R \dfrac{k[x]}{(x^3)}\right]_n$ with $\dfrac{k[x]}{(x^3)}$.

If $n=0$, then $\ker(\alpha_n \otimes 1) = k[x]/(x^3)$ and $\operatorname{im}(\alpha_n \otimes 1) = (x^2 + (x^3))$.

Therefore $H_0\left(A \otimes_R \dfrac{k[x]}{(x^3)}\right) \cong \dfrac{k[x]}{(x^2)} \cong \underline{k \oplus k}$

as k-modules.

If $n \neq 0$, $\ker(\alpha_n \otimes 1) = (x + (x^3))$

and $\operatorname{im}(\alpha_{n+1} \otimes 1) = (x^2 + (x^3))$,

hence $H_n\left(A \otimes_R \dfrac{k[x]}{(x^3)}\right) \cong \dfrac{(x^2)}{(x^3)} \cong \underline{k}$.

(2) (i) For convenience define $\alpha_{-1} = \beta_{-1} = h_{-1} = \theta_{-2} = \phi_{-2} = 0$.

(i) We use induction, so suppose $n \in \mathbb{Z}$, $n \geq -1$, and we have constructed R-maps $\theta_i : Q_i \to I_i$ such that $\beta_i \theta_{i-1} = \phi_i \alpha_i \quad \forall\, i \leq n$. We can obviously do this for $n = -1$. We now do this for $n+1$. Note that

$$\beta_n \theta_{n-1} = \theta_n \alpha_n$$

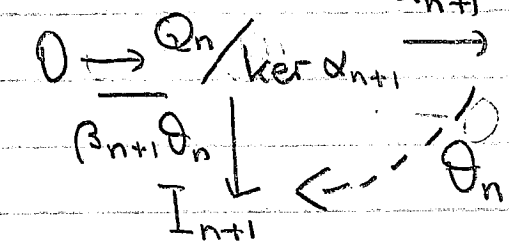
yields $0 = \beta_{n+1} \beta_n \theta_{n-1} = \beta_{n+1} \theta_n \alpha_n$, so

$\beta_{n+1} \theta_n (\operatorname{im} \alpha_n) = 0$. Since (Q, α_0) is exact, $\operatorname{im} \alpha_n = \ker \alpha_{n+1}$, hence $\beta_{n+1} \theta_n (\ker \alpha_{n+1}) = 0$.

Therefore $\beta_{n+1} \theta_n$ induces an R-map

$$\overline{\beta_{n+1} \theta_n} : Q_n / \ker \alpha_{n+1} \to I_{n+1}.$$

If $\overline{\alpha}_{n+1} : Q_n / \ker \alpha_{n+1} \to I_{n+1}$ is the R-map induced by α_{n+1}, then $\exists$ R-map $\theta_{n+1} : Q_{n+1} \to I_{n+1}$ such that $\overline{\beta_{n+1} \theta_n} = \theta_{n+1} \overline{\alpha}_{n+1}$ because I_{n+1} is injective. Then $\beta_{n+1} \theta_n = \theta_{n+1} \alpha_{n+1}$ and the induction step is complete.



Note It is important in the induction step to show $\beta_{n+1} \theta_n (\ker \alpha_{n+1}) = 0$, so it induces a map $Q_n / \ker \alpha_{n+1} \to I_{n+1}$.

(ii) Again use induction, so suppose $n \in \mathbb{Z}$, $n \geq -1$, and we have constructed R-maps $h_i : Q_i \to I_{i-1}$ such that

$$\theta_i - \phi_i = \beta_i h_i + h_{i+1} \alpha_{i+1}$$

$\forall i \leq n$. We can do this for $n = -1$ by taking $h_0 = 0$. Now

$$\theta_n - \phi_n - \beta_n h_n = h_{n+1} \alpha_{n+1}$$

yields

$$\beta_{n+1} \theta_n - \beta_{n+1} \phi_n = \beta_{n+1} h_{n+1} \alpha_{n+1},$$

hence

$$\beta_{n+2}(\theta_{n+1} - \phi_{n+1} - \beta_{n+1} h_{n+1}) \alpha_{n+1} =$$

$$\beta_{n+1} \theta_n - \beta_{n+1} \phi_n - \beta_{n+1} h_{n+1} \alpha_{n+1} = 0,$$

so $(\theta_{n+1} - \phi_{n+1} - \beta_{n+1} h_{n+1}) \operatorname{im} \alpha_{n+1} = 0$. Since $\operatorname{im} \alpha_{n+1} = \ker \alpha_{n+2}$, we deduce

$$(\theta_{n+1} - \phi_{n+1} - \beta_{n+1} h_{n+1}) \ker \alpha_{n+2} = 0.$$

Thus $\theta_{n+1} - \phi_{n+1} - \beta_{n+1} h_{n+1}$ and α_{n+2} induce maps $\theta_{n+1} - \phi_{n+1} - \beta_{n+1} h_{n+1}$ and $\overline{\alpha}_{n+2}$ on

$$Q_{n+1} / \ker \alpha_{n+2}.$$

Since I_{n+1} is injective, there exists an R-map $h_{n+2}: Q_{n+2} \to I_{n+1}$ such that

$$0 \to \frac{Q_{n+1}}{\ker \alpha_{n+2}} \xrightarrow{\overline{\alpha}_{n+2}} Q_{n+2}$$

$$\theta_{n+1} - \phi_{n+1} - \beta_{n+1} h_{n+1} : \frac{Q_{n+1}}{\ker \alpha_{n+2}} \to I_{n+1}$$

$$h_{n+2} : Q_{n+2} \to I_{n+1}$$

$$h_{n+2}\, \overline{\alpha}_{n+2} = \theta_{n+1} - \phi_{n+1} - \beta_{n+1} h_{n+1}.$$

Then $h_{n+2}\, \alpha_{n+2} = \theta_{n+1} - \phi_{n+1} - \beta_{n+1} h_{n+1}$.

(3) Let $\theta_{-1} : M \to M$ denote the identity map. We know $\exists$ R-maps $\theta_i : P_i \to Q_i$ such that $\theta_{i-1} \alpha_i = \beta_i \theta_i \quad \forall i \in \mathbb{N}$. I claim $\theta_i \otimes 1 : P_i \otimes_R L \to Q_i \otimes_R L$ induces a group homomorphism $\theta_{i*} : H_i(P \otimes_R L) \to H_i(Q \otimes_R L)$. We define $\theta_{i*}(u + \ker(\alpha_i \otimes 1)) = (\theta_i \otimes 1)u + \ker(\beta_i \otimes 1)$.

$\underline{\theta_{i*} \text{ well defined}}$ $u + \operatorname{im}(\alpha_{i+1} \otimes 1) = v + \operatorname{im}(\alpha_{i+1} \otimes 1)$

$\Rightarrow u - v \in \operatorname{im}(\alpha_{i+1} \otimes 1) \Rightarrow u - v = \alpha_{i+1} \otimes 1\, w$ for some $w \in P_{i+1} \otimes_R L \Rightarrow \theta_i \otimes 1\,(u - v) = (\theta_i \otimes 1)(\alpha_{i+1} \otimes 1) w = (\beta_{i+1} \otimes 1)(\theta_{i+1} \otimes 1) w \Rightarrow$

$$\theta_i \otimes 1\, u + \operatorname{im}(\beta_i \otimes 1) = \varphi_i \otimes 1\, v + \operatorname{im}(\beta_i \otimes 1),$$

so θ_{i*} does not depend on the choice of u mod $\operatorname{im}(\alpha_{i+1} \otimes 1)$.

θ_{i*} maps into $H_i(Q \otimes_R L)$

$u \in \ker \alpha_i \otimes 1 \Rightarrow \alpha_i \otimes 1\, u = 0 \Rightarrow$
$(\theta_{i-1} \otimes 1)(\alpha_i \otimes 1)\, u = 0 \Rightarrow (\theta_{i-1}\alpha_i \otimes 1)u = 0 \Rightarrow$

$(\beta_i \theta_i \otimes 1)u = 0 \Rightarrow (\beta_i \otimes 1)(\theta_i \otimes 1)\, u = 0,$
so $(\theta_i \otimes 1)u \in \ker(\beta_i \otimes 1)$.

θ_{i*} does not depend on the choice of θ_i

Suppose $\phi_{-1} = \theta_{-1}$ and $\phi_i : P_i \longrightarrow Q_i$ are R-maps such that $\phi_{i-1}\alpha_i = \beta_i \phi_i \ \forall i \in \mathbb{N}$. We want to show that

$$\theta_{i*} = \phi_{i*} : H_i(P \otimes_R L) \to H_i(Q \otimes_R L).$$

We know $\exists$ R-maps $h_i : P_i \longrightarrow Q_{i+1}$, $h_{-1} = 0$, such that

$$\theta_i - \phi_i = h_{i-1}\alpha_i + \beta_{i+1} h_i \quad \forall i \in \mathbb{N}.$$

Let $u \in \ker(\alpha_i \otimes 1)$, so

$\theta_{i*}(u + \operatorname{im}\alpha) = \theta_i u + \operatorname{im}\alpha$,

$\phi_{i*}(u + \operatorname{im}\alpha) = \phi_i u + \operatorname{im}\alpha$.

Then $(\theta_i - \phi_i)u = R_{i-1}\alpha_i u + \beta_{i+1} R_i u$

$= 0 + \beta_{i+1} R_i u$

$\in \operatorname{im}\beta_{i+1}$.

Therefore $\theta_{i*}(u + \operatorname{im}\alpha) = \phi_{i*}(u + \operatorname{im}\alpha)$,

so $\theta_{i*} = \phi_{i*}$.

Suppose now $\psi_i : Q_i \to P_i$, $\psi_{-1} : M \to M$

are R-maps such that $\psi_{-1} = \text{identity}$ and

$\psi_{i-1}\beta_i = \alpha_i \psi_i \quad \forall\, i \in \mathbb{N}$.

Then $\psi_{i-1}\theta_{i-1}\alpha_i = \psi_{i-1}\beta_i\theta_i = \alpha_i \psi_i \theta_i$.

Since $\psi_{-1}\theta_{-1} = 1 : M \to M$, it follows

that $(\psi_i \theta_i)_* = \text{identity}$, hence

$\psi_{i*}\theta_{i*} = \text{identity}$.

Similarly $\theta_{i*}\psi_{i*} = \text{identity}$. It follows that θ_{i*} is an isomorphism, as required.

(4) Write $n = n/q$ and let $[b]$ denote the residue class of b modulo q. We can define a projective resolution of $\mathbb{Z}/q\mathbb{Z}$ by setting

$$A_i = \mathbb{Z}/n\mathbb{Z} \quad \forall\, i$$

$$\alpha_0 : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/q\mathbb{Z} \qquad \alpha_0[1] = [1]$$

$$\alpha_{2i+1} : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \qquad \alpha_{2i+1}[1] = [q]$$

$$\alpha_{2i+2} : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \qquad \alpha_{2i+2}[1] = [n]$$

for all $i \in \mathbb{N}$.

Then $\mathrm{Hom}_{\mathbb{Z}/n\mathbb{Z}}(A_i, \mathbb{Z}/q\mathbb{Z}) \cong \mathbb{Z}/q\mathbb{Z}$ $\forall\, i$, and we shall identify $\mathrm{Hom}_{\mathbb{Z}/n\mathbb{Z}}(A_i, \mathbb{Z}/q\mathbb{Z})$ with $\mathbb{Z}/q\mathbb{Z}$.

$\alpha_{2i+1}^{*} : \mathrm{Hom}_{\mathbb{Z}/n\mathbb{Z}}(A_{2i}, \mathbb{Z}/q\mathbb{Z}) \longrightarrow$

$\mathrm{Hom}_{\mathbb{Z}/n\mathbb{Z}}(A_{2i+1}, \mathbb{Z}/q\mathbb{Z})$ is the zero map,

$$\alpha^*_{2i+2} : \mathrm{Hom}_{\mathbb{Z}/n\mathbb{Z}}(A_{2i+1}, \mathbb{Z}/q\mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}/n\mathbb{Z}}(A_{2i+2}, \mathbb{Z}/q\mathbb{Z})$$

is on identifying $\mathrm{Hom}_{\mathbb{Z}}(A_i, \mathbb{Z}/q\mathbb{Z})$ with $\mathbb{Z}/q\mathbb{Z}$

$$[1] \longmapsto [m].$$

Thus $\mathrm{Ext}^0_{\mathbb{Z}/n\mathbb{Z}}(\mathbb{Z}/q\mathbb{Z}, \mathbb{Z}/q\mathbb{Z}) \cong$

$$\ker \alpha_1^* : \mathbb{Z}/q\mathbb{Z} \to \mathbb{Z}/q\mathbb{Z} = \underline{\mathbb{Z}/q\mathbb{Z}}.$$

$$\mathrm{Ext}^{2i+1}_{\mathbb{Z}/n\mathbb{Z}}(\mathbb{Z}/q\mathbb{Z}, \mathbb{Z}/q\mathbb{Z}) \cong$$

$$\frac{\ker \alpha^*_{2i+2}}{\mathrm{im}\, \alpha^*_{2i+1}} = \ker \alpha^*_{2i+2} =$$

$$\ker([1] \mapsto [m]) : \mathbb{Z}/q\mathbb{Z} \longrightarrow \mathbb{Z}/q\mathbb{Z}$$

$$= \left(\frac{q}{(m,q)}\mathbb{Z}\right)/q\mathbb{Z} \cong \mathbb{Z}/(m,q)\mathbb{Z} = \underline{\mathbb{Z}/l\mathbb{Z}}.$$

$$\mathrm{Ext}^{2i+2}_{\mathbb{Z}/n\mathbb{Z}}(\mathbb{Z}/q\mathbb{Z}, \mathbb{Z}/q\mathbb{Z}) \cong \frac{\ker \alpha^*_{2i+3}}{\mathrm{im}\, \alpha^*_{2i+2}}$$

$$\cong \frac{\mathbb{Z}/q\mathbb{Z}}{(m+q\mathbb{Z})} \cong \frac{\mathbb{Z}/q\mathbb{Z}}{(m,q)\mathbb{Z}/q\mathbb{Z}} \cong \mathbb{Z}/(m,q)\mathbb{Z}$$

$$= \underline{\mathbb{Z}/l\mathbb{Z}}.$$

(5) For each $i \in \mathcal{J}$, let (P^i, α^i) be a projective resolution for M_i. Then

$\left(\bigoplus_{i \in \mathcal{J}} P^i , \bigoplus_{i \in \mathcal{J}} \alpha^i \right)$ is a projective resolution for $\bigoplus M_i$ (direct sums of projective modules are projective, direct sums of exact sequences are exact. Of course, here P^i, α^i mean P superscript i, α superscript i respectively, not to the power i). Then (abbreviate $\bigoplus_{i \in \mathcal{J}}$ to $\bigoplus$)

$$\mathrm{Ext}^n_R \left(\bigoplus_{i \in \mathcal{J}} M_i , A \right) = \frac{\ker \left(\bigoplus \alpha^i_{n+1} \right)^*}{\mathrm{im} \left(\bigoplus \alpha^i_n \right)^*}$$

(where we need to take $\alpha_0 = 0$ in the case $n = 0$)

Now $\bigoplus \alpha^i_{n+1}$ is a map $\bigoplus P^i_{n+1} \longrightarrow \bigoplus P^i_n$

so $\left(\bigoplus \alpha^i_{n+1} \right)^*$ is the map

$$\mathrm{Hom}_R \left(\bigoplus P^i_n , A \right) \longrightarrow \mathrm{Hom}_R \left(\bigoplus P^i_{n+1} , A \right).$$

Now (cf. Exercise 6(i)), $\mathrm{Hom}_R \left(\bigoplus P^i_n , A \right)$

$\cong \prod \mathrm{Hom}_R \left(P^i_n , A \right)$; if the map giving this isomorphism is denoted θ_n, then θ_n is described as follows: let $f \in \mathrm{Hom}_R \left(\bigoplus P^i_n , A \right)$.

Then f defines (by restricting f to P_n^i)
$f^i \in \mathrm{Hom}_R(P_n^i, A)$, and $\theta_n f$

is the element of $\mathrm{Hom}_R(P_n^i, A)$ whose ith component is f^i.

Since $f \in \ker(\oplus \alpha_{n+1}^i)^*$ iff
$f^i \in \ker(\alpha_{n+1}^i)^* \quad \forall\, i \in \mathcal{J}$,
it follows that $\theta_n \ker(\oplus \alpha_{n+1}^i)^*$
$= \prod_{i \in \mathcal{J}} \ker(\alpha_{n+1}^i)^*$.

Similarly $\theta_n \operatorname{im}(\oplus \alpha_n^i)^* = \prod_{i \in \mathcal{J}} \operatorname{im}(\oplus \alpha_n^i)^*$

Therefore $\mathrm{Ext}_R^n(\bigoplus_{i \in \mathcal{J}} M_i, A) =$

$$\frac{\prod_{i \in \mathcal{J}} \ker(\alpha_{n+1}^i)^*}{\prod_{i \in \mathcal{J}} \operatorname{im}(\alpha_n^i)^*} \cong \prod_{i \in \mathcal{J}} \frac{\ker(\alpha_{n+1}^i)^*}{\operatorname{im}(\alpha_n^i)^*} =$$

$$\prod_{i \in \mathcal{J}} \mathrm{Ext}_R^n(M_i, A).$$

Alternative proof for the $\mathrm{Ext}^0_{\mathbb{Z}/n\mathbb{Z}}$ part of (4).

Remember $\mathrm{Ext}^0_R(M,N) \cong \mathrm{Hom}_R(M,N)$,

so $\mathrm{Ext}^0_{\mathbb{Z}/n\mathbb{Z}}(\mathbb{Z}/q\mathbb{Z}, \mathbb{Z}/q\mathbb{Z}) \cong$

$\mathrm{Hom}_{\mathbb{Z}/n\mathbb{Z}}(\mathbb{Z}/q\mathbb{Z}, \mathbb{Z}/q\mathbb{Z}) \cong \mathbb{Z}/q\mathbb{Z}$.

Third Homework Solutions

(1) (i) If I is an injective R-module and $0 \to A \to B \to C \to 0$ is an exact sequence of R-modules, then

$$0 \to \mathrm{Hom}_R(C, I) \to \mathrm{Hom}_R(B, I) \to \mathrm{Hom}_R(A, I) \to 0$$

is also exact (see Exercise 5 of Policy sheet and the definition of injective module). It follows that if

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0 \qquad (1)$$

is any long exact sequence, then

$$0 \to \mathrm{Hom}_R(M, I) \to \mathrm{Hom}_R(P_0, I) \to \mathrm{Hom}_R(P_1, I) \to \cdots \quad (2)$$

is also exact. By choosing (1) to be a projective resolution of M, we will have (2) exact. This proves (i).

(ii) Let $0 \to A \to B \to C \to 0$ be an exact of R-modules. Then the long exact sequence for Ext yields an exact sequence

$$0 \to \mathrm{Hom}_R(C, I) \to \mathrm{Hom}_R(B, I) \to \mathrm{Hom}_R(A, I) \to 0$$

because $\mathrm{Ext}^1_R(C, I) = 0$ and $\mathrm{Ext}^0_R(C, I) \cong \mathrm{Hom}_R(C, I)$. This establishes the result.

(2) We calculate the Ext groups from the projective resolution

$$(P,\varepsilon): 0 \to \mathbb{Z} \xrightarrow{\mu} \mathbb{Z} \xrightarrow{\varepsilon} \mathbb{Z}/q\mathbb{Z} \to 0$$

where μ denotes multiplication by q and ε is the natural epimorphism. Thus $P_0 = P_1 = \mathbb{Z}$, $P_i = 0$ for $i \geq 2$, $\mu_1 = \mu$, $\mu_i = 0$ for $i > 2$, and then

$\operatorname{Ext}^n_{\mathbb{Z}}(\mathbb{Z}/q\mathbb{Z}, A)$ is the nth cohomology group of the complex

$$0 \to \operatorname{Hom}_{\mathbb{Z}}(P_0, A) \xrightarrow{\mu_1^*} \operatorname{Hom}_{\mathbb{Z}}(P_1, A) \to 0.$$

(i) $\operatorname{Ext}^0_{\mathbb{Z}}(\mathbb{Z}/q\mathbb{Z}, A) \cong \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/q\mathbb{Z}, A)$

$\cong \{a \in A \mid aq = 0\}$ via the map $f \mapsto f(1)$.

(ii) $\operatorname{Hom}_{\mathbb{Z}}(P_0, A) \cong \operatorname{Hom}_{\mathbb{Z}}(P_1, A) \cong \mathbb{Z}$ and μ_1^* is multiplication by q. Therefore

$$\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/q\mathbb{Z}, A) = \frac{\ker \mu_2^*}{\operatorname{im} \mu_1^*} = A/qA$$

(iii) $\operatorname{Ext}^n_{\mathbb{Z}}(\mathbb{Z}/q\mathbb{Z}, A) = 0$ for $n \geq 2$ because $\operatorname{Hom}_{\mathbb{Z}}(P_n, A) = 0$.

(3) First consider the case $A = \mathbb{Z}/q\mathbb{Z}$ where $q \in \mathbb{N}$.

If $q \in \mathbb{P}$, then $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/q\mathbb{Z}, B) \cong B/Bq$, $\mathrm{Ext}^n_{\mathbb{Z}}(\mathbb{Z}/q\mathbb{Z}, B) = 0$ for $n \geq 2$ by (2).

If $q = 0$, then $\mathrm{Ext}^n_{\mathbb{Z}}(\mathbb{Z}/q\mathbb{Z}, B) = 0 \ \forall n \geq 1$ because $\mathbb{Z}$ is a projective $\mathbb{Z}$-module.

Therefore $\mathrm{Ext}^n_{\mathbb{Z}}(\mathbb{Z}/q\mathbb{Z}, B)$ is a torsion group $\forall n \in \mathbb{P}, \forall q \in \mathbb{N}$. Since A is a finite direct sum of modules of the form $\mathbb{Z}/q\mathbb{Z}$, the result follows from (5) of previous Homework.

(iii) For $q \in \mathbb{P}$, we have $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/q\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}/q\mathbb{Z}$.

Set $C = \bigoplus_{q=2}^{\infty} \mathbb{Z}/q\mathbb{Z}$. Then using (5) of previous homework again, $\mathrm{Ext}^1_{\mathbb{Z}}(C, \mathbb{Z}) \cong \prod_{q=2}^{\infty} \mathbb{Z}/q\mathbb{Z}$: this has elements of infinite order, eg. $([1]_2, [1]_3, [1]_4, [1]_5, \ldots)$ where $[\cdot]_q$ denote the residue class mod q.

(4) We need to use the fact that $\mathbb{Z}$-submodules of free $\mathbb{Z}$-submodules are free. Map a free $\mathbb{Z}$-module P_0 onto A. Then we have a projective resolution

$$(P, \varepsilon): \quad 0 \longrightarrow P_1 \longrightarrow \cdots \longrightarrow P_0 \xrightarrow{\varepsilon} A \longrightarrow 0$$

of A, where $P_1 = \ker \varepsilon$ and $P_n = 0 \ \forall\, n \geq 2$.

Since $\mathrm{Ext}^n_{\mathbb{Z}}(A, B)$ is a subquotient of $\mathrm{Hom}_{\mathbb{Z}}(P_n, B)$, it follows that $\mathrm{Ext}^n_{\mathbb{Z}}(A, B) = 0$ $\forall\, n \geq 2$.

Since $\mathrm{Tor}^{\mathbb{Z}}_n(A, B)$ is a subquotient of $P_n \otimes_{\mathbb{Z}} B$, it follows that $\mathrm{Tor}^{\mathbb{Z}}_n(A, B) = 0$ $\forall\, n \geq 2$.

(5) Let

$$(P, \alpha_0): \longrightarrow P_2 \xrightarrow{\alpha_2} P_1 \xrightarrow{\alpha_1} P_0 \longrightarrow M \longrightarrow 0$$

be a projective resolution for M.

(i) Let $r \in I$. We have a commutative diagram

$$\begin{array}{ccccccccc}
\longrightarrow & P_2 & \xrightarrow{\alpha_2} & P_1 & \xrightarrow{\alpha_1} & P_0 & \xrightarrow{\alpha_0} & M & \longrightarrow 0 \\
 & \theta_2 \downarrow & \phi_2 & \theta_1 \downarrow & \phi_1 & \theta_0 \downarrow & \phi_0 & \theta_{-1} \downarrow & \phi_{-1} \\
\longrightarrow & P_2 & \xrightarrow{\alpha_2} & P_1 & \xrightarrow{\alpha_1} & P_0 & \xrightarrow{\alpha_0} & M & \longrightarrow 0
\end{array}$$

where all the θ_i are multiplication by r, and all the ϕ_i are 0. Since $\theta_{-1} = \phi_{-1}$, it follows that $\theta_i^* = \phi_i^* = 0$:

$$H^i(\mathrm{Hom}(P, N)) \longrightarrow H^i(\mathrm{Hom}(P, N)).$$

Therefore $\mathrm{Ext}^i_R(M, N)\, r = 0$ as required.

(ii) Let $r \in J$ and let $\rho: N \to N$ denote multiplication by r. Then

$$\rho_{i*} = 0: \mathrm{Hom}(P_n, N) \longrightarrow \mathrm{Hom}(P_n, N),$$

hence $\mathrm{Ext}^n_R(M, N)\, r = 0$.

(iii) We have $\mathrm{Ext}^n_R(M, N)\,(I+J) = 0$ from (i) and (ii), hence $\mathrm{Ext}^n_R(M, N)\, R = 0$.

(6) (i) This is routine checking. To verify that M and N are R-modules, we need only do this for M. Clearly M is an abelian, $(m_1+m_2)r = (m_1+m_2)\theta r$ $= m_1\theta r + m_2\theta r = m_1 r + m_2 r$, $m(r_1+r_2) = m\theta(r_1+r_2) = m(\theta r_1 + \theta r_2) = m\theta r_1 + m\theta r_2 = mr_1 + mr_2$, $(mr_1)r_2 = (m\theta r_1)\theta r_2 = m(\theta r_1 \theta r_2) = m\theta(r_1 r_2) = m(r_1 r_2)$.

$f(mr) = f(m\theta r) = (fm)\theta r = (fm)r$ shows that f is an R-module map.

(ii) Apply Lemma 1 of the Chain Complexes handout (2nd handout).

(iii) $\mathrm{Ext}^n_S(M,N) = H^n(\mathrm{Hom}_S(Q,N))$

$\mathrm{Ext}^n_R(M,N) = H^n(\mathrm{Hom}_R(P,N))$

$f + \mathrm{im}\,\beta_n^* = g + \mathrm{im}\,\beta_n^* \Rightarrow f - g \in \mathrm{im}\,\beta_n^* \Rightarrow f - g = h\beta_n$ for some $h \in \mathrm{Hom}_R(Q_{n-1}, N) \Rightarrow (f-g)\theta_n = h\beta_n\theta_n$ $= h\theta_{n-1}\alpha_n \Rightarrow (f-g)\theta_n \in \mathrm{im}\,\alpha_n^* \Rightarrow f + \mathrm{im}\,\alpha_n^* =$ $g + \mathrm{im}\,\alpha_n^*$, so θ_n^* well defined.

$\beta_{n+1}^* f = 0 \Rightarrow f\beta_{n+1} = 0 \Rightarrow f\theta_n\alpha_{n+1} = f\beta_{n+1}\theta_{n+1} = 0$, so $f \in \ker \alpha_{n+1}^*$.

(iv) Lemma 1 of Chain Complexes handout yields R-maps $h_i : P_i \to Q_{i+1}$ such that $\theta_i - \phi_i = h_{i-1}\alpha_i + \beta_{i+1}h_i$ $\forall$ $i \in \mathbb{N}$ where $h_{-1} = 0$. The proof is now identical to that of Lemma 2 (ii) of the Chain Complexes handout.

(7) (i) Clearly the map $i \longmapsto (1+J)$ defines a group homomorphism $\theta : I \to I \otimes_R R/J$. Note that for $i \in I$, $j \in J$,

$$\theta(ij) = ij \otimes (1+J) = i \otimes (j+J) = 0,$$

hence $\theta(IJ) = 0$, so θ induces a map $\bar{\theta} : I/IJ \to I \otimes_R R/J$.

The map $(i, r+J) \longmapsto ir + IJ$ is a well defined balanced map $I \times R/J \to I$, hence it induces a group homomorphism $\phi : I \otimes_R R/J \to I/IJ$. It is easily checked that $\phi\bar{\theta}$ is the identity, so $\bar{\theta}$ is a monomorphism, hence $\ker\theta = IJ$ as required.

(ii) The exact sequence $0 \to I \to R \to R/I \to 0$ yields an exact sequence

$$0 \to \mathrm{Tor}_1^R(R/I, R/J) \to I \otimes_R R/J \to R \otimes_R R/J$$

$$\to R/I \otimes_R R/J \to 0. \quad \text{The result follows.}$$

(iii) From (i), $I \otimes_R R/J \cong I/IJ$ via $i \otimes 1+J \longmapsto i+IJ$. Note that every element of $I \otimes_R R/J$ can be written in the form $i \otimes 1+J$. Now $i \otimes 1+J \to 0$ in $R \otimes_R R/J$ iff $1 \otimes i+J = 0$ in $R \otimes_R R/J$, i.e. iff $i \in J$, i.e. iff $i \in I \cap J$. It follows that $\ker I \otimes_R R/J \to R \otimes_R R/J \cong \dfrac{I \cap J}{IJ}$.

Fourth Homework Solutions

(1)

$$\begin{array}{ccccccccc}
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & A_2 & \xrightarrow{\theta_2} & B_2 & \xrightarrow{\phi_2} & C_2 & \longrightarrow & 0 \\
 & & \alpha_2\uparrow & & \beta_2\uparrow & & \gamma_2\uparrow & & \\
0 & \longrightarrow & A_1 & \xrightarrow{\theta_1} & B_1 & \xrightarrow{\phi_1} & C_1 & \longrightarrow & 0 \\
 & & \alpha_1\uparrow & & \beta_1\uparrow & & \gamma_1\uparrow & & \\
0 & \longrightarrow & A_0 & \xrightarrow{\theta_0} & B_0 & \xrightarrow{\phi_0} & C_0 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

First we define $\partial_n : H^{n-1}(C) \to H^n(A)$. If $\bar{c} \in H^{n-1}(C)$, write $\bar{c} = c + \operatorname{im}\gamma_{n-1}$ where $c \in \ker \gamma_n$. Since γ_n is onto, $\exists\ b \in B_{n-1}$ such that $\phi_{n-1} b = c$. Now $\phi_n \beta_n b = \gamma_n \phi_{n-1} b = \gamma_n c = 0$ and the rows are exact, so $\exists\ a \in A_n$ such that $\theta_n a = \beta_n b$. We define $\partial_n \bar{c} = \bar{a}$ where $\bar{a} = a + \operatorname{im}\alpha_n \in A_n / \operatorname{im}\alpha_n$.

First show ∂_n is well defined, so suppose we make choices c_1, b_1, a_1 above instead of a, b, c. Then $c - c_1 \in \operatorname{im}\gamma_{n-1}$, so $c - c_1 = \gamma_{n-1} z$ where $z \in C_{n-2}$. Since ϕ_{n-2} is onto, $\exists\ y \in B_{n-2}$ such that $\phi_{n-2} y = z$, so $c - c_1 = \gamma_{n-1}\phi_{n-2} y = \phi_{n-1}\beta_{n-1} y$

If $b_1 \in B_{n-1}$ and $\phi_{n-1} b_1 = c_1$, then $\phi_{n-1}(b - b_1) = c - c_1$ and so $\phi_{n-1}(b - b_1 - \beta_{n-1} y) = 0$. By exactness of row $n-1$ at B_{n-1}, $\exists\ x \in A_{n-1}$ such that

$\theta_{n-1} x = b - b_1 - \beta_{n-1} y$. If $a_1 \in A_n$ and

$\partial_n a_1 = \beta_n b_1$, then $\partial_n(a-a_1) = \beta_n(b-b_1) = \beta_n(\partial_{n-1}x + \beta_{n-1}y) = \beta_n\partial_{n-1}x = \partial_n\alpha_n x$.
Since ∂_n is mono, we deduce that $a - a_1 = \alpha_n x$ and hence $a + \operatorname{im}\alpha_n = a_1 + \operatorname{im}\alpha_n$ as required.

Next we show that $a + \operatorname{im}\alpha_n$ defines an element of $H^n(A)$; equivalently $a \in \ker\alpha_{n+1}$. Since ∂_{n+1} is injective, it will be sufficient to prove $\partial_{n+1}\alpha_{n+1}a = 0$.
But $\partial_{n+1}\alpha_{n+1}a = \beta_{n+1}\partial_n a = \beta_{n+1}\beta_n b = 0$.

Now we need to show exactness.
At B_n: since $\phi_n\partial_n = 0$, it is clear that $\operatorname{im}\partial_{n*} \subseteq \ker\phi_{n*}$, so we need to prove $\ker\phi_{n*} \subseteq \operatorname{im}\partial_{n*}$. If

$b \in \ker\beta_{n+1}$ and $\phi_{n*}(b + \operatorname{im}\beta_n) = 0$, then $\beta_{n+1}b = 0$ and $\phi_n b \in \operatorname{im}\gamma_n$, so $\phi_n b = \gamma_n z$ where $z \in C_{n-1}$. Since ϕ_{n-1} is onto, we may write $z = \phi_{n-1}y$ where $y = B_{n-1}$ and then $\phi_n b = \gamma_n\phi_{n-1}y = \phi_n\beta_n y$, hence $\phi_n(b - \beta_n y) = 0$. By exactness of row n, we may write $b - \beta_n y = \partial_n x$ where $x \in A_n$, so $b = \partial_n x + \beta_n y$ and thus $b + \operatorname{im}\beta_n = \partial_{n*}(x + \operatorname{im}\alpha_n)$; also $\partial_{n+1}\alpha_{n+1}x = \beta_{n+1}\partial_n x = 0$, so $x \in \ker\alpha_{n+1}$ and $x + \operatorname{im}\alpha_n \in H^n(A)$.

Exactness at C_n: Spse. $b \in B_n$ where $\beta_{n+1}b = 0$. If $a \in A_{n+1}$ and $a = 0$, then $\partial_{n+1}a = \beta_{n+1}b$ and it follows by definition of ∂_{n+1} that $\partial_{n+1}\phi_{n*}(b + \operatorname{im}\beta_n) = a + \operatorname{im}\alpha_{n+1}$. Thus $\partial_{n+1}\phi_{n*} = 0$. Conversely spse $c \in C_n$, $\gamma_{n+1}c = 0$

and $\partial_{n+1}(c + \operatorname{im}\gamma_n) = 0$. This means $\exists\ b \in B_n$ and $a \in A_{n+1}$ such that $a \in \operatorname{im}\alpha_{n+1}$, $\vartheta_{n+1} a = \beta_{n+1} b$ and $\phi_n b = c$. Write $a = \alpha_{n+1} x$ where $x \in A_n$. Then $\beta_{n+1} b = \vartheta_{n+1} a = \vartheta_{n+1}\alpha_{n+1} x = \beta_{n+1}\vartheta_n x$, so $\beta_{n+1}(b - \vartheta_n x) = 0$ and then $\phi_{n*}(b - \vartheta_n x) + \operatorname{im}\beta_n = c + \operatorname{im}\gamma_n$.

Exactness at A_n: Spse. $c \in C_{n-1}$ and $\gamma_n c = 0$. Choose $b \in B_{n-1}$ so that $\phi_{n-1} b = c$, and $a \in A_n$ so that $\vartheta_n a = \beta_n b$. Then $\vartheta_{n*}\partial_n(c + \operatorname{im}\gamma_{n-1}) = \vartheta_n a + \operatorname{im}\beta_n = \beta_n b + \operatorname{im}\beta_n = 0$. Conversely spse.

$a \in A_n$, $\alpha_{n+1} a = 0$ and $\vartheta_{n*}(a + \operatorname{im}\alpha_n) = 0$. This means $\vartheta_n a \in \operatorname{im}\beta_n$, so $\vartheta_n a = \beta_n b$ where $b \in B_{n-1}$. If $c = \phi_{n-1} b$, then by definition $\partial_n(c + \operatorname{im}\gamma_{n-1}) = a + \operatorname{im}\alpha_n$; also $\gamma_n c = \gamma_n \phi_{n-1} b = \phi_n \beta_n b = \phi_n \vartheta_n a = 0$, so $c + \operatorname{im}\gamma_{n-1} \in H^{n-1}(C)$. This completes the proof.

Note When checking $\ker ? \subseteq \operatorname{im} ?$, eg. $\ker \phi_{n*} \subseteq \operatorname{im}\vartheta_{n*}$, when we have obtained $x \in A_n$ so that $\vartheta_n x + \operatorname{im}\alpha_n =$ the given element of $\ker \phi_{n*}$, it is important to show $\alpha_{n+1} x = 0$, so $x + \operatorname{im}\alpha_n$ is actually an element of $H^n(A)$.

② For the first Tor sequence, let

$$(P,\alpha_0): \cdots \xrightarrow{\alpha_2} P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} M \to 0$$

be a projective resolution for M (with right R-modules). Since P_n is projective, the sequence

$$0 \to P_n \otimes_R A \xrightarrow{1\otimes\alpha} P_n \otimes_R B \xrightarrow{1\otimes\beta} P_n \otimes_R C \to 0$$

is exact. Now apply Lemma 1.

For the second Tor sequence, let

$$(P,\alpha_0): \cdots \xrightarrow{\alpha_2} P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} A \to 0$$
$$(T,\gamma_0): \cdots \xrightarrow{\gamma_2} T_1 \xrightarrow{\gamma_1} T_0 \xrightarrow{\gamma_0} C \to 0$$

be projective resolutions for A and C respectively. By the Horseshoe Lemma, we obtain a commutative diagram with exact rows and columns.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \xrightarrow{\alpha_3} & P_2 & \xrightarrow{\alpha_2} & P_1 & \xrightarrow{\alpha_1} & P_0 & \xrightarrow{\alpha_0} & A \to 0 \\
 & & \downarrow{\scriptstyle\theta_2} & & \downarrow{\scriptstyle\theta_1} & & \downarrow{\scriptstyle\theta_0} & & \downarrow{\scriptstyle\alpha} \\
\cdots & \xrightarrow{\beta_3} & Q_2 & \xrightarrow{\beta_2} & Q_1 & \xrightarrow{\beta_1} & Q_0 & \xrightarrow{\beta_0} & B \to 0 \\
 & & \downarrow{\scriptstyle\phi_2} & & \downarrow{\scriptstyle\phi_1} & & \downarrow{\scriptstyle\phi_0} & & \downarrow{\scriptstyle\beta} \\
\cdots & \xrightarrow{\gamma_3} & T_2 & \xrightarrow{\gamma_2} & T_1 & \xrightarrow{\gamma_1} & T_0 & \xrightarrow{\gamma_0} & C \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0 & & 0
\end{array}$$

where (Q,β_0) is a projective resolution for B. Since T is projective, the sequence $0 \to P_n \otimes_R M \xrightarrow{\theta_n\otimes 1} Q_n \otimes_R M \xrightarrow{\phi_n\otimes 1} T_n \otimes_R M \to 0$ is exact $\forall n \in \mathbb{N}$. Now apply Lemma 1.

(3) Let $\bar{X}$ and $\bar{Y}$ denote the images of X and Y in R, and let k denote the R-module which is isomorphic to k as a k-module and has $\bar{X}$ and $\bar{Y}$ acting trivially (so $a\bar{X} = a\bar{Y} = 0$ $\forall\ a \in k$). We have an exact sequence of R-modules

$$0 \to k \oplus k \longrightarrow R \longrightarrow k \to 0.$$

Applying the long exact sequence for Ext in the first variable, we obtain a long exact sequence of k-modules

$$\begin{aligned} 0 \to \mathrm{Hom}_R(k,k) &\to \mathrm{Hom}_R(R,k) \to \mathrm{Hom}_R(k\oplus k, k) \\ &\to \mathrm{Ext}^1_R(k,k) \to \mathrm{Ext}^1_R(R,k) \to \cdots \\ &\cdots \to \mathrm{Ext}^n_R(R,k) \to \mathrm{Ext}^n_R(k\oplus k, k) \\ &\to \mathrm{Ext}^{n+1}_R(k,k) \to \mathrm{Ext}^{n+1}_R(R,k) \to \cdots \end{aligned}$$

Now use $\mathrm{Ext}^n_R(R,k) = 0 \quad \forall\ n \geq 1$ and $\mathrm{Hom}_R(R,k) \cong k$. We obtain $\mathrm{Ext}^0_R(k,k) \cong k$, $\mathrm{Ext}^1_R(k,k) \cong k \oplus k$ and $\mathrm{Ext}^n_R(k\oplus k, k) \cong \mathrm{Ext}^{n+1}_R(k,k)$ $\forall\ n \geq 1$ (as k-modules). The last isomorphism is $\mathrm{Ext}^n_R(k,k) \oplus \mathrm{Ext}^n_R(k,k) \cong \mathrm{Ext}^{n+1}_R(k,k)$: the result follows.

(4) We have an exact sequence of R-modules

$$0 \to V \to R \to U \oplus V \to 0.$$

Applying the long exact sequence for Ext in the first variable, we obtain an exact sequence of R-modules

$$0 \to \mathrm{Hom}_R(U\oplus V, V) \to \mathrm{Hom}_R(R, V) \to \mathrm{Hom}_R(V,V)$$
$$\to \mathrm{Ext}^1_R(U\oplus V, V) \to \mathrm{Ext}^1_R(R, V) \to \cdots$$
$$\cdots \to \mathrm{Ext}^n_R(R,V) \to \mathrm{Ext}^n_R(V,V)$$
$$\to \mathrm{Ext}^{n+1}_R(U\oplus V, V) \to \mathrm{Ext}^{n+1}_R(R,V) \to \cdots$$

Since $\mathrm{Ext}^n_R(R,V) = 0 \quad \forall\, n > 0$, we obtain

$\mathrm{Ext}^1_R(U\oplus V, V) \cong k$ and $\mathrm{Ext}^n_R(V,V) \cong \mathrm{Ext}^{n+1}_R(U\oplus V, V) \quad \forall\, n \geq 1$. Now V is a projective R-module because

$$R = e_{11}R \oplus e_{22}R$$

and $V \cong e_{22}R$, so $\mathrm{Ext}^n_R(V,V) = 0 \quad \forall\, n \in \mathbb{R}$. The result follows.

(5) (i) $S^{-1}M$ is an $S^{-1}\mathbb{Z}$-submodule of $S^{-1}A$, and $S^{-1}M = M$. Since every subspace of a vector space has a direct complement, we may write

$$S^{-1}A = M \oplus N$$

for some $S^{-1}\mathbb{Z}$-submodule N of $S^{-1}A$. Then $A = M \oplus N \cap A$.

(ii) Let B be an arbitrary $\mathbb{Z}$-module, let T be the torsion submodule of B, and suppose M is $\mathbb{Z}$-submodule of B. Since $M \cap T = 0$, we see that $(M+T)/T \cong \mathbb{Q}$; also B/T is torsion free, so by (i) $\exists$ a $\mathbb{Z}$-submodule N/T of B/T such that $B/T = \frac{(M+T)}{T} \oplus N/T$. Then $B = M \oplus N$, which establishes that M is injective.

(6) (i) Application of the Ext sequence in the second variable to $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ yields an exact sequence

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(C_{p^\infty}, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(C_{p^\infty}, \mathbb{Q}) \to \mathrm{Hom}_{\mathbb{Z}}(C_{p^\infty}, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Ext}^1_{\mathbb{Z}}(C_{p^\infty}, \mathbb{Z}) \to \mathrm{Ext}^1_{\mathbb{Z}}(C_{p^\infty}, \mathbb{Q}).$$

We have $\mathrm{Hom}_{\mathbb{Z}}(C_{p^\infty}, \mathbb{Q}) = 0$, and $\mathrm{Ext}^1_{\mathbb{Z}}(C_{p^\infty}, \mathbb{Q}) = 0$ because $\mathbb{Q}$ is injective. Therefore $\mathrm{Ext}^1_{\mathbb{Z}}(C_{p^\infty}, \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(C_{p^\infty}, \mathbb{Q})$ and the result follows (in fact $\mathrm{Hom}_{\mathbb{Z}}(C_{p^\infty}, \mathbb{Q}) \cong \mathbb{Z}_p$, the p-adic integers).

(ii) Let $S = \mathbb{Z} \setminus 0$, so $S^{-1}\mathbb{Z} \cong \mathbb{Q}$, $S^{-1}C_{p^\infty} = 0$, and $S^{-1}\mathrm{Hom}_{\mathbb{Z}}(C_{p^\infty}, \mathbb{Q}) \neq 0$ (because $\mathrm{Hom}_{\mathbb{Z}}(C_{p^\infty}, \mathbb{Q})$ is nonzero, torsion free). Then $S^{-1}\mathrm{Ext}^1_{\mathbb{Z}}(C_{p^\infty}, \mathbb{Z}) \neq 0$ yet $\mathrm{Ext}^1_{S^{-1}\mathbb{Z}}(S^{-1}C_{p^\infty}, S^{-1}\mathbb{Z}) = \mathrm{Ext}^1_{S^{-1}\mathbb{Z}}(0, \mathbb{Q}) = 0$.

(iii) Let A be any $\mathbb{Z}$-module. Application of the Ext sequence in the second variable to $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ yields an exact sequence

$$\cdots \to \mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{Q}) \to \mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Ext}^2_{\mathbb{Z}}(A, \mathbb{Z}) \to \cdots$$

By Third HW prob 4, $\mathrm{Ext}^2_{\mathbb{Z}}(A, \mathbb{Z}) = 0$. Therefore

$\mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}) = 0$. Now $\mathbb{Q}/\mathbb{Z} \cong C_{p^\infty} \oplus \bigoplus_{q \neq p} C_{q^\infty}$, Hence $\mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Ext}^1_{\mathbb{Z}}(A, C_{p^\infty}) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(A, \bigoplus_{q \neq p} C_{q^\infty})$ and we deduce that $\mathrm{Ext}^1_{\mathbb{Z}}(A, C_{p^\infty}) = 0$. Now apply Third HW prob 1.

(7) Let F be a field and let M, N be $F[X]$-modules. Since $F[X]$ is a PID, every $F[X]$-submodule of a free $F[X]$-module is free. By choosing an epimorphism from a free $F[X]$-module P_0 to M, we see that there is a projective resolution of M of the form

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

It follows immediately that $\operatorname{Tor}_n^R(M, N) = 0 \ \forall n \geq 2$.

We apply the above to the present problem. Let S be the multiplicatively closed subset $K \setminus 0$. Then

$$S^{-1}\operatorname{Tor}_n^R(A, B) \cong \operatorname{Tor}_n^{S^{-1}R}(S^{-1}A, S^{-1}B)$$
$$\cong \operatorname{Tor}_n^{F[X]}(S^{-1}A, S^{-1}B)$$

where F is the quotient field of K. Therefore

$S^{-1}\operatorname{Tor}_n^R(A, B) = 0 \quad \forall \ n \geq 2$ and the result follows.

Proof of (6) on sixth handout

Since $M \otimes_R -$ is right exact, applying this to the exact sequence

$$P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} A \longrightarrow 0$$

yields an exact sequence

$$M \otimes_R P_1 \xrightarrow{1 \otimes \alpha_1} M \otimes_R P_0 \xrightarrow{1 \otimes \alpha_0} M \otimes_R A \longrightarrow 0$$

so $$\frac{M \otimes_R P_0}{\operatorname{im}(1 \otimes \alpha_1)} \cong M \otimes_R A.$$

Therefore $H_0(M \otimes_R P) \cong \operatorname{Tor}_0(M, A)$ (this part doesn't use the hypothesis that the P_i are flat).

Now the exact sequence

$$0 \longrightarrow P_2/\ker \alpha_2 \xrightarrow{\bar{\alpha}_2} P_1 \xrightarrow{\pi} P_1/\ker \alpha_1 \longrightarrow 0$$

(where $\bar{\alpha}_2$ is the map induced by α_2, and π is the natural surjection) yields an exact sequence

$$M \otimes_R P_2/\ker \alpha_2 \xrightarrow{1 \otimes \bar{\alpha}_2} M \otimes_R P_1 \xrightarrow{1 \otimes \pi} M \otimes_R P_1/\ker \alpha_1 \longrightarrow 0,$$

Clearly $\operatorname{im}(1 \otimes \bar{\alpha}_2) = \operatorname{im}(1 \otimes \alpha_2)$,

Since $\overline{1 \otimes \alpha_1} = (1 \otimes \bar{\alpha}_1)\,\theta$

(both sides induced by $m \otimes p_1 + \operatorname{im}(1 \otimes \alpha_2) \longmapsto m \otimes \alpha_1 p_0$)

and θ is an isomorphism, $\ker \overline{1 \otimes \alpha_1} \cong \ker 1 \otimes \bar{\alpha}_1$, hence

$$H_1(M \otimes_R P) \cong \operatorname{Tor}_1^R(M, A).$$

We now show $H_n(M \otimes_R P) \cong \operatorname{Tor}_n^R(M, A)$ $\forall n$ by dimension shifting.

Let $B = \operatorname{im} \alpha_1$. Then

$$(Q, \alpha_1) \dashrightarrow P_n \to P_{n-1} \to \cdots \to P_1 \xrightarrow{\alpha_1} B \to 0$$

is a flat resolution of B, and by induction

$$\operatorname{Tor}_{n-1}^R(M, B) \cong H_{n-1}(M \otimes_R Q) = H_n(M \otimes_R P).$$

The exact sequence $0 \to B \to P_0 \to A \to 0$ yields an exact sequence

$$\operatorname{Tor}_n^R(M, P_0) \to \operatorname{Tor}_n^R(M, A) \to \operatorname{Tor}_{n-1}^R(M, B) \to \operatorname{Tor}_{n-1}^R(M, P_0).$$

Since $\operatorname{Tor}_n^R(M, P_0) = \operatorname{Tor}_{n-1}^R(M, P_0) = 0$ for $n-1 \geq 1$ ($\because P_0$ flat), the result follows.

so we have an isomorphism

$$\theta : \frac{M \otimes_R P_1}{\operatorname{im}(1 \otimes \alpha_2)} \longrightarrow M \otimes_R P_1/\ker \alpha_1$$

which is induced by

$$m \otimes p_1 + \operatorname{im}(1 \otimes \alpha_2) \longmapsto m \otimes (p_1 + \ker \alpha_1).$$

Now the exact sequence

$$0 \longrightarrow P_1/\ker \alpha_1 \xrightarrow{\bar{\alpha}_1} P_0 \xrightarrow{\alpha_0} A \longrightarrow 0$$

(where $\bar{\alpha}_1$ is the map induced by α_1) yields an exact sequence

$$0 \longrightarrow \operatorname{Tor}_1^R(M, A) \longrightarrow M \otimes_R P_1/\ker \alpha_1 \xrightarrow{1 \otimes \bar{\alpha}_1} M \otimes_R P_0$$

$$\xrightarrow{1 \otimes \alpha_0} M \otimes_R A \longrightarrow 0$$

(because $\operatorname{Tor}_1^R(M, P_0) = 0$ — P_0 is flat, see (1) on sixth handout),

so $\operatorname{Tor}_1^R(M, A) \cong \ker 1 \otimes \bar{\alpha}_1$

Let $\overline{1 \otimes \alpha_1} : \dfrac{M \otimes_R P_1}{\operatorname{im}(1 \otimes \alpha_2)} \longrightarrow M \otimes_R P_0$

denote the map induced by $1 \otimes \alpha_1$. By definition $H_1(M \otimes_R P) \equiv \ker \overline{1 \otimes \alpha_1}$.

Fifth Homework Solutions

(1) (i) The exact sequence $0 \to A \xrightarrow{\alpha_0} I_0 \xrightarrow{\alpha_1} I_1$ yields an exact sequence

$$0 \to \mathrm{Hom}_R(M, A) \xrightarrow{\alpha_{0*}} \mathrm{Hom}_R(M, I_0) \xrightarrow{\alpha_{1*}} \mathrm{Hom}_R(M, I_1)$$

Now $\mathrm{Ext}^0_R(M, A) \cong \mathrm{Hom}_R(M, A)$ and

$$H^0(\mathrm{Hom}_R(M, I)) = \ker \alpha_{1*}$$

$$= \mathrm{im}\, \alpha_{0*} \quad \text{(by exactness of the sequence)}$$

$$\cong \mathrm{Hom}_R(M, A) \quad \text{as required.}$$

(ii) The exact sequence $0 \to \ker \alpha_1 \to I_0 \to \ker \alpha_2 \to 0$ comes from the given exact sequence; the map $\ker \alpha_1 \to I_0$ is the natural inclusion which we shall denote by ι, and the map $I_0 \to \ker \alpha_2$ is $x \mapsto \alpha_1 x$, which we shall denote by θ. Then the long exact sequence for Ext yields

$$0 \to \mathrm{Hom}_R(M, \ker \alpha_1) \xrightarrow{\iota_*} \mathrm{Hom}_R(M, I_0) \xrightarrow{\theta_*}$$

$$\mathrm{Hom}_R(M, \ker \alpha_2) \xrightarrow{\partial} \mathrm{Ext}^1_R(M, \ker \alpha_1) \to 0$$

because $\mathrm{Ext}^1_R(M, I_1) = 0$ (see 3rd HW prob 1(i))

Now $\ker \alpha_1 \cong A$, so $\mathrm{Ext}^1_R(M, A) \cong$

$$\mathrm{Hom}_R(M, \ker \alpha_2) / \mathrm{im}\, \theta_*$$

Let $\sigma : \ker \alpha_2 \longrightarrow I$ denote the natural inclusion. Since $\mathrm{Hom}_R(M, -)$ is left exact, the exact sequence $0 \to \ker \alpha_2 \xrightarrow{\sigma} I_1 \xrightarrow{\alpha_2} I_2$ yields an exact sequence

$$0 \to \mathrm{Hom}_R(M, \ker \alpha_2) \xrightarrow{\sigma_*} \mathrm{Hom}_R(M, I_1) \xrightarrow{\alpha_{2*}} \mathrm{Hom}_R(M, I_2)$$

Thus $\mathrm{Hom}_R(M, \ker \alpha_2) / \mathrm{im}\, \theta_* \cong$

$$\frac{\mathrm{im}\, \sigma_*}{\mathrm{im}\, \sigma_* \theta_*} = \frac{\ker \alpha_{2*}}{\mathrm{im}\, (\sigma\theta)_*} = \frac{\ker \alpha_{2*}}{\mathrm{im}\, \alpha_{1*}} =$$

$H^0(\mathrm{Hom}_R(M, I))$.

(iii) $A \cong \ker \alpha_1$, so using induction it will be sufficient to prove $\mathrm{Ext}_R^{n+1-t}(M, \ker \alpha_t) \cong \mathrm{Ext}_R^{n-t}(M, \ker \alpha_{t+1})$ $\forall t \geq 1$. The exact sequence

$$\longrightarrow I_{t-1} \xrightarrow{\alpha_t} I_t \xrightarrow{\alpha_{t+1}} I_{t+1}$$

$$\ker \alpha_t \nearrow \qquad \searrow \ker \alpha_{t+1}$$

shows we have an exact sequence

$0 \to \ker \alpha_t \longrightarrow I_{t-1} \longrightarrow \ker \alpha_{t+1} \longrightarrow 0$, so the long exact sequence for Ext yields

$\mathrm{Ext}_R^{n-t}(M, I_{t-1}) \to \mathrm{Ext}_R^{n-t}(M, \ker \alpha_{t+1}) \to$
$\mathrm{Ext}_R^{n+1-t}(M, \ker \alpha_t) \longrightarrow \mathrm{Ext}_R^{n+1-t}(M, I_{t-1})$. But
$\mathrm{Ext}_R^{n-t}(M, I_{t-1}) = \mathrm{Ext}_R^{n+1-t}(M, I_{t-1}) = 0$ because I_{t-1} is injective, and the result follows.

(2) (i) We have a projective resolution of k as $k[X]$-modules

$$0 \to k[X] \xrightarrow{\theta} k[X] \longrightarrow k \to 0$$

where $\theta 1 = X$. Thus $\mathrm{Ext}^n_{k[X]}(k, k)$ will be the cohomology of the complex

$$0 \to \mathrm{Hom}_{k[X]}(k[X], k) \xrightarrow{\theta^*} \mathrm{Hom}_{k[X]}(k[X], k) \to 0;$$

this is $0 \to k \xrightarrow{\theta^*} k \to 0$

where θ^* is multiplication by X, i.e. the zero map.

(ii) $k[X] \cong k \otimes_{k[Y]} k[X, Y]$ and

$\mathrm{Ext}^n_{k[X,Y]}(k \otimes_{k[Y]} k[X,Y], k) \cong$

$\mathrm{Ext}^n_{k[Y]}(k, k)$.

(iii) θ^* is mult^n by X. However mult^n by X on k is zero on k and induces multiplication by X on $\mathrm{Ext}^n_{k[X,Y]}(k[X], k)$, so θ^* is the zero map.

(iv) The long exact for Ext in the first variable for $0 \to k[X] \xrightarrow{\theta} k[X] \to k \to 0$ yields

$$0 \longrightarrow \mathrm{Ext}^0_{k[x,y]}(k,k) \to \mathrm{Ext}^0_{k[x,y]}(k[x],k) \xrightarrow{\theta^*} \mathrm{Ext}^0_{k[x,y]}(k[x],k)$$
$$\to \mathrm{Ext}^1_{k[x,y]}(k,k) \to \mathrm{Ext}^1_{k[x,y]}(k[x],k) \xrightarrow{\theta^*}$$

Since θ^* is the zero map, in view of (i) the sequence is

$$0 \longrightarrow k \longrightarrow k \xrightarrow{\theta^*} k \longrightarrow \mathrm{Ext}^1_{k[x,y]}(k,k)$$
$$\to k \xrightarrow{\theta^*} k \longrightarrow \mathrm{Ext}^2_{k[x,y]}(k,k) \longrightarrow 0 \longrightarrow \cdots$$

and the result follows.

(3) Let $S = R \setminus 0$. Since A is fin. gen. and $R[X]$ is Noetherian,

$$\mathrm{Ext}^n_{K[X]}(S^{-1}A, S^{-1}M) \cong S^{-1}\mathrm{Ext}^n_{R[X]}(A, M)$$

Now if B, C are any $K[X]$-modules, then $\mathrm{Ext}^n_{K[X]}(B, C) = 0 \quad \forall n \geq 2$ (because B has a projective resolution of the form $0 \to P \to Q \to B \to 0$).

Therefore $S^{-1}\mathrm{Ext}^n_{R[X]}(A, M) = 0 \quad \forall n \geq 2$,

so $\mathrm{Ext}^n_{R[X]}(A, M)$ is a torsion R-module.

But the hypothesis on M implies that multiplication by s on M is an isomorphism $\forall s \in S$, so multiplication by s on $\mathrm{Ext}^n_{R[X]}(A, M)$ is also an isomorphism.

Therefore $\mathrm{Ext}^n_{R[X]}(A, M) = 0 \quad \forall n \geq 2$.

(4) One way to prove this is to first show that any module is a direct limit of finitely presented modules and then use the fact that Tor commutes with direct limits. To prove that a countable module is a direct limit of finitely presented ones is not difficult, but the general case is awkward to write out, so we'll proceed differently.

(1) $\mathrm{Tor}_1^R(A,B)$ is a torsion group for all finitely generated left R-modules B.

Write $B = F/K$ where F is a fin. gen. free R-module, and let $\{K_i \mid i \in \mathcal{S}\}$ be the set of finitely generated submodules of F. Then $B \cong \varinjlim F/K_i$, so

$$\mathrm{Tor}_1^R(A,B) \cong \varinjlim \mathrm{Tor}_1^R(A, F/K_i).$$

The F/K_i are finitely presented and a direct limit of torsion groups is a torsion group.

(2) $\mathrm{Tor}_1^R(A,B)$ is a torsion group for all left R-modules B.

Let $\{B_i \mid i \in \mathcal{S}\}$ be the set of finitely generated submodules of B (where $\mathcal{S}$ is an indexing set). Then $B \cong \varinjlim B_i$, so

$$\mathrm{Tor}_1^R(A,B) \cong \varinjlim \mathrm{Tor}_1^R(A, B_i).$$

Since $\mathrm{Tor}_1^R(A, B_i)$ is a torsion group (by (1)) and a direct limit of torsion groups is a torsion group, it follows that $\mathrm{Tor}_1^R(A, B)$ is a torsion group in this case as well.

(3) $\mathrm{Tor}_n^R(A, B)$ is a torsion group $\forall\, n \in \mathbb{P}$.

If $0 \to B_1 \to F \to B_2 \to 0$ is an exact sequence of R-modules with F free, then the long exact sequence for Tor in the second variable shows $\mathrm{Tor}_{n+1}^R(A, B_2) \cong \mathrm{Tor}_n^R(A, B_1)$ $\forall\, n \in \mathbb{P}$. It follows that $\mathrm{Tor}_n^R(A, B) \cong \mathrm{Tor}_1^R(A, C)$ for some left R-module C, hence $\mathrm{Tor}_n^R(A, B)$ is a torsion module $\forall\, n \in \mathbb{P}$.

(5) Let $f_i^j : M_i \longrightarrow M_j$ $(i \leq j)$ be the associated maps. Write $M = \varinjlim M_i$ and let $f_i : M_i \longrightarrow M$ be the associated maps.

Write $g_i^j = f_i^j{}_* : \mathrm{Hom}_R(A, M_i) \longrightarrow \mathrm{Hom}_R(A, M_j)$

$g_i = f_i{}_* : \mathrm{Hom}_R(A, M_i) \longrightarrow \mathrm{Hom}_R(A, M)$

Then $\varinjlim \mathrm{Hom}_R(A, M_i)$ is the direct limit of the system $(\mathrm{Hom}_R(A, M_i), g_i^j)$.

Let $\mu_i : \mathrm{Hom}_R(A, M_i) \longrightarrow \varinjlim \mathrm{Hom}_R(A, M_i)$ be the corresponding maps.

The maps $g_i : \mathrm{Hom}_R(A, M_i) \longrightarrow \mathrm{Hom}_R(A, M)$

yield a unique map $\theta : \varinjlim \mathrm{Hom}_R(A, M_i) \longrightarrow \mathrm{Hom}_R(A, M)$ such that $\theta \mu_i = g_i$. We want to show θ is an isomorphism.

(1) Case $A = R$. $\mathrm{Hom}_R(R, M_i) \cong M_i$ naturally via $f \longmapsto f(1)$, so the direct system $(\mathrm{Hom}_R(R, M_i))$ is $\cong (M_i)$ and it follows that θ is an isomorphism in this case.

(2) Case $A = R^n$. Since $\varinjlim$ commutes with direct sums and $\mathrm{Hom}_R(-, X)$ commutes with finite direct sums for any R-module X (see Exercise 6(i) on first handout), the result follows in this case as well.

(3) A finitely presented. Choose an exact sequence

$$F_1 \longrightarrow F_0 \longrightarrow A \longrightarrow 0$$

where F_1 and F_0 are free. Then we have a commutative diagram

$$\begin{array}{ccccccc} 0 \to & \varinjlim \mathrm{Hom}_R(A, M_i) & \longrightarrow & \varinjlim \mathrm{Hom}_R(F_0, M_i) & \to & \varinjlim \mathrm{Hom}_R(F_1, M_i) \\ & \downarrow & & \downarrow \cong & & \downarrow \cong \\ 0 \longrightarrow & \mathrm{Hom}_R(A, M) & \longrightarrow & \mathrm{Hom}_R(F_0, M) & \to & \mathrm{Hom}_R(F_1, M) \end{array}$$

The rows are exact because $\mathrm{Hom}_R(-, X)$ is left exact and $\varinjlim$ is exact.

The two vertical arrows on the right are isomorphisms (already proved). It follows by a routine piece of diagram chasing that the vertical arrow on the left is also and isomorphism, as required.

(4) $\mathrm{Ext}^n_R(B, \varinjlim M_i) \cong \varinjlim \mathrm{Ext}^n_R(B, M_i)$

This follows from the $n=0$ case and the fact that $\varinjlim$ is exact. In detail, write

$$h_i^j = f_{i*}^j : \mathrm{Ext}^n_R(A, M_i) \longrightarrow \mathrm{Ext}^n_R(A, M_j)$$
$$h_i = f_{i*} : \mathrm{Ext}^n_R(A, M_i) \longrightarrow \mathrm{Ext}^n_R(A, M)$$

Then $\varinjlim \mathrm{Ext}^n_R(A, M_i)$ is the direct limit of the system $(\mathrm{Ext}^n_R(A, M_i), h_i^j)$. Let

$\nu_i : \mathrm{Ext}^n_R(A, M_i) \longrightarrow \varinjlim \mathrm{Ext}_R(A, M_i)$ be the corresponding maps.

The maps h_i yield a unique map

$\vartheta_n(A) : \varinjlim \mathrm{Ext}^n_R(A, M_i) \longrightarrow \mathrm{Ext}^n_R(A, M)$

such that $\vartheta_n(A)\, \nu_i = h_i$.

Now choose an exact sequence $0 \to K \to F \to B \to 0$ where F is a finitely generated free R-module. Since R is right Noetherian, K is also finitely presented. The long exact sequence for Ext in the first variable together with the naturality of the maps involved yield a

commutative diagram

$$\begin{array}{ccccccc}
\varinjlim \mathrm{Ext}^n_R(F, M_i) & \to & \varinjlim \mathrm{Ext}^n_R(K, M_i) & \to & \varinjlim \mathrm{Ext}^{n+1}_R(B, M_i) & \to & \varinjlim \mathrm{Ext}^{n+1}_R(F, M_i) \\
\downarrow \theta_n(F) & & \downarrow \theta_n(K) & & \downarrow \theta_{n+1}(B) & & \downarrow \theta_{n+1}(F) \\
\mathrm{Ext}^n_R(F, M) & \longrightarrow & \mathrm{Ext}^n_R(K, M) & \longrightarrow & \mathrm{Ext}^{n+1}_R(B, M) & \to & \mathrm{Ext}^{n+1}_R(F, M)
\end{array}$$

in which the rows are exact (the top row is exact because $\varinjlim$ is exact). Now $\mathrm{Ext}^{n+1}_R(F, M_i) =$

$\mathrm{Ext}^{n+1}_R(F, M) = 0$, and $\theta_n(F)$, $\theta_n(K)$ are isomorphisms by induction. It follows that $\theta_{n+1}(B)$ is an isomorphism as required.

(6) Let $S = K \setminus 0$. Since K is a flat R-module, (5) yields $S^{-1}\mathrm{Ext}^n_R(A,M) \cong \mathrm{Ext}^n_K(S^{-1}A, M) = 0$ $\forall n \in \mathbb{P}$. Therefore if $\xi \in \mathrm{Ext}^n_R(A,M)$ where $n \in \mathbb{P}$, $\exists$ $s \in S$ such that $s\xi = 0$. Let $\sigma : M \to M$ denote multiplication by s. Since M is a K-module, σ is an isomorphism, hence so is $\sigma_{n*} = \mathrm{Ext}^n_R(A,M) \to \mathrm{Ext}^n_R(A,M)$. Therefore $\xi = 0$ and it follows $\xi = 0$. Thus $\mathrm{Ext}^n_R(A,M) = 0$ $\forall n > 0$.

Since every R-submodule of a free R-module is free, there is a projective resolution of A of the form $0 \to F_1 \to F_0 \to A \to 0$. This shows $\mathrm{Ext}^2_R(A,N) = 0$. The long exact sequence for Ext in the second variable for $0 \to N \to M \to M/N \to 0$ yields

$\cdots \to \mathrm{Ext}^1_R(A,M) \to \mathrm{Ext}^1_R(A, M/N) \to \mathrm{Ext}^2_R(A,N)$,

hence $\mathrm{Ext}^1_R(A, M/N) = 0$ and the result follows.

Fifth HW, Problem 6 Wednesday, Sept 28

The solution was written out incorrectly because (5) of Handout 6 was misquoted. The solution should read as follows.

Let $S = K \setminus 0$. Since K is a flat R-module, (5) yields

$$\mathrm{Ext}^n_R(A, M) \cong \mathrm{Ext}^n_K(S^{-1}A, M)$$

because $S^{-1}A \cong A \otimes_R K$. Now K is a field, so

$\mathrm{Ext}^n_K(S^{-1}A, M) = 0 \quad \forall n \in \mathbb{P}$. Thus $\mathrm{Ext}^n_R(A, M) = 0$ $\forall n \in \mathbb{P}$.

Rest of solution OK.

[(5) of Handout 6 was:
Let R, T be rings, let $\theta : R \to T$ be a ring homomorphism which makes T into a flat left R-module, let A be a right R-module and let B be a right T-module. Then there is a natural isomorphism between $\mathrm{Ext}^n_R(A, B)$ and $\mathrm{Ext}^n_T(A \otimes_R T, B)$.]

Sixth Homework Solutions

(1) (i) We shall use the universal property of inverse limit. Define $f_i : \bigcap M_i \to M_i$ to be the natural inclusion. Then $f_i^j f_j = f_i$ whenever $j \leq i$ is clear.

Suppose we're given an R-module N and R-maps $g_i : N \to M_i$ such that $f_i^j g_j = g_i$. Define $\theta : N \to M$ by $\theta(n) = g_i n$. where $i \in \mathcal{P}$. Note that $\theta(n)$ does not depend on the choice of i because of $f_{ij} g_j = g_i$. Furthermore $\theta(n) \in g_i N \ \forall i$, so

$\theta(n) = \bigcap M_i$, so we have proved the existence of an R-map $\theta : N \to \bigcap M_i$, which clearly satisfies $f_i \theta = g_i$. If $\phi : N \to \bigcap M_i$ is another R-map satisfying $f_i \phi = g_i$, then $f_i \theta(n) = f_i \phi(n) \ \forall i$, hence (using this just for a single i), $\theta(n) = \phi(n) \ \forall n \in N$, so $\theta = \phi$ and so θ is unique. This shows that $\bigcap M_i$ satisfies the universal property for inverse limits.

(ii) Let $M = \prod_{i=1}^{\infty} \mathbb{Z}$ (the cartesian product of a countably infinite number of $\mathbb{Z}$), let $M_i = \{ (m_j) \mid m_2 = \cdots = m_i = 0 \}$ $(i \in \mathbb{P})$, and for $i \leq j$, let $f_i^j : M_j \to M_i$ denote the natural inclusion. Then M_i is uncountable $\forall i$ and by (i), $\varprojlim M_i \cong \bigcap_{i \in \mathbb{P}} M_i \cong \mathbb{Z}$.

(2) Let $A = \varinjlim A_i$, and let $f_i^j : A_i \to A_j$, $f_i : A_i \to A$ be the corresponding maps. Set $g_i^j = (f_i^j)^* : \mathrm{Hom}_R(A_j, M) \to \mathrm{Hom}_R(A_i, M)$. Then $(\mathrm{Hom}_R(A_i, M), g_i^j)$ is an inverse system: let the inverse limit be B, and let $g_i : B \to \mathrm{Hom}_R(A_i, M)$ be the corresponding maps.

The maps $f_i^* : \mathrm{Hom}_R(A, M) \to \mathrm{Hom}_R(A_i, M)$ yield a unique map $\theta : \mathrm{Hom}_R(A, M) \to B$ such that $g_i \theta = f_i^*$. We need to construct a map in the opposite direction.

Given $\alpha \in B$, set $\alpha_i = g_i \alpha : A_i \to M$. Since

$$\alpha_j f_i^j = (g_j \alpha) f_i^j = g_i^j (g_j \alpha) = (g_i^j g_j) \alpha = g_i \alpha = \alpha_i,$$

by the universal property of direct limit, the α_i define a unique R-map $\phi(\alpha) : A \to M$ satisfying $\phi(\alpha) f_i = \alpha_i$. Then it is easily checked that $\theta\phi$ and $\phi\theta$ are the identity maps.

$g_i \theta(\phi\alpha) = f_i^*(\phi\alpha) = \phi\alpha f_i = \alpha_i = g_i \alpha \ \forall i$, so $\theta\phi\alpha = \alpha$.

If $\beta \in \mathrm{Hom}_R(A, M)$, then $\phi(\theta\beta) f_i = f_i^*(\phi(\theta\beta)) = g_i \theta\phi\theta\beta = g_i \theta\beta$ ($\because \theta\phi$ = identity, already proved) $= f_i^* \beta = \beta f_i \ \forall i$, so $\phi\theta\beta = \beta$.

(3) In both these problems, we need to establish the Mittag-Leffler condition for the inverse system (A_i). Let $f_i^j : A_j \to A_i$ for $i \leq j$ be the associated maps.

(i) If $k \geq j$, then $\operatorname{im} f_i^k \subseteq \operatorname{im} f_i^j$, so $\operatorname{im} f_i^j$ as j increases is a decreasing sequence of abelian groups. Since A_i is finite, this sequence must become stationary, i.e. $\exists\ j \geq i$ such that $\operatorname{im} f_i^k = \operatorname{im} f_i^j \quad \forall\ k \geq j$, which shows that (A_i) satisfies the Mittag-Leffler condition.

(ii) The proof of this is almost identical. The $\operatorname{im} f_i^j$ (for fixed i and j increasing) form a decreasing sequence of subspaces. Since $\dim_{\mathbb{R}} A_i < \infty$, this sequence must become stationary, and again $(A_i,\ f_i^j)$ satisfies the Mittag-Leffler condition.

(4) (i) Let $(\mathbb{Z}, g_i^j)$ denote the inverse system where for $i \leq j$, g_i^j is the identity map, and let $(\mathbb{Z}/p^i\mathbb{Z}, h_i^j)$ denote the inverse system where for $i \leq j$, h_i^j denotes the natural epimorphism. Then we have an exact sequence of inverse systems

$$0 \to (p^i\mathbb{Z}, f_i^j) \to (\mathbb{Z}, g_i^j) \to (\mathbb{Z}/p^i\mathbb{Z}, h_i^j) \to 0,$$

hence an exact sequence

$$0 \to \varprojlim (p^i\mathbb{Z}) \to \varprojlim (\mathbb{Z}) \to \varprojlim (\mathbb{Z}/p^i\mathbb{Z}) \to \varprojlim{}^1 (p^i\mathbb{Z}) \to 0$$

because $\varprojlim{}^1 (\mathbb{Z}) = 0$. Also $\varprojlim p^i\mathbb{Z} \cong \bigcap_i p^i\mathbb{Z} = 0$, so we have an exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}_p \to \varprojlim{}^1 (p^i\mathbb{Z}) \to 0.$$

(ii) We follow the above exactly, replacing $\mathbb{Z}$ with $\mathbb{Z}_p$, to obtain an exact sequence

$$0 \to \mathbb{Z}_p \xrightarrow{\theta} \varprojlim \mathbb{Z}_p/p^i\mathbb{Z}_p \to \varprojlim{}^1 p^i\mathbb{Z}_p \to 0.$$

But $\varprojlim \mathbb{Z}_p/p^i\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^i\mathbb{Z} \cong \mathbb{Z}_p$, and it is easily checked that θ is the identity map.

(5) (i) Write $\mathbb{Z}[1/p] = \bigcup p^{-i}\mathbb{Z}$. Then we have an exact sequence (by stated Theorem near the end)

$$0 \longrightarrow \varprojlim{}^1 \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) \longrightarrow \operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}[\tfrac{1}{p}],\mathbb{Z}) \longrightarrow \varprojlim \operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) \longrightarrow 0.$$

Now $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) = 0$, so $\varprojlim \operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) = 0$.

Also $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z})$ is the inverse system $(p^i\mathbb{Z})$, so the result follows from 4 (i).

(ii) First compute $\operatorname{Ext}^1_{\mathbb{Z}}(C_{p^\infty},\mathbb{Z})$ where $C_{p^\infty} = \mathbb{Z}[1/p]/\mathbb{Z}$ (see 4th HW no. 6). We have $C_{p^\infty} = \bigcup p^{-i}\mathbb{Z}/\mathbb{Z}$, so we have an exact sequence

$$0 \longrightarrow \varprojlim{}^1 \operatorname{Hom}_{\mathbb{Z}}(p^{-i}\mathbb{Z}/\mathbb{Z},\mathbb{Z}) \longrightarrow \varprojlim{}^1 \operatorname{Ext}^1_{\mathbb{Z}}(C_{p^\infty},\mathbb{Z}) \longrightarrow \varprojlim{}^1 \operatorname{Ext}^1_{\mathbb{Z}}(p^{-i}\mathbb{Z}/\mathbb{Z},\mathbb{Z}) \longrightarrow 0.$$

Now $p^{-i}\mathbb{Z}/\mathbb{Z} \cong \mathbb{Z}/p^i\mathbb{Z}$, so $\operatorname{Hom}_{\mathbb{Z}}(p^{-i}\mathbb{Z}/\mathbb{Z},\mathbb{Z}) = 0$ and $\operatorname{Ext}^1_{\mathbb{Z}}(p^{-i}\mathbb{Z}/\mathbb{Z},\mathbb{Z}) \cong \mathbb{Z}/p^i\mathbb{Z}$. Moreover the exact sequence $0 \to p^{-i}\mathbb{Z}/\mathbb{Z} \to p^{-i-1}\mathbb{Z}/\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$ yields an exact sequence

$$\cdots \longrightarrow \operatorname{Ext}^1_{\mathbb{Z}}(p^{-i-1}\mathbb{Z}/\mathbb{Z},\mathbb{Z}) \longrightarrow \operatorname{Ext}^1_{\mathbb{Z}}(p^{-i}\mathbb{Z}/\mathbb{Z}) \to 0$$

(because $\operatorname{Ext}^2_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z},\mathbb{Z}) = 0$, and it follows that

$\mathrm{Ext}^1_{\mathbb{Z}}(C_{p^\infty}, \mathbb{Z}) \cong \varprojlim \mathbb{Z}/p^i\mathbb{Z}$

(where the $f_i^i : \mathbb{Z}/p^i\mathbb{Z} \to \mathbb{Z}/p^i\mathbb{Z}$ are the natural epimorphisms) $\cong \mathbb{Z}_p$.

Now $\mathbb{Q}/\mathbb{Z} \cong \bigoplus_q \mathbb{Z}[1/q]/\mathbb{Z}$, so

$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}) \cong \mathrm{Ext}^1_{\mathbb{Z}}(\bigoplus \mathbb{Z}[1/q]/\mathbb{Z}, \mathbb{Z}) \cong$

$\prod_q \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}[1/q]/\mathbb{Z}, \mathbb{Z}) = \prod_q \mathbb{Z}_q$.

(iii) The exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ yields (using the long exact sequence for Ext in the second variable)

$0 \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})$

$\to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}) \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})$

Since $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}) = 0 = \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})$ and $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$, it follows we have an exact sequence (on using (ii))

$$0 \to \mathbb{Z} \xrightarrow{\theta} \prod_q \mathbb{Z}_q \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) \to 0.$$

(It is not difficult to verify that $\theta 1 = (1, 1, 1, \ldots)$.)

(6) (i) Let $\mathcal{F} = \{ \frac{1}{x}R \mid x \in R \setminus 0 \}$.
Clearly $(\mathcal{F}, \leq)$ is a partially ordered set where $\leq$ means "is a subset of", and $K = \bigcup_{M \in \mathcal{F}} M$,

because every element of K is of the form r/x where $r \in R$, $x \in R \setminus 0$. We need to show

$\mathcal{F}$ is directed. If $\frac{1}{x}R, \frac{1}{y}R \in \mathcal{F}$ $(x, y \in R \setminus 0)$, then $\frac{1}{xy}R \supseteq \frac{1}{x}R, \frac{1}{y}R$ as required.

(ii) Let $\mathcal{F} = \mathbb{P}$, $R = \mathbb{Z}$,

$$M_i = \frac{\bigoplus_{j=1}^{\infty} \mathbb{Q}}{\bigoplus_{j=i}^{\infty} \mathbb{Q}}.$$

Then $M_i \cong \bigoplus_{j=i+1}^{\infty} \mathbb{Q}$ $\forall i$, which is not free,

yet $\varinjlim M_i = 0$.

(7) Write the direct system as (R_i, f_i^j) over Φ, and let $f_i : R_i \to R$ be the associated maps.

(i) We shall use the original definition of tensor product. Let F be the free abelian group with basis $\{(a,b) \mid a \in A,\ b \in B\}$. Let

$$C = \{ (a_1,b)+(a_2,b)-(a_1+a_2,b) \quad (a,b_1)+(a,b_2)-(a,b_1+b_2) \} \qquad a,a_1,a_2 \in A,\ b,b_1,b_2 \in B$$

$$D_i = \{ (a\, g_i s, b) - (a, (g_i s)b \}, \qquad s \in R_i,\ r \in R$$

$$D = \{ (ar, b) - (a, rb) \}.$$

Then $$A \otimes_{R_i} B \cong \frac{F}{\langle C, D_i \rangle}$$

$$A \otimes_R B \cong \frac{F}{\langle C, D \rangle}$$

Clearly $D_i \subseteq D_j \subseteq D$ whenever $i \leq j$, and the maps $A \otimes_{R_i} B \longrightarrow A \otimes_{R_j} B$ which make $(A \otimes_{R_i} B)$ into a direct system are just the maps $\frac{F}{\langle C, D_i \rangle} \longrightarrow \frac{F}{\langle C, D_j \rangle}$ induced by the identity on F. Note $D = \bigcup D_i$ because $R = \bigcup f_i R_i$. It follows that $\varinjlim \frac{F}{\langle C, D_i \rangle} \cong \frac{F}{\langle C, D \rangle}$ and the result follows.

(ii) First we prove R is a flat left R_i-module $\forall i \in \mathcal{J}$. It will be sufficient to show $\mathrm{Tor}_1^{R_i}(M, R) = 0$ $\forall$ R_i-modules M. But

$$\begin{aligned}\mathrm{Tor}_1^{R_i}(M, R) &\cong \mathrm{Tor}_1^{R_i}(M, \varinjlim R_j) \\ &\cong \varinjlim \mathrm{Tor}_1^{R_i}(M, R_j) \\ &= 0\end{aligned}$$

(where the limits are taken over the directed set $\{j \in \mathcal{J} \mid j \geq i\}$).

Thus R is a flat left R_i-module.

Now $\varinjlim \mathrm{Tor}_n^{R_i}(A, B) \cong$

$\varinjlim \mathrm{Tor}_n^{R}(A \otimes_{R_i} R, B)$ because R is a flat left R_i-module

$\cong \mathrm{Tor}_n^{R}(\varinjlim A \otimes_{R_i} R, B)$

$\cong \mathrm{Tor}_n^{R}(A \otimes_{R} R, B)$

$\cong \mathrm{Tor}_n^{R}(A, B)$.

Alternative method for (1)(i)

Considering $\varprojlim M_i$ as a submodule of

$\prod_{i \in \mathcal{I}} M_i$, we have

$$\varprojlim M_i = \{ (m_i) \mid m_j = m_i \text{ whenever } M_j \subseteq M_i \}.$$

Since $\mathcal{I}$ is a directed set, we have $m_i = m \ \forall i$ for some m, and $m \in \bigcap M_i$. Thus

$$\varprojlim M_i = \{ (m_i) \mid m_i = m \in \bigcap M_j \ \ \forall i \},$$

so $\varprojlim M_i \cong \bigcap M_j$.

Seventh Homework Solutions

(1) (i) For $i \in \mathbb{P}$, let $\overline{\ } : \mathbb{Z}/p^{i+1}\mathbb{Z} \to \mathbb{Z}/p^i\mathbb{Z}$ denote the natural epimorphism induced by the identity on $\mathbb{Z}$. We have $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^i\mathbb{Z}$, so we may consider $\mathbb{Z}_p$ to be the set of sequences $\alpha = (a_i) = (a_1, a_2, a_3, \dots)$ where $a_i \in \mathbb{Z}/p^i\mathbb{Z}$ and $\overline{a}_{i+1} = a_i \ \forall\, i \in \mathbb{P}$. We may consider the elements of $\mathbb{Z}/p^i\mathbb{Z}$ to be $\{0, 1, \dots, p^{i-1}\}$, and then we may write

$$a_i = b_{i1} + b_{i2}p + \cdots + b_{ii}p^{i-1}$$

where $0 \le b_{ij} \le p-1$. The condition $\overline{a}_i = a_{i-1}$ translates to $b_{ij} = b_{ij-1} \ \forall\, j < i$, so

$$a_i = b_{11} + b_{22}p + \cdots + b_{ii}p^{i-1}.$$

Conversely given $b_0 + b_1 p + b_2 p^2 + \cdots$, we can form the p-adic integer

$$(b_0,\ b_0 + b_1 p,\ b_0 + b_1 p + b_2 p^2,\ \dots).$$

This establishes the (1-1) correspondence between $\mathbb{Z}_p$ and $\{b_0 + b_1 p + \cdots \mid 0 \le b_i \le p-1\}$.

(ii) From (proof of) (i), every infinite sequence (a_i) where $a_i \in \{0, 1, \ldots, p-1\}$ can be associated to a unique element of $\mathbb{Z}_p$. Since the set of all such infinite sequences is uncountable, the result follows.

(iii) $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Q}$ is a vector space over $\mathbb{Q}$, so let $\{e_i \mid i \in \mathcal{I}\}$ be a $\mathbb{Q}$-basis. For each $i \in \mathcal{I}$, we may choose $n_i \in \mathbb{Z} \setminus 0$ such that $e_i n_i \in \mathbb{Z}_p$ $(= \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z})$, in other words we may assume that $e_i \in \mathbb{Z}_p \ \forall\, i$. Since $\mathbb{Z}_p$ is uncountable, so is $\mathcal{I}$, hence the e_i form an uncountable basis for a free abelian subgroup of $\mathbb{Z}_p$. This proves (iii).

(iv) By (iii), we may consider A as a subgroup of $\mathbb{Z}_p$. Set $C_i = p^i \mathbb{Z}_p \cap A$. Then

$$A/C_i \cong \text{subgp. of } \mathbb{Z}_p/p^i\mathbb{Z}_p \cong \mathbb{Z}/p^i\mathbb{Z}.$$

By taking a subsequence of the C_i if necessary, we may assume that $A/C_i \cong \mathbb{Z}/p^i\mathbb{Z}\ \forall\, i$. (at least if $A \neq 1$: the problem should have had this hypothesis). This completes the proof

(2) Let A be a free abelian normal subgroup of finite index in G.

(i) Set $G_i = A^{(i!)}$

(ii) Clearly $G_i \cap H$ is a normal subgroup of finite index in H, so we want to prove that the $G_i \cap H$ form a cofinal subset in the set of normal subgroups of finite index in H. In view of (i), it will be sufficient to show that if $K \lhd H$, $[H:K] < \infty$, then $\exists$ a normal subgroup of finite index in G whose intersection with H is contained in K.

Let $B/A\cap K$ be a torsion free subgroup of finite index in $A/A\cap K$, and then let C be a normal subgroup of finite index in G. Then $(H\cap B)/(A\cap K)$ is torsion free ($\because$ $B/A\cap K$ is torsion free) and is torsion ($\because$ $K/A\cap K$ is finite), so $H\cap B = A\cap K$ and we're done.

The fact that the subgroups G_iH/H form a cofinal system in the group G/H follows from the subgroup correspondence theorem and (i).

Since replacing the directed set with a cofinal one does not affect $\varprojlim$, the result follows.

(iii) We have an exact sequence of inverse systems

$$0 \to (H/G_\lambda \cap H) \to (G/G_\lambda) \to (G/G_\lambda H) \to 0,$$

and $\varprojlim^1 H/G_\lambda \cap H = 0$ because $H/G_\lambda \cap H$ is finite and the indexing set is $\mathbb{P}$ (see HW6 no. 3).

Therefore we have an exact sequence

$$1 \to \varprojlim H/G_\lambda \cap H \to \varprojlim G/G_\lambda \to \varprojlim G/G_\lambda H \to 1,$$

and the result follows from (ii).

Remark There are various results (maybe some to be proved) on identifying normalizers and centralizers in G with ones in $\hat{G}$. Sometimes it can be easier to work with $\hat{G}$ instead of G.

(3) Let L_n denote the subfield of $\mathbb{C}$ generated by the p^nth roots of unity, let $G = Gal(K/\mathbb{Q})$, and let $H_n = \{ g \in G \mid g\,l = l \ \ \forall\, l \in L_n \}$. Since every field automorphism of L_n can be extended to a field automorphism of K (any automorphism of L_n can be extended to an automorphism of L_{m+1} $\forall$ m), we that

$$G/H_n \cong Gal(L_n/\mathbb{Q}) \cong \mathbb{Z}/p^{n-1}\mathbb{Z} \times \mathbb{Z}/(p-1)\mathbb{Z}.$$

Under these isomorphisms, the natural epimorphism $G \longrightarrow G/H_n$ corresponds to restricting an automorphism of K to L_n, and the natural epimorphism $G/H_{n+1} \longrightarrow G/H_n$ correspond to the natural epimorphism $\mathbb{Z}/p^n\mathbb{Z} \times \mathbb{Z}/(p-1)\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z} \times \mathbb{Z}/(p-1)\mathbb{Z}$ induced by the identity on $\mathbb{Z}$. By the universal property of direct limits it follows that we have a natural map

$$G \longrightarrow \varprojlim G/H_n \cong \left(\varprojlim \mathbb{Z}/p^{n-1}\mathbb{Z}\right) \times \mathbb{Z}/(p-1)\mathbb{Z}$$
$$= \mathbb{Z}_p \times \mathbb{Z}/(p-1)\mathbb{Z}$$

such that

$$\begin{array}{ccc} G & \longrightarrow & \varprojlim G/H_n \\ & \searrow & \downarrow \\ & & G/H_n \end{array}$$

is commutative.

To get a map going the other way, suppose we are given $\theta_n \in G/H_n$ such that the image of θ_{n+1} in G/H_n is θ_n $\forall n$. This means that viewing $\theta_n \in Gal(L_n/\mathbb{Q})$, the restriction of θ_{n+1} to L_n is θ_n, so the θ_n define an automorphism θ $\bigcup_{n=1}^{\infty} L_n = K$, and the restriction of θ to L_n will be θ_n. Thus we have a map $\varprojlim G/H_n \longrightarrow G$, and the diagram

$$\begin{array}{ccc} G & \longleftarrow & \varprojlim G/H_n \\ & \searrow & \downarrow \\ & & G/H_n \end{array}$$

is commutative. Thus $G \cong \varprojlim G/H_n$ and the result follows.

(4) (i) The exact sequence of inverse systems

$$0 \to (x^i k[x]) \to (k[x]) \to (k[x]/(x^i)) \to 0$$

yields

$$0 \to \varprojlim x^i k[x] \to \varprojlim k[x] \to \varprojlim k[x]/(x^i)$$
$$\to \varprojlim{}^1 x^i k[x] \to \varprojlim{}^1 k[x].$$

But $\varprojlim x^i k[x] = 0$, $\varprojlim k[x] \cong k[x]$, $\varprojlim \frac{k[x]}{(x^i)} \cong k[[x]]$, $\varprojlim{}^1 k[x] = 0$; the result follows.

The exact sequence of inverse systems

$$0 \to (x^i k[[x]]) \to (k[[x]]) \to (k[[x]]/(x^i)) \to 0$$

yields

$$0 \to \varprojlim x^i k[[x]] \to \varprojlim k[[x]] \to \varprojlim \frac{k[[x]]}{(x^i)}$$
$$\to \varprojlim{}^1 x^i k[[x]] \to \varprojlim{}^1 k[[x]].$$

But $\varprojlim x^i k[[x]] = 0$, $\varprojlim k[[x]] \cong k[[x]]$,

$\varprojlim \frac{k[[x]]}{(x^i)} \cong k[[x]]$ and $\varprojlim{}^1 k[[x]] = 0$.

Also it is easily checked that the map in the

above exact sequence from $\varprojlim k[[X]]$ to $\varprojlim k[[X]]$ is an isomorphism, hence $\varprojlim^1 X^i k[[X]] = 0$.

(ii) We apply the theorem quoted at the end of the last handout in a similar way to HW6 no. 5.

We may write $k[X, X^{-1}] = \bigcup_{i=0}^{\infty} X^{-i} k[X]$, so we have an exact sequence

$$0 \longrightarrow \varprojlim{}^1 \operatorname{Ext}^0_{k[X]}(X^{-i}k[X], k[X]) \longrightarrow \operatorname{Ext}^1_{k[X]}(k[X,X^{-1}], k[X]) \longrightarrow \varprojlim \operatorname{Ext}^1_{k[X]}(X^{-i}k[X], k[X]) \longrightarrow 0$$

Now $\operatorname{Ext}^0_{k[X]}(X^{-i}k[X], k[X]) \cong \operatorname{Hom}_{k[X]}(X^{-i}k[X], k[X]) \cong X^i k[X]$ via $f \mapsto f(1)$, and $\operatorname{Hom}_{k[X]}(X^{-j}k[X], k[X]) \longrightarrow \operatorname{Hom}_{k[X]}(X^{-i}k[X], k[X])$ corresponds to the natural inclusion $X^j k[X] \to X^i k[X]$. Therefore

$$\varprojlim{}^1 \operatorname{Ext}^0_{k[X]}(X^{-i}k[X], k[X]) \cong \varprojlim{}^1 X^i k[X]$$

$\cong k[[X]]/k[X]$ by (i). Since $X^{-i}k[X]$ is a projective $k[X]$-module, it follows that

$\mathrm{Ext}^1_{k[x]}(x^{-i}k[x], k[x]) = 0$, hence

$\mathrm{Ext}^1_{k[x]}(k[x,x^{-1}], k[x]) \cong k[[x]]/k[x]$.

We may write $k[[x,x^{-1}]] = \bigcup x^{-i}k[[x,x^{-1}]]$,
so we have an exact sequence

$$0 \to \varprojlim{}^1 \mathrm{Ext}^0_{k[[x]]}(x^{-i}k[[x]], k[[x]]) \to$$
$$\mathrm{Ext}^1_{k[[x]]}(k[[x,x^{-1}]], k[[x]]) \to \varprojlim \mathrm{Ext}^1_{k[[x]]}(x^{-i}k[[x]], k[[x]])$$
$$\to 0.$$

By the same argument as above,

$$\varprojlim{}^1 \mathrm{Ext}^0_{k[[x]]}(x^{-i}k[[x]], k[[x]]) \cong$$
$$\varprojlim x^{-i}k[[x]] \text{ and } \varprojlim \mathrm{Ext}^1_{k[[x]]}(x^{-i}k[[x]], k[[x]])$$
$$= 0.$$

From (i), $\varprojlim{}^1 x^{-i}k[[x]] = 0$ and we deduce that

$$\mathrm{Ext}^1_{k[[x]]}(k[[x,x^{-1}]], k[[x]]) = 0.$$

(5) Note that if R is a commutative ring, P is a projective R-module and Q is a flat R-module, then $P \otimes_R Q$ is a flat R-module; indeed since P is a direct summand of a direct sum of copies of R, we need only do it for the case $P = R$ when it is obvious.

Since $K[X]$ is a flat $k[X]$-module, $(A \otimes_{k[X]} K[X], \alpha_0 \otimes 1)$ is a flat resolution of $U \otimes_{k[X]} K[X]$ (as $K[X]$-modules).

Therefore by the Künneth formula (and the note above)

$(A \otimes_{k[X]} K[X] \otimes_{K[X]} B, \alpha_0 \otimes 1 \otimes \beta_0)$

is a flat resolution of $U \otimes_{k[X]} K[X] \otimes_{K[X]} V$.

Using $(M \otimes N) \otimes L \cong M \otimes (N \otimes L)$, it follows that $(A \otimes_{k[X]} B, \alpha_0 \otimes \beta_0)$ is a flat resolution of $U \otimes_{k[X]} V$ (as $K[X]$-modules). Using Lemma 7 of Handout 5, we see that (since $K[X]$ is a flat $k[X]$-module) $(A \otimes_{k[X]} B, \alpha_0 \otimes \beta_0)$ is a flat resolution of $U \otimes_{k[X]} V$ as $k[X]$-modules.

(6) (i) We have an exact sequence of inverse systems

$$0 \longrightarrow (\mathbb{Z}/p\mathbb{Z}) \longrightarrow (\mathbb{Z}/p^i\mathbb{Z}) \xrightarrow{\times p} (\mathbb{Z}/p^i\mathbb{Z}) \longrightarrow 0,$$

where all the associated maps f_i^i for $(\mathbb{Z}/p\mathbb{Z})$ are the zero maps. This yields an exact sequence

$$0 \to \varprojlim \mathbb{Z}/p\mathbb{Z} \to \varprojlim \mathbb{Z}/p^i\mathbb{Z} \xrightarrow{\times p} \varprojlim \mathbb{Z}/p^i\mathbb{Z}$$
$$\to \varprojlim{}^1 \mathbb{Z}/p\mathbb{Z} \to \cdots. \qquad \mathbb{Z}_p \qquad\qquad \mathbb{Z}_p$$

But $\varprojlim \mathbb{Z}/p\mathbb{Z} = 0$ (because all the associated maps are the zero map), so multiplication by p on $\mathbb{Z}_p$ is a monomorphism.

(ii) mult^n by q : $(\mathbb{Z}/p^i\mathbb{Z}) \longrightarrow (\mathbb{Z}/p^i\mathbb{Z})$ is an automorphism, hence so is multiplication by q on $\varprojlim \mathbb{Z}/p^i\mathbb{Z} = \mathbb{Z}_p$.

(iii) Set $R = \mathbb{Z}$, $S = \{1, p, p^2, \dots\}$.
Then $\varprojlim S^{-1}\mathbb{Z}/p^i\mathbb{Z} = \varprojlim 0 = 0$

and $S^{-1}\varprojlim \mathbb{Z}/p^i\mathbb{Z} = S^{-1}\mathbb{Z}_p \neq 0$ by (i).

(iv) Let p, q be distinct primes, let $R = \mathbb{Z}$, $A = \mathbb{Z}/q\mathbb{Z}$.
Then $\varprojlim{}^1 \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/q\mathbb{Z}, p^i\mathbb{Z}) = \varprojlim{}^1 0 = 0$

and $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/q\mathbb{Z}, \varprojlim{}^1 p^i\mathbb{Z}) = \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/q\mathbb{Z}, \mathbb{Z}_p/\mathbb{Z})$
$\neq 0$ by (ii).

Eighth Homework Solutions

(1) (i) This has already been done (see third HW)
In fact $\mathrm{Ext}^0_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, M_i) = 0$
$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, M_i) \cong \mathbb{Z}$,
$\mathrm{Ext}^n_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, M_i) = 0$ for $n \geq 2$.

(ii) $\varprojlim M_i = \bigcap q^i\mathbb{Z}$ (see HW6 1(i)).
$= 0$,
Hence $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, \varprojlim M_i)$
$= \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, 0) = 0$.

(iii) $\varprojlim \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, M_i)$
$= \varprojlim \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z})$
$\cong \varprojlim \mathbb{Z}/p\mathbb{Z}$.

Now multiplication by q on $\mathbb{Z}$ induces multiplication by q on $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z})$, so $(\mathbb{Z}/p\mathbb{Z})$ is the inverse system (N_i, f_i^{i+1}) where $N_i = \mathbb{Z}/p\mathbb{Z}$ and f_i^{i+1} is multiplication by q. By considering $\varprojlim N_i$ as a submodule of $\prod_{i=1}^{\infty} N_i$, $\varprojlim N_i$ consists of elements $(a_1, a_2, a_3, \ldots)$ where $a_i \in \mathbb{Z}/p\mathbb{Z}$ and $q\,a_{i+1} = a_i$ $\forall\, i \in \mathbb{P}$. Thus $a_i = q^{i-1} a_1$, so a_i is determined by a_1 and hence $\varprojlim \mathbb{Z}/p\mathbb{Z} \cong \mathbb{Z}/p\mathbb{Z}$. The result follows.

(2) The result is true if $n=0$ because Hom commutes with inverse limits (see Handout 6).

Now we consider the case $n=1$. The exact sequence for Ext in the first variable applied to $0 \to K \to F \to A \to 0$ yields an exact sequence

$$0 \to \mathrm{Ext}^0_R(A, M_i) \to \mathrm{Ext}^0_R(F, M_i) \to \mathrm{Ext}^0_R(K, M_i) \xrightarrow{\alpha_i} \mathrm{Ext}^1_R(A, M_i) \to 0$$

because $\mathrm{Ext}^1_R(F, M_i) = 0$, for each $i \in \mathbb{P}$. We would like to conclude that the induced map

$$\alpha : \varprojlim \mathrm{Ext}^0_R(K, M_i) \to \varprojlim \mathrm{Ext}^1_R(A, M_i)$$

is surjective, but we cannot say this immediately because $\varprojlim$ is left exact but not right exact in general. Write $X_i = \ker \alpha_i$. Then we have an exact sequence of inverse systems

$$0 \to (X_i) \to \underset{\cong \mathrm{Hom}_R(K, M_i)}{\mathrm{Ext}^0_R(K, M_i)} \to \mathrm{Ext}^1_R(A, M_i) \to 0.$$

Since X_i is finite $\forall i \in \mathbb{P}$, (X_i) satisfies the Mittag-Leffler condition (cf. HW6 3(i)), hence

$$0 \to \varprojlim X_i \to \varprojlim \mathrm{Ext}^0_R(K, M_i) \to \varprojlim \mathrm{Ext}^1_R(A, M_i) \to 0$$

is exact, in particular α is onto. As in the seventh

handout, for each $i \in P$ we have an R-map $f_i : \varprojlim M_i \longrightarrow M_i$, and this induces a map

$(f_i)_{n*} = \mathrm{Ext}^n_R(A, \varprojlim M_i) \longrightarrow \mathrm{Ext}^n_R(A, M_i)$ which

in turn induces a map $g_n(A) : \mathrm{Ext}^n_R(A, \varprojlim M_i) \longrightarrow$

$\varprojlim \mathrm{Ext}^n_R(A, M_i)$. Now consider the commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Ext}^0_R(F, \varprojlim M_i) & \longrightarrow & \mathrm{Ext}^0_R(K, \varprojlim M_i) & \longrightarrow & \mathrm{Ext}^1_R(A, \varprojlim M_i) & \longrightarrow & 0 \\
\downarrow g_0(F) & & \downarrow g_0(K) & & \downarrow g_1(A) & & \\
\varprojlim \mathrm{Ext}^0_R(F, M_i) & \longrightarrow & \varprojlim \mathrm{Ext}^0_R(K, M_i) & \xrightarrow{\alpha} & \varprojlim \mathrm{Ext}^1_R(A, M_i) & \longrightarrow & 0
\end{array}$$

The top row is exact (because it is the long exact sequence for Ext in the first variable applied to the exact sequence $0 \to K \to F \to A \to 0$), $g_0(F)$ and $g_0(K)$ are isomorphisms (see sixth handout), and α is onto (just proved). A piece of diagram chasing now shows that $g_1(A)$ is an isomorphism and the case $n=1$ is proven.

(3) Let (P, α_0) be a projective resolution for A
Let (Q, β_0) be a projective resolution for B
Let (T, γ_0) be a projective resolution for C.

The map $p \otimes (q \otimes t) \longmapsto (p \otimes q) \otimes t$ induces an isomorphism $P \otimes_R (Q \otimes_R T) \to (P \otimes_R Q) \otimes_R T$ which commutes with the boundary maps, hence

$$H_n(P \otimes_R (Q \otimes_R T)) \cong H_n((P \otimes_R Q) \otimes_R T)$$

$\forall n$. We use the Künneth formula to compute both sides of the above isomorphism.

$$\text{LHS} \cong A \otimes_R H_2(Q \otimes_R T) \oplus \operatorname{Tor}_1^R(A, H_1(Q \otimes_R T))$$
$$\cong A \otimes_R 0 \oplus \operatorname{Tor}_1^R(A, \operatorname{Tor}_1^R(B, C))$$
$$= \operatorname{Tor}_1^R(A, \operatorname{Tor}_1^R(B, C))$$

(Remember $H_0(P) \cong A$ and $H_i(P) = 0$ for $i > 0$.)

Similarly

$$\text{RHS} \cong H_2(P \otimes_R Q) \otimes_R C \oplus \operatorname{Tor}_1^R(H_1(P \otimes_R Q), C)$$
$$\cong 0 \otimes_R C \oplus \operatorname{Tor}_1^R(\operatorname{Tor}_1^R(H_0(P) \otimes_R H_0(Q)), C)$$
$$\cong \operatorname{Tor}_1^R(\operatorname{Tor}_1^R(A, B), C).$$

The result follows.

(4) Embed $H_0(A)$ in an injective R-module B_0, and then choose a monomorphism $\gamma_1 : B_0 \to I_1$ where I_1 is an injective R-module and $\ker \gamma_1 = H_0(A)$. Set $\beta_0 : 0 \to B_0$ to be the zero map.

Suppose we have constructed a sequence of injective R-modules

$$0 \xrightarrow{\beta_0} B_0 \xrightarrow{\beta_1} B_1 \to \cdots \xrightarrow{\beta_n} B_n$$

such that $\beta_i \beta_{i-1} = 0$, $\ker \beta_i / \operatorname{im} \beta_{i-1} \cong H_{i-1}(A)$,

injective R-modules $I_1, \ldots, I_{n+1}$, R-maps $\gamma_{i+1} : B_i \to I_{i+1}$ such that $\gamma_{i+1} \beta_i = 0$, and

$\ker \gamma_{i+1} / \operatorname{im} \beta_i \cong H_i(A) \quad (1 \le i \le n)$.

The above is certainly OK for $n = 0$; we want to show that we can do it for $n+1$, for then the result would follow by induction.

Embed $H_{n+1}(A)$ in an injective module B'_{n+1}

and set $B_{n+1} = B'_{n+1} \oplus I_{n+1}$. Define $\beta_{n+1} : B_n \to B_{n+1}$ by $\beta_{n+1} b = (0, \gamma_{n+1} b)$.

Choose an R-map $\gamma_{n+2} : B_{n+1} \to I_{n+2}$ such that

I_{n+2} is injective and $\ker \gamma_{n+2} = H_{n+1}(A) \oplus \operatorname{im} \gamma_{n+1}$.

Then $\beta_{n+1}\beta_n = 0$, $\dfrac{\ker \beta_{n+1}}{\operatorname{im} \beta_n} \cong H_n(A)$,

$\gamma_{n+2}\beta_{n+1} = 0$, and $\dfrac{\ker \gamma_{n+2}}{\operatorname{im} \beta_{n+1}} \cong H_{n+1}(A)$.

This means that the induction step is complete and the result is proven.

(5) First note that in the proof of the Künneth formula, the property of k being left hereditary is only used to show that the short exact sequence splits; the existence of the short exact sequence is OK. Thus by naturality, we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 \to & \bigoplus_{r+s=n} H_r(A) \otimes_k H_s(B) & \longrightarrow & H_n(A \otimes_k B) & \longrightarrow & \bigoplus_{r+s=n-1} \mathrm{Tor}_1^k(H_r(A), H_s(B)) & \to 0 \\
& \downarrow \bigoplus 1 \otimes \theta_{s*} & & \downarrow (1 \otimes \theta)_* & & \bigoplus \theta_{s*} & \\
0 \to & \bigoplus_{r+s=n} H_r(A) \otimes_k H_s(C) & \longrightarrow & H_n(A \otimes_k C) & \longrightarrow & \bigoplus_{r+s=n-1} \mathrm{Tor}_1^k(H_r(A), H_s(C)) & \to 0
\end{array}$$

The two outside vertical maps are isomorphisms, hence by the five lemma the middle map is an isomorphism as required.

[Remark: the five lemma is just a very special case of Lemma 1 of handout 5: it says if

$$\begin{array}{ccccccccc}
0 & \to & A_1 & \to & B_1 & \to & C_1 & \to & 0 \\
 & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\
0 & \to & A_2 & \to & B_2 & \to & C_2 & \to & 0
\end{array}$$

is a commutative diagram with exact rows, and α, γ are isomorphisms, then β is also an isomorphism.

(6) Define $\theta : kG \times kH \to k[G \times H]$ by $\theta(\alpha, \beta) = \alpha\beta$. Then θ is easily seen to be a k-balanced map, so it induces k-map $\hat{\theta} : kG \otimes_k kH \longrightarrow k[G \times H]$ satisfying $\hat{\theta}(\alpha \otimes \beta) = \alpha\beta$. Let $U = \{g \otimes h \mid g \in G, h \in H\}$. Then U is a subgroup of the group of units of $kG \otimes_k kH$. It follows that the map

$(g, h) \longmapsto g \otimes h$ (a group homomorphism)

induces a k-algebra homomorphism

$\phi : k[G \times H] \longrightarrow kG \otimes_k kH$

satisfying $\phi(g, h) = g \otimes h$. It is easy to see that θ and ϕ are the identity maps and the result follows.

(7) (i) $\mathrm{Ext}^0_{\mathbb{Z}}(B_i, A) = \mathrm{Hom}_{\mathbb{Z}}(B_i, A)$ and if $f \in \mathrm{Hom}_{\mathbb{Z}}(B_i, A)$, then $fB_i \subseteq$ elements of order dividing p^i. It follows that $\mathrm{Ext}^0_{\mathbb{Z}}(B_i, A)$ is finite $\forall i \in \mathbb{P}$, so

$\mathrm{Ext}^0_{\mathbb{Z}}(B_i, A)$ satisfies the Mittag Leffler condition.

(ii) Write $B = \varinjlim B_i = \varinjlim \mathbb{Z}/p^i\mathbb{Z}$. Then the exact sequence of the theorem at the end of handout 7 yields an exact sequence

$$0 \to \varprojlim{}^1 \mathrm{Ext}^0_{\mathbb{Z}}(B_i, A) \to \mathrm{Ext}^1_{\mathbb{Z}}(B, A) \to \varprojlim \mathrm{Ext}^1_{\mathbb{Z}}(B_i, A) \to 0.$$

Now $\varprojlim{}^1 \mathrm{Ext}^0_{\mathbb{Z}}(B_i, A) = 0$ by (i), and $\mathrm{Ext}^1_{\mathbb{Z}}(B_i, A) \cong$

A/Ap^i by HW3 prob. 2(ii); moreover the map $\mathrm{Ext}^1_R(B_{i+1}, A) \to \mathrm{Ext}^1_R(B_i, A)$ is the natural epimorphism $A/Ap^{i+1} \to A/Ap^i$; thus $\mathrm{Ext}^1_{\mathbb{Z}}(B, A) \cong$ $\varprojlim A/Ap^i$. Since $\mathbb{Q}/\mathbb{Z} \cong \bigoplus_q \varinjlim \mathbb{Z}/q^i\mathbb{Z}$ where the direct sum is over all primes, we may apply HW2 prob 5 to deduce that

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}) \cong \prod_q \varprojlim A/q^iA.$$

Ninth Homework Solutions

(1) (i) Since P is projective, $\exists$ R-module Q such that $P \oplus Q \cong F$ where F is a free R-module. Now $R \otimes_R B \cong B$, so $F \otimes_R B \cong \bigoplus_{i \in \mathcal{J}} B$ (as $\mathbb{Z}$-modules) for some indexing set $\mathcal{J}$. Since B is free as a $\mathbb{Z}$-module, $\bigoplus_{i \in \mathcal{J}} B$ is also free and it follows that $P \otimes_R B$ is a free $\mathbb{Z}$-module.

(ii) Let (P, α_0) be a projective resolution for A. Since $B/pB \cong B \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$ as R-modules, it follows that

$$\begin{aligned} \operatorname{Tor}_R^n(A, B/pB) &\cong H_n(P \otimes_R B/pB) \\ &\cong H_n((P \otimes_R B) \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}). \end{aligned}$$

Now $P \otimes_R B$ is a projective $\mathbb{Z}$-chain complex by (i) and $\mathbb{Z}$ is hereditary, so we can apply the Universal coefficient theorem to obtain

$$\begin{aligned} \operatorname{Tor}_n^R(A, B/pB) &\cong H_n(P \otimes_R B) \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \oplus \\ &\qquad \operatorname{Tor}_1^{\mathbb{Z}}(H_{n-1}(P \otimes_R B), \mathbb{Z}/p\mathbb{Z}) \\ &\cong \operatorname{Tor}_n^R(A, B) \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \oplus \\ &\qquad \operatorname{Tor}_1^{\mathbb{Z}}(\operatorname{Tor}_{n-1}^R(A, B), \mathbb{Z}/p\mathbb{Z}). \end{aligned}$$

(2) (i) This is routine checking. Clearly the maps

$\hat{X} : u \otimes v \mapsto uX \otimes v$ and $\hat{Y} : u \otimes v \mapsto u \otimes vY$

are k-endomorphism of $U \otimes_k V$, and since $\hat{X}\hat{Y} = \hat{Y}\hat{X}$, the assignment $X \mapsto \hat{X}$, $Y \mapsto \hat{Y}$ define a k-algebra homomorphism of $k[X, Y]$ into $\mathrm{End}_k(U \otimes_k V)$: this is equivalent to saying that

$U \otimes_k V$ is a $k[X, Y]$-module with the required property $(u \otimes v)X = uX \otimes v$, $(u \otimes v)Y = u \otimes (vY)$.

We have a k-balanced map $k[X] \times k[Y] \to k[X, Y]$ defined by $(\alpha, \beta) \mapsto \alpha\beta$ which induces a k-map $k[X] \otimes_k k[Y] \to k[X, Y]$. This is clearly a map of $k[X, Y]$-modules. It has an inverse defined by $X^i Y^j \mapsto X^i \otimes Y^j$, so the map is a $k[X, Y]$-isomorphism. It follows if U and V are free $k[X]$ and $k[Y]$-modules respectively, then $U \otimes_k V$ is a free $k[X, Y]$-module. If U and V a projective, we may choose $k[X]$, $k[Y]$-modules U_1, V_1 respectively such that $U \oplus U_1$ and $V \oplus V_1$ are free. Then the isomorphism

$$(U \oplus U_1) \otimes_k (V \oplus V_1) \cong U \otimes_k V \oplus U_1 \otimes_k V \oplus U \otimes_k V_1 \oplus U_1 \otimes_k V_1$$

shows that $U \otimes_k V$ is a projective $k[X, Y]$-module.

(ii) Let (P, α_0) be a projective resolution for U with $k[X]$-modules, and let (Q, β_0) be a projective resolution for V with $k[Y]$-modules. Then $(P \otimes_k Q, \alpha_0 \otimes \beta_0)$ is a $k[X,Y]$-resolution of $U \otimes_k V$ by the Künneth formula, and it is projective by (i). Write $R = k[X,Y]$. We regard P as a $k[X,Y]$-module with Y acting trivially, and Q as a $k[X,Y]$-module with X acting trivially.

$$\mathrm{Tor}_n^R(U \otimes_k V, k) \cong H_n(P \otimes_k Q \otimes_R k)$$

$$\cong H_n(P \otimes_k Q \otimes_R k \otimes_R k) \qquad (\text{since } k \otimes_R k \cong k)$$

$$\cong H_n(k \otimes_R P \otimes_k Q \otimes_R k) \qquad (\text{since } A \otimes_R k \cong k \otimes_R A)$$

$$\cong H_n([P \otimes_R k] \otimes_k [Q \otimes_R k])$$

$$\cong \bigoplus_{r+s=n} H_r(P \otimes_R k) \otimes_k H_s(Q \otimes_R k) \qquad (\text{By Künneth formula})$$

$$\cong \bigoplus_{r+s=n} \mathrm{Tor}_r^{k[X]}(U, k) \otimes_k \mathrm{Tor}_s^{k[Y]}(V, k).$$

This is a k-isomorphism, but it is easily seen to be also a $k[X,Y]$-isomorphism.

(3) (i) Let U be an R-module. Then for $n \in \mathbb{N}$, the exact sequence

$$0 \to \ker \alpha_n \to A_n \to \operatorname{im} \alpha_{n-1} \to 0$$

yields an exact sequence

$$\cdots \to \operatorname{Tor}_2^R(\operatorname{im} \alpha_{n-1}, U) \to \operatorname{Tor}_1^R(\ker \alpha_n, U) \to \operatorname{Tor}_1^R(A_n, U) \to \cdots,$$

Since $\operatorname{Tor}_2^R(\operatorname{im} \alpha_{n-1}, U) = 0 = \operatorname{Tor}_1^R(A_n, U)$, it follows that $\operatorname{Tor}_1^R(\ker \alpha_n, U) = 0$. Therefore $\ker \alpha_n$ is a flat R-module.

(ii) The proof is identical to the proof of the Künneth formula in the notes: here is a sketch for what one is suppose to do.

First one considers the case A has trivial boundary, i.e. $\alpha_n = 0 \ \forall n$. In the case given in the notes one does not use the hypothesis that k is commutative hereditary, only that A is projective. The reason for this is that we wanted $H_s(A_r \otimes_R B) \cong A_r \otimes_R H_s(B)$. However this is true even if A_r is only assumed to be flat. Thus the proof is OK in the case A has trivial boundary.

Now the general case (i.e. when the boundary of A is not necessarily trivial). As in the notes write $C_n = \ker \alpha_n : A_n \to A_{n-1}$ and $D_n = \operatorname{im} \alpha_n : A_n \to A_{n-1}$. Then

$0 \to C \to A \to D \to 0$ is an exact sequence of chain complexes, and hence so is

$0 \to C \otimes_R B \to A \otimes_R B \to D \otimes_R B \to 0$ because D is flat (for each $i, j \in \mathbb{N}$, we want

$$0 \to C_i \otimes_R B_j \to A_i \otimes_R B_j \to D_i \otimes_R B_j \to 0$$

exact; this is in fact the case because D_i flat implies $\mathrm{Tor}_1^R(D_i, B_j) = 0$. Finally C and D are flat chain complexes with trivial boundary: D is flat by hypothesis and C is flat by (i). Thus we may apply the Künneth formula to $C \otimes_R B$ and $D \otimes_R B$. From the exact sequence $0 \to D_{r+1} \to C_r \to H_r(A) \to 0$, the long exact for Tor in the first argument yields an exact sequence

$0 \to \mathrm{Tor}_1^R(H_r(A), H_s(B)) \to D_{r+1} \otimes_R H_s(B) \to$
$C_r \otimes_R H_s(B) \to H_r(A) \otimes_R H_s(B) \to 0$, hence we have a commutative diagram with exact row

$$\begin{array}{ccccccccc}
0 \to \bigoplus_{r+s=n} \mathrm{Tor}_1^R(H_r(A), H_s(B)) & \to & \bigoplus_{r+s=n} D_{r+1} \otimes_R H_s(B) & \to & \bigoplus_{r+s=n} C_r \otimes_R H_s(B) & & & & \\
 & & \delta \downarrow & & \downarrow \gamma & & & & \\
\to H_{n+1}(A \otimes_R B) & \xrightarrow{\phi_{n+1}} & H_{n+1}(D \otimes_R B) & \xrightarrow{\partial_{n+1}} & H_n(C \otimes_R B) & & & & \\
\to \bigoplus_{r+s=n} H_r(A) \otimes_R H_s(B) & \to & 0 & & & & & & \\
\pi \downarrow & & & & & & & & \\
\to H_n(A \otimes_R B) & \xrightarrow{\phi_n} & \cdots & & & & & &
\end{array}$$

where δ and γ are isomorphisms (by the special case of trivial boundary). We can now follow the argument as in the notes.

(4) (i) Write $G = t_1 H \cup \cdots \cup t_n H$. Then $\mathbb{Z}G = t_1 \mathbb{Z}H + \cdots + t_n \mathbb{Z}H$ which shows that $\mathbb{Z}G$ is a finitely generated right $\mathbb{Z}H$-module. Since $\mathbb{Z}H$ is right Noetherian, it follows that $\mathbb{Z}G$ is right Noetherian. Similarly $\mathbb{Z}G$ is left Noetherian.

(ii) If G has $\mathbb{Z}$-basis (= free generators) $\{x_1, \ldots, x_n\}$, then $\mathbb{Z}G \cong \mathbb{Z}[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$ and it is well known that this latter ring is Noetherian (proof; the polynomial ring in $2n$ variables $\mathbb{Z}[Y_1, Z_1, \ldots, Y_n, Z_n]$ is Noetherian by Hilbert's basis theorem, and maps onto $\mathbb{Z}[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$ via $Y_i \mapsto X_i$, $Z_i \mapsto X_i^{-1}$. Since homomorphic images of Noetherian rings are Noetherian, it follows that $\mathbb{Z}[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$ is Noetherian). That $\mathbb{Z}G$ is Noetherian now follows from (i).

(5) Since A is finitely generated abelian, it has a torsion free subgroup B of finite index in A. Now $[G:B] < \infty$, so $\exists\ C \lhd G$, $C \subseteq B$ such that $[G:C] < \infty$ (we could take $C = \bigcap_{t \in T} t^{-1}Bt$ where T is a right transversal for B in G). Since C is a torsion free subgroup of the finitely generated abelian group A, it follows that C is finitely generated free abelian. By prob. 4, we deduce that $\mathbb{Z}G$ is Noetherian.

Now $H^n(G, M_i) = \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, M_i)$ and $\mathbb{Z}$ is certainly a finitely generated $\mathbb{Z}G$-module, so the result follows from HW5 no. 5.

(6) Let $\mathcal{I}$ be the relevant indexing set, and let the direct system be (G_i, f_i^j). The f_i^j induce $\mathbb{Z}$-algebra homomorphisms $\mathbb{Z}G_i \to \mathbb{Z}G_j$ which we shall also denote by f_i^j.

(i) For each $i \in \mathcal{I}$, there is a group homomorphism $f_i : G_i \to \varinjlim G_i$ which induces a ring homomorphism

$\mathbb{Z}G_i \to \mathbb{Z}\varinjlim G_i$, which we shall also denote by f_i.

Let $g_i : \mathbb{Z}G_i \to \varinjlim \mathbb{Z}G_i$ be the corresponding maps

to the direct system $(\mathbb{Z}G_i, f_i^j)$. Since $f_j f_i^j = f_i$ for $i \leq j$, it follows that there is a ring homomorphism $\theta : \varinjlim \mathbb{Z}G_i \to \mathbb{Z}\varinjlim G_i$ such that

$\theta g_i = f_i \quad \forall i \in \mathcal{I}$.

For each $i \in \mathcal{I}$, there is a group homomorphism $g_i : G_i \to \varinjlim \mathbb{Z}G_i$ (the restriction of g_i to G_i)

which induce a group homomorphism $\phi : \varinjlim G_i \to \varinjlim \mathbb{Z}G_i$ such that $\phi f_i = g_i \ \forall i \in \mathcal{I}$. Then ϕ

will in turn induce a ring homomorphism $\mathbb{Z}\varinjlim G_i \to \varinjlim \mathbb{Z}G_i$ which we shall also denote

by ϕ, and $\phi f_i = g_i \ \forall i \in \mathcal{J}$ again.

Since $\theta\phi f_i = \theta g_i = f_i \ \forall i \in \mathcal{J}$, $\theta\phi$ is the identity on $\mathbb{Z}\varinjlim G_i$. Since $\phi\theta g_i = g_i$ $\forall i \in \mathcal{J}$, $\phi\theta$ is the identity on $\varinjlim \mathbb{Z}G_i$. Therefore

$$\mathbb{Z}\varinjlim G_i \cong \varinjlim \mathbb{Z}G_i .$$

(ii) $$H_n(\varinjlim G_i, \mathbb{Z}) = \mathrm{Tor}_n^{\mathbb{Z}\varinjlim G_i}(\mathbb{Z}, \mathbb{Z})$$
$$\cong \mathrm{Tor}_n^{\varinjlim \mathbb{Z}G_i}(\mathbb{Z}, \mathbb{Z})$$

By the Proposition in Shapiro's lemma, $\mathbb{Z}G_j$ is a free (and hence flat) $\mathbb{Z}G_i$-module. We may now apply HW6 prob 7 (ii) to deduce that

$$\varinjlim \mathrm{Tor}_n^{\mathbb{Z}G_i}(\mathbb{Z}, \mathbb{Z}) \cong \mathrm{Tor}_n^{\varinjlim \mathbb{Z}G_i}(\mathbb{Z}, \mathbb{Z}),$$

hence $$H_n(\varinjlim G_i, \mathbb{Z}) \cong \varinjlim H_n(G_i, \mathbb{Z}).$$

(iii) $$H^1(\varinjlim G_i, \mathbb{Z}) \cong \mathrm{Hom}(\varinjlim G_i, \mathbb{Z})$$
$$\cong \varprojlim \mathrm{Hom}(G_i, \mathbb{Z})$$

(same proof as HW6 no. 2)

$$\cong \varprojlim H^1(G_i, \mathbb{Z}).$$

Tenth Homework Solutions

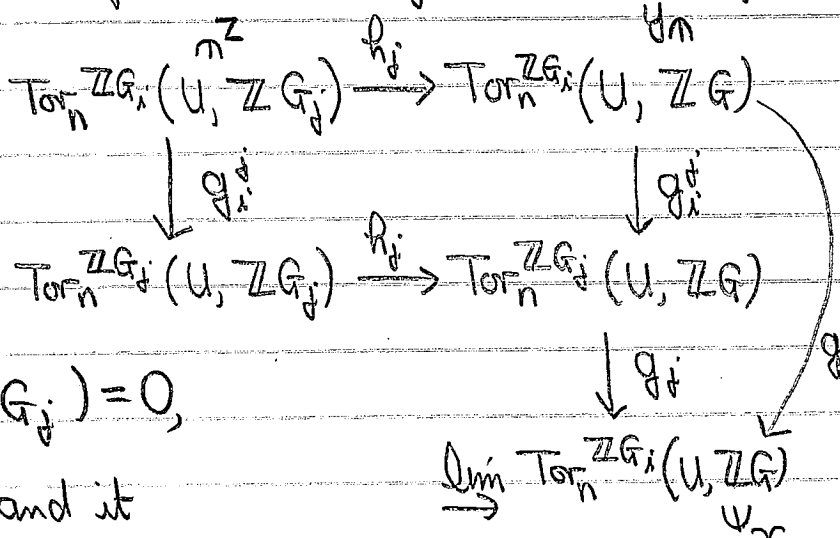
(1) Write the direct system as (G_i, f_i^j), so we have corresponding maps $f_i : G_i \to G_j$.

Let $(\mathrm{Tor}_n^{\mathbb{Z}G_i}(U, \mathbb{Z}G))$ have corresponding maps g_i^j and

$g_i : \mathrm{Tor}_n^{\mathbb{Z}G_i}(U, \mathbb{Z}G) \to \varinjlim \mathrm{Tor}_n^{\mathbb{Z}G_i}(U, \mathbb{Z}G)$

(so $g_i^j = f_{i*}^j$). If $x \in \varinjlim \mathrm{Tor}_n^{\mathbb{Z}G_i}(U, \mathbb{Z}G)$

$x = g_i y$ for some i and some $y \in \mathrm{Tor}_n^{\mathbb{Z}G_i}(U, \mathbb{Z}G)$.
Now $\mathbb{Z}G \cong \varinjlim \mathbb{Z}G_i$ (HW9, 6(i)) and Tor commutes with direct limits, so $\exists\ j \geq i$ and $z \in \mathrm{Tor}_n^{\mathbb{Z}G_i}(U, \mathbb{Z}G_j)$ such that $y = h_j z$ where $h_j : \mathrm{Tor}_n^{\mathbb{Z}G_i}(U, \mathbb{Z}G_j) \to$ $\mathrm{Tor}_n^{\mathbb{Z}G_i}(G_i, \mathbb{Z}G)$ is f_{j*}. The diagram on the right is commutative,



so $x = g_i h_j z$
$= g_j h_j (g_i^j z)$.

But $\mathrm{Tor}_n^{\mathbb{Z}G_j}(U, \mathbb{Z}G_j) = 0$,
hence $g_i^j z = 0$ and it follows that $x = 0$. Thus $\varinjlim \mathrm{Tor}_n^{\mathbb{Z}G_i}(U, \mathbb{Z}G) = 0$.

(2) As in HW3 prob. 6, let

$(P, \alpha_0) : \cdots \xrightarrow{\alpha_2} P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} M \to 0$ and

$(Q, \beta_0) : \cdots \xrightarrow{\beta_2} Q_1 \xrightarrow{\beta_1} Q_0 \xrightarrow{\beta_0} M \to 0$

be projective resolutions for M as $\mathbb{Z}H$ and $\mathbb{Z}G$-modules respectively. Then by HW3 prob. 6(ii) $\exists$ $\mathbb{Z}H$-maps $\theta_n : P_n \to Q_n$ such that $\beta_n \theta_n = \theta_{n-1} \alpha_n$ $\forall n \in \mathbb{N}$, where θ_{-1} is the identity map on M.

Note Given $\mathbb{Z}G$-modules U, V, then there is a well defined group homomorphism

$$U \otimes_{\mathbb{Z}H} V \longrightarrow U \otimes_{\mathbb{Z}G} V$$

given by the formula $u \otimes v \longmapsto u \otimes v$

(we have a $\mathbb{Z}H$-balanced map $U \times V \to U \otimes_{\mathbb{Z}G} V$ given by $(u, v) \longmapsto u \otimes v$, so it induces a group homomorphism $U \otimes_{\mathbb{Z}H} V \to U \otimes_{\mathbb{Z}G} V$: it is clear that this homomorphism is in fact onto, but we do not need this fact here).

Proceeding as in the proof of HW3 prob 6(ii), the θ_n induce well defined group homomorphisms $\theta_n^* : \mathrm{Tor}_n^{\mathbb{Z}H}(M, N) \to \mathrm{Tor}_n^{\mathbb{Z}G}(M, N)$ which are induced by $P_n \otimes_{\mathbb{Z}H} N \to P_n \otimes_{\mathbb{Z}G} N$.

We will let $\theta : \mathbb{Z}H \to \mathbb{Z}G$ also denote the map induced by $\theta : H \to G$. By naturality, the exact sequences

$$0 \to \mathcal{I} \to \mathbb{Z}H \to \mathbb{Z} \to 0$$

$$0 \to \mathcal{J} \to \mathbb{Z}G \to \mathbb{Z} \to 0$$

yield a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \to & \mathrm{Tor}_1^{\mathbb{Z}H}(\mathbb{Z},\mathbb{Z}) & \to & \mathcal{I} \otimes_{\mathbb{Z}H} \mathbb{Z} & \xrightarrow{\alpha_1 \otimes 1} & \mathbb{Z}H \otimes_{\mathbb{Z}H} \mathbb{Z} & \\ & \theta_{1*} \downarrow & & \downarrow \theta \otimes 1 & & \downarrow \theta \otimes 1 & \\ 0 \to & \mathrm{Tor}_1^{\mathbb{Z}G}(\mathbb{Z},\mathbb{Z}) & \to & \mathcal{J} \otimes_{\mathbb{Z}G} \mathbb{Z} & \xrightarrow{\alpha_2 \otimes 1} & \mathbb{Z}G \otimes_{\mathbb{Z}G} \mathbb{Z} & \end{array}$$

where the α_i are the inclusion maps. Note that for $h \in H$, $(\alpha_1 \otimes 1)((h-1) \otimes 1) = (h-1) \otimes 1 = h \otimes 1 - 1 \otimes 1 = 1 \otimes 1 - 1 \otimes 1 = 0$, so $\mathrm{im}\, \alpha_1 \otimes 1 = 0$. Similarly $\mathrm{im}\, \alpha_2 \otimes 1 = 0$. Thus

$\theta_{1*} : H_1(H, \mathbb{Z}) \to H_1(G, \mathbb{Z})$ can be identified with $\mathcal{I} \otimes_{\mathbb{Z}H} \mathbb{Z} \xrightarrow{\theta \otimes 1} \mathcal{J} \otimes_{\mathbb{Z}G} \mathbb{Z}$

i.e. $H/H' \xrightarrow{\bar{\theta}} G/G'$,

where $\bar{\theta}(H'h) = G'\theta h$.

(3) (i) Using the description in handout 6 that a direct limit is the quotient of a direct sum D/F, it is easy to see that direct limits commute with arbitrary direct sums. Also Tor commutes with arbitrary direct sums so if $F = \bigoplus F_j$ where $F_i \cong \mathbb{Z}G$ $\forall i$, then

$$\varinjlim \operatorname{Tor}_n^{\mathbb{Z}G_i}(U, F) = \varinjlim \operatorname{Tor}_n^{\mathbb{Z}G_i}(U, \bigoplus F_j)$$

$$\cong \varinjlim \bigoplus \operatorname{Tor}_n^{\mathbb{Z}G_i}(U, F_j)$$

$$\cong \bigoplus \varinjlim \operatorname{Tor}_n^{\mathbb{Z}G_i}(U, \mathbb{Z}G)$$

$$= 0 \quad \text{by prob. 1}$$

(ii) Choose an exact sequence

$$0 \to K \to F \to M \to 0$$

where F is a free $\mathbb{Z}G$-module. Then the long exact sequence for Tor in the second variable yields a commutative diagram with exact rows

$$\begin{array}{ccccccc}
\varinjlim \operatorname{Tor}_1^{\mathbb{Z}G_i}(U,F) & \to & \varinjlim \operatorname{Tor}_1^{\mathbb{Z}G_i}(U,M) & \to & \varinjlim U \otimes_{\mathbb{Z}G_i} K & \to & \varinjlim U \otimes_{\mathbb{Z}G_i} F \\
\downarrow 0 \text{ by (i)} & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\
0 = \operatorname{Tor}_1^{\mathbb{Z}G}(U,F) & \to & \operatorname{Tor}_1^{\mathbb{Z}G}(U,M) & \to & U \otimes_{\mathbb{Z}G} K & \to & U \otimes_{\mathbb{Z}G} F
\end{array}$$

The top row is exact because $\varinjlim$ is exact, and β, γ are isomorphisms by HW6 prob 7(i). The map α is defined as follows: by prob 2 we have maps

$\mathrm{Tor}_1^{\mathbb{Z}G_i}(U,M) \to \mathrm{Tor}_1^{\mathbb{Z}G}(U,M)$ such that the diagram

$$\begin{array}{ccc} \mathrm{Tor}_1^{\mathbb{Z}G_i}(U,M) & \longrightarrow & \mathrm{Tor}_1^{\mathbb{Z}G}(U,M) \\ & \searrow \quad \nearrow & \\ & \mathrm{Tor}_1^{\mathbb{Z}G_j}(U,M) & \end{array}$$

is commutative whenever $i \le j$. Hence they induce a map $\alpha : \varinjlim \mathrm{Tor}_1^{\mathbb{Z}G_i}(U,M) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}G}(U,M)$.

A piece of diagram chasing now shows that α is also an isomorphism.

(iii) This is similar to (ii). The case $n=0$ is HW6 prob 7(i), while the case $n=1$ is (ii). If $n>1$, choose as in (ii) an exact sequence of $\mathbb{Z}G$-modules $0 \to K \to F \to M \to 0$ with F free. Then the long exact sequence for Tor in the second variable yields a commutative diagram

$$\begin{array}{ccccccccc} \varinjlim \mathrm{Tor}_n^{\mathbb{Z}G_i}(U,F) & \to & \varinjlim \mathrm{Tor}_n^{\mathbb{Z}G_i}(U,M) & \to & \varinjlim \mathrm{Tor}_{n-1}^{\mathbb{Z}G_i}(U,K) & \to & \varinjlim \mathrm{Tor}_{n-1}^{\mathbb{Z}G_i}(U,F) & & \\ \| 0 & & \downarrow & & \cong \downarrow \text{by induction.} & & \downarrow & & \| 0 \\ 0 = \mathrm{Tor}_n^{\mathbb{Z}G}(U,F) & \longrightarrow & \mathrm{Tor}_n^{\mathbb{Z}G}(U,M) & \to & \mathrm{Tor}_{n-1}^{\mathbb{Z}G}(U,K) & \longrightarrow & \mathrm{Tor}_{n-1}^{\mathbb{Z}G}(U,F) & & \| 0 \end{array}$$

It follows that $\varinjlim \mathrm{Tor}_n^{\mathbb{Z}G_i}(U,M) \cong \mathrm{Tor}_n^{\mathbb{Z}G}(U,M)$

as required.

(4) $H^1(G, \mathbb{Z}) \cong \mathrm{Hom}(G, \mathbb{Z})$.

If $f \in \mathrm{Hom}(G, \mathbb{Z})$ and $g \in G$, then g has finite order because G is finite, hence $f(g)$ has finite order. Since $\mathbb{Z}$ is torsion free, $f(g) = 0$ and it follows that $H^1(G, \mathbb{Z}) = 0$.

The exact sequence $0 \to \sigma\mathcal{J} \to \mathbb{Z}G \to \mathbb{Z} \to 0$ yields an exact sequence

$$H^1(G, \mathbb{Z}G) \to H^1(G, \mathbb{Z}) \to H^2(G, \sigma\mathcal{J}) \to H^2(G, \mathbb{Z}G).$$

Since G is finite, $H^1(G, \mathbb{Z}G) = H^2(G, \mathbb{Z}G) = 0$.
Therefore $H^1(G, \mathbb{Z}) \cong H^2(G, \sigma\mathcal{J})$, so $H^2(G, \sigma\mathcal{J}) = 0$.

(5) $G = \dot{\bigcup}_{t \in T} Ht$ ($\dot{\cup}$ means disjoint union)

(i) $\mathbb{Z}G = \sum_{t \in T} \mathbb{Z}Ht$, so

$I\mathbb{Z}G = I \sum \mathbb{Z}Ht = \sum (I\mathbb{Z}H)t = \sum It.$

The sum is direct because $\mathbb{Z}G = \bigoplus_{t \in T} \mathbb{Z}Ht$ and $I \subseteq \mathbb{Z}H$.

<u>Note</u> If $g \in G \setminus H$, then $g - 1 \notin \mathfrak{f}\mathbb{Z}G$. Indeed if $g - 1 \in \mathfrak{f}\mathbb{Z}G$, then write (using the above) $g - 1 = \sum_{t \in T} \alpha_t t$ where $\alpha_t \in \mathfrak{f}$ $\forall t \in T$. We may assume that $1 \in T$. Since $\mathbb{Z}G = \bigoplus_{t \in T} \mathbb{Z}Ht$, we must have $-1 = \alpha_1$, so $-1 \in \mathfrak{f}$ which is not true.

(ii) Suppose $\mathfrak{o}\mathfrak{f} = \alpha_1 \mathbb{Z}G + \cdots + \alpha_n \mathbb{Z}G$. Write $\alpha_i = \sum a_{ij} g_j$ where $a_{ij} \in \mathbb{Z}$, $g_j \in G$; we may take this to be a finite sum, so if H is the subgroup of G generated by all the g_j above, the H is finitely generated. Furthermore $\alpha_i \in \mathbb{Z}H$ $\forall i$. Let $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$ denote the augmentation map. Since $\alpha_i \in \mathfrak{o}\mathfrak{f}$, it follows that $\varepsilon(\alpha_i) = 0$ and hence $\alpha_i \in \mathfrak{f}$ $\forall i$. Set $I = \alpha_1 \mathbb{Z}H + \cdots + \alpha_n \mathbb{Z}H$. Then $I\mathbb{Z}G = \mathfrak{o}\mathfrak{f}$ (since $\alpha_i \in I$ $\forall i$) and $I \subseteq \mathfrak{f}\mathbb{Z}G$, so $\mathfrak{o}\mathfrak{f} \subseteq \mathfrak{f}\mathbb{Z}G$. By the Note above, $H = G$, so G is finitely generated.

(iii) Suppose we have a strictly ascending chain of subgroups $H_1 < H_2 < \cdots \leq G$. Let $\mathcal{I}_i$ denote the augmentation ideal of $\mathbb{Z}H_i$. Then $\mathcal{I}_1\mathbb{Z}G \subseteq \mathcal{I}_2\mathbb{Z}G \subseteq \cdots \subseteq \mathbb{Z}G$ is an ascending chain of right ideals of $\mathbb{Z}G$. If $g \in H_{i+1} \setminus H_i$, then $g-1 \notin \mathcal{I}_{i+1}\mathbb{Z}G \setminus \mathcal{I}_i\mathbb{Z}G$ by the Note, so the chain is <u>strictly</u> ascending. This contradicts the hypothesis that $\mathbb{Z}G$ is right Noetherian.

(6) If $\mathfrak{I}$ is a finitely generated $\mathbb{Z}G$-module, then G is a finitely generated group by prob. 5(ii).

Conversely suppose $G = \langle g_1, g_2, \dots \rangle$ and let $H = \{ g \in G \mid g-1 \in (g_1-1)\mathbb{Z}G + (g_2-1)\mathbb{Z}G + \cdots \}$. We prove that H is subgroup of G. If this is established, then since $g_i \in H \ \forall i$, we will have $H = G$ and the result will follow.

Obviously $1 \in H$, and if $a \in H$ then $a^{-1} \in H$ (since $a^{-1} - 1 = -(a-1)a^{-1}$). So let $a, b \in H$; we want to prove $ab \in H$. This can be seen from the equality $ab - 1 = (a-1)b + (b-1)$.

(7) Let $\theta : H \to G$ denote the natural inclusion, and let $\phi : G \to H$ denote the natural epimorphism. Then θ and ϕ induce group homomorphisms

$$\theta_n^* : H^n(G, M) \to H^n(H, M) \text{ and}$$

$$\phi_n^* : H^n(H, M) \to H^n(G, M) \qquad (1)$$

[θ_n^* is usually called the restriction map, and ϕ_n^* the inflation map].

Remark Why was the original problem wrong when M was an arbitrary $\mathbb{Z}G$-module? It is in (1) above: M needs to be a $\mathbb{Z}H$-module here, and it becomes a $\mathbb{Z}G$-module via the map ϕ. i.e. $mg := m(\phi g)$, so in particular $ma = m$ $\forall\, a \in A$ because $\ker \phi = A$. This means that the required $\mathbb{Z}G$-module structure on M agrees with the original $\mathbb{Z}G$-module structure on M if and only if A acts trivially on M. The question would have been OK with the weaker hypothesis of just A acting trivially on M.

Now $\theta_n^* \phi_n^* = (\phi_n \theta_n)^* = (\text{identity})^* :$

$$H^n(H, M) \to H^n(G, M) \to H^n(H, M).$$

This shows that $H^n(H, M)$ is naturally isomorphic to a subgroup of $H^n(G, M)$.

Example to show $H^n(A, M)$ not isomorphic to a subgroup of $H^n(G, M)$.

Take $A = \mathbb{Z}/3\mathbb{Z} = \langle (123) \rangle$, $G = S_3$ (symmetric group on $\{1,2,3\}$), $M = \mathbb{Z}/3\mathbb{Z}$ with G acting trivially. Then $H^1(A, M) \cong \mathrm{Hom}(A, \mathbb{Z}/3\mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}$, yet $H^1(G, M) \cong \mathrm{Hom}(G, \mathbb{Z}/3\mathbb{Z}) \cong \mathrm{Hom}(G/G', \mathbb{Z}/3\mathbb{Z}) \cong \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) = 0$.

(8) $H^0(G, \mathbb{Z}G) = \mathbb{Z}G^G$ where G is acting on $\mathbb{Z}G$ by right multiplication.

Let $\alpha = \sum_{g \in G} a_g g \in \mathbb{Z}G$ where $a_g \in \mathbb{Z}$ $\forall g \in G$. Then

$$\alpha \in \mathbb{Z}G^G \iff \sum_{g \in G} a_g gx = \sum_{g \in G} a_g g \quad \forall x \in G$$

$$\iff \sum_{g \in G} a_g x^{-1} g = \sum_{g \in G} a_g g$$

$$\iff a_{gx^{-1}} = a_g \quad \forall x \in G$$

$$\iff a_h = a_g \quad \forall h, g \in G. \quad (*)$$

<u>G infinite</u> Since $a_g \neq 0$ for only finitely many $g \in G$, we must have $a_x = 0$ for some $x \in G$, and then it follows from $(*)$ that $a_g = 0$ $\forall g \in G$. Thus $\alpha = 0$ and $H^0(G, \mathbb{Z}G) = 0$.

<u>G finite</u> Using $(*)$, we see that $\mathbb{Z}G^G = \mathbb{Z}\left(\sum_{g \in G} g\right)$. Thus $H^0(G, \mathbb{Z}G) \cong \mathbb{Z}$.

(9) Write $G = \langle g \rangle$.

(i) Define a $\mathbb{Z}G$-map $\theta : \mathbb{Z}G \to {}_gJ$ by $\theta(1) = g - 1$ (so $\theta(\alpha) = (g-1)\alpha \ \forall \alpha \in \mathbb{Z}G$).

θ onto. $\theta(1 + g + \cdots + g^{n-1}) = g^n - 1$, so $\operatorname{im}\theta$ contains a $\mathbb{Z}$-generating set of ${}_gJ$, hence θ is onto.

$\ker\theta = 0$. $\theta(\alpha) = 0 \Rightarrow (g-1)\alpha = 0 \Rightarrow \alpha g = \alpha$
$\Rightarrow \alpha g^n = \alpha \ \forall n \in \mathbb{Z} \Rightarrow \alpha x = \alpha \ \forall x \in G$
$\Rightarrow \alpha = 0$ by prob. (8).
Thus θ is an isomorphism and hence $\mathbb{Z}G \cong {}_gJ$.

(ii) Let $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$ denote the augmentation map. By (i), we have an exact sequence $0 \to \mathbb{Z}G \to \mathbb{Z}G \to \mathbb{Z} \to 0$. It follows immediately that $H^n(G, M) = 0 \ \forall n \geq 2$.

(iii) If $g \in G$, then multiplication by g on $\mathbb{Z}G$ induces the same map on $\operatorname{Ext}^1_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}G)$ as multiplication by g on $\mathbb{Z}$. The latter is the identity map. Therefore G acts trivially on $H^1(G, \mathbb{Z}G)$ when acting by right multiplication on $\mathbb{Z}G$.

(iv) The resolution $0 \to \mathbb{Z}G \to \mathbb{Z}G \to \mathbb{Z} \to 0$ (from (i)) shows that $H^1(G, \mathbb{Z}G)$ is H^1 of the cochain complex $0 \to \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, \mathbb{Z}G) \to \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, \mathbb{Z}G) \to 0$. Therefore as a $\mathbb{Z}G$-module, $H^1(G, \mathbb{Z}G)$ is a quotient of $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, \mathbb{Z}G) \cong \mathbb{Z}G$. But G acts trivially on $H^1(G, \mathbb{Z}G)$ by (iii), and it follows that $H^1(G, \mathbb{Z}G)$ is a quotient of $\mathbb{Z}$.

(v) The exact sequence $0 \to \mathcal{I} \to \mathbb{Z}G \to \mathbb{Z} \to 0$ yields a long exact sequence

$$H^1(G, \mathbb{Z}G) \to H^1(G, \mathbb{Z}) \to H^2(G, \mathcal{I})$$
$$\| \qquad\qquad \| \qquad\qquad \| 0 \text{ by (ii)}$$
$$\mathrm{Hom}(G, \mathbb{Z}) \cong \mathbb{Z}$$

Thus $H^1(G, \mathbb{Z}G)$ maps onto $\mathbb{Z}$, hence $H^1(G, \mathbb{Z}G) \cong \mathbb{Z}$ by (iv).

ELEVENTH HOMEWORK SOLUTIONS

(1) (i) HW1 prob. 2(iii) yields a natural isomorphism of chain complexes

$$\mathrm{Hom}_{\mathbb{Z}H}(P \otimes_{\mathbb{Z}G} \mathbb{Z}G, M) \cong \mathrm{Hom}_{\mathbb{Z}G}(P, \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M)).$$

Now $P \otimes_{\mathbb{Z}G} \mathbb{Z}G \cong P$ and P is projective as a $\mathbb{Z}H$-module (any projective $\mathbb{Z}G$-module is projective as a $\mathbb{Z}H$-module). Therefore

$$H^n(\mathrm{Hom}_{\mathbb{Z}H}(P \otimes_{\mathbb{Z}G} \mathbb{Z}G, M)) \cong H^n(H, M)$$

and the result follows.

(ii) $H^n(\mathrm{Hom}_{\mathbb{Z}G}(P, \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M))) \cong H^n(G, \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M))$, so the result follows (i).

12th HW Delete prob. 7 (since it was 11th HW prob. 9).

prob. 6: "If P is the Sylow p-subgroup" should read "If P is a Sylow p-subgroup"

(2) (i) $\mathfrak{g} \subseteq I\mathfrak{g}$. If $\alpha, \beta \in \mathfrak{g}$, then $[\alpha,\beta] = \alpha\beta - \beta\alpha \in (I\mathfrak{g})^2$. Therefore $\iota[\mathfrak{g},\mathfrak{g}] \subseteq (I\mathfrak{g})^2$, hence ι induces a k-map $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}] \to (I\mathfrak{g})/(I\mathfrak{g})^2$.

(ii) Let $\sigma : \mathfrak{g} \to \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ be the natural epimorphism, and let $A = k \oplus \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ be the associative k-algebra with multiplication $(a,x)(b,y) = (ab, 0)$ for $a, b \in k$ and $x, y \in \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$. Then σ extends to a k-algebra homomorphism $\hat{\sigma} : T(\mathfrak{g}) \to A$ (where $T(\mathfrak{g})$ denotes the tensor algebra). Obviously if $\alpha, \beta \in \mathfrak{g} \subseteq T(\mathfrak{g})$, then $\hat{\sigma}([\alpha,\beta] - \alpha\beta - \beta\alpha) = 0$, so $\hat{\sigma}$ induces a k-algebra homomorphism $\hat{\tau} : U\mathfrak{g} \to A$ (recall that $U\mathfrak{g} = T(\mathfrak{g})/J$, where $\mathfrak{g}$ is identified with the degree one terms of $T(\mathfrak{g})$, and J is the 2-sided ideal of $T(\mathfrak{g})$ generated by $\{[\alpha,\beta] - \alpha\beta - \beta\alpha \mid \alpha, \beta \in \mathfrak{g}\}$). Also $\hat{\tau}(I\mathfrak{g}) \subseteq \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ and $\hat{\tau}(I\mathfrak{g})^2 = 0$. It follows that $\hat{\tau}$ induces a k-module map $\tau : I\mathfrak{g} \to \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ such that $\tau(I\mathfrak{g})^2 = 0$.

(iii) If $\bar{\iota} : \mathfrak{g}/[\mathfrak{g},\mathfrak{g}] \to I\mathfrak{g}/(I\mathfrak{g})^2$ and $\bar{\tau} : I\mathfrak{g}/(I\mathfrak{g})^2 \to \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ are the maps induced by ι and τ respectively, then it is clear that $\bar{\iota}\bar{\tau} = \bar{\tau}\bar{\iota} = \text{identity}$; this proves the result.

(3) The long exact sequence for Ext in the first variable applied to $0 \to I_{\mathcal{O}_J} \xrightarrow{\iota} U_{\mathcal{O}_J} \to k \to 0$ (where ι is the natural inclusion) yields an exact sequence

$$0 \to \mathrm{Hom}_{U_{\mathcal{O}_J}}(k, M) \to \mathrm{Hom}_{U_{\mathcal{O}_J}}(U_{\mathcal{O}_J}, M) \xrightarrow{\iota^*} \mathrm{Hom}_{U_{\mathcal{O}_J}}(I_{\mathcal{O}_J}, M)$$

$$\to H^1(\mathcal{O}_J, M) \to 0$$

because $\mathrm{Ext}^1_{U_{\mathcal{O}_J}}(U_{\mathcal{O}_J}, M) = 0$. Here ι^* is just the restriction map. If $f \in \mathrm{Hom}_{U_{\mathcal{O}_J}}(U_{\mathcal{O}_J}, M)$ and

$x \in I_{\mathcal{O}_J}$, then $f(x) = f(1)x = 0$ because $I_{\mathcal{O}_J}$ acts trivially on M and hence $\iota^* f = 0$. Therefore $H^1(\mathcal{O}_J, M) \cong \mathrm{Hom}_{U_{\mathcal{O}_J}}(I_{\mathcal{O}_J}, M)$. Now if

if $f \in \mathrm{Hom}_{U_{\mathcal{O}_J}}(I_{\mathcal{O}_J}, M)$, then $f((I_{\mathcal{O}_J})^2) = 0$

because $M I_{\mathcal{O}_J} = 0$, so $H^1(\mathcal{O}_J, M) \cong$

$\mathrm{Hom}_{U_{\mathcal{O}_J}}((I_{\mathcal{O}_J})/(I_{\mathcal{O}_J})^2, M) \cong \mathrm{Hom}_k((I_{\mathcal{O}_J})/(I_{\mathcal{O}_J})^2, M)$.

Now apply prob. 2(iii).

Remark prob. 2(iii) is stated for k a field, whereas prob 3 is stated for k an arbitrary commutative ring, so prob. 3 should have said k a field. However the same proof works for prob. 2 if k is an arbitrary commutative ring (though in this case the map ι is not an injection in general); i.e. prob 3 is correct.

(4) Write $G = \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ and $H = \mathbb{Z}/6\mathbb{Z}$. We will consider G as $H \times \mathbb{Z}/3\mathbb{Z}$ and apply the Künneth formula to obtain a split exact sequence of abelian groups

$$0 \to \bigoplus_{r+s=4} H^r(H,\mathbb{Z}) \otimes_{\mathbb{Z}} H^s(\mathbb{Z}/3\mathbb{Z},\mathbb{Z}) \to$$

$$H^4(G,\mathbb{Z}) \to \bigoplus_{r+s=5} \mathrm{Tor}_1^{\mathbb{Z}}(H^r(H,\mathbb{Z}), H^s(\mathbb{Z}/3\mathbb{Z},\mathbb{Z})) \to 0$$

Now $H^1(\mathbb{Z}/3\mathbb{Z},\mathbb{Z}) = 0 = H^3(\mathbb{Z}/3\mathbb{Z},\mathbb{Z}) = H^1(H,\mathbb{Z})$ (see first page of handout 11) and $H^0(F,\mathbb{Z}) \cong \mathbb{Z}$ for any group F, so $H^4(G,\mathbb{Z}) \cong$

$$H^4(\mathbb{Z}/3\mathbb{Z},\mathbb{Z}) \oplus H^2(H,\mathbb{Z}) \otimes_{\mathbb{Z}} H^2(\mathbb{Z}/3\mathbb{Z},\mathbb{Z}) \oplus$$
$$H^4(H,\mathbb{Z}) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H^3(H,\mathbb{Z}), H^2(\mathbb{Z}/3\mathbb{Z},\mathbb{Z})).$$

where we have used $\mathrm{Tor}_1^{\mathbb{Z}}(?,\mathbb{Z}) = 0$. Now $H^2(\mathbb{Z}/3\mathbb{Z},\mathbb{Z}) \cong H^4(\mathbb{Z}/3\mathbb{Z},\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}$ (see first page on handout 11) and $H^4(H,\mathbb{Z}) \cong (\mathbb{Z}/3\mathbb{Z})^2 \oplus \mathbb{Z}/6\mathbb{Z}$ (see the Exercise on handout 11), so it remains to calculate $H^2(H,\mathbb{Z})$ and $H^3(H,\mathbb{Z})$. By the Künneth formula

$$H^3(H,\mathbb{Z}) \cong \bigoplus_{r+s=3} H^r(\mathbb{Z}/6\mathbb{Z},\mathbb{Z}) \otimes_{\mathbb{Z}} H^s(\mathbb{Z}/3\mathbb{Z},\mathbb{Z}) \oplus$$
$$\bigoplus_{r+s=4} \mathrm{Tor}_1^{\mathbb{Z}}(H^r(\mathbb{Z}/6\mathbb{Z},\mathbb{Z}), H^s(\mathbb{Z}/3\mathbb{Z},\mathbb{Z})) \cong$$
$$\mathrm{Tor}_1^{\mathbb{Z}}(H^2(\mathbb{Z}/6\mathbb{Z},\mathbb{Z}), H^2(\mathbb{Z}/3\mathbb{Z},\mathbb{Z})) \cong \mathbb{Z}/3\mathbb{Z} \text{ and}$$

$$H^2(H,\mathbb{Z}) \cong \bigoplus_{r+s=2} H^r(\mathbb{Z}/6\mathbb{Z},\mathbb{Z}) \otimes_{\mathbb{Z}} H^s(\mathbb{Z}/3\mathbb{Z},\mathbb{Z})$$

$$\oplus \bigoplus_{r+s=3} \mathrm{Tor}_1^{\mathbb{Z}}(H^r(\mathbb{Z}/6\mathbb{Z},\mathbb{Z}),\ H^s(\mathbb{Z}/3\mathbb{Z},\mathbb{Z}))$$

$$\cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}.$$

Therefore $H^4(G,\mathbb{Z}) \cong$

$$(\mathbb{Z}/3\mathbb{Z})^2 \oplus \mathbb{Z}/3\mathbb{Z} \oplus (\mathbb{Z}/3\mathbb{Z})^2 \oplus \mathbb{Z}/6\mathbb{Z} \oplus$$

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/3\mathbb{Z},\ \mathbb{Z}/3\mathbb{Z}) \cong \underline{(\mathbb{Z}/3\mathbb{Z})^6 \oplus \mathbb{Z}/6\mathbb{Z}.}$$

Remark If H is a finite cyclic group, then $H^r(H,\mathbb{Z}) = 0$ $\forall$ odd r. Also $H^1(H,\mathbb{Z})$ if H is any finite group. However (for example)

$H^3(H,\mathbb{Z}) \neq 0$ for "most" finite groups H.

(5) Let P and Q be projective resolutions for k as kG and kH-modules respectively. Then by the Künneth formula $P \otimes_k Q$ is a $k[G \times H]$-projective resolution for k as $k[G \times H]$-modules. Given kG, kH-modules U, V, then there is a natural isomorphism of $k[G \times H]$-modules

$$(U \otimes_{kG} k) \otimes_k (V \otimes_{kH} k) \cong (U \otimes_k V) \otimes_{k[G \times H]} k$$

given by $(u \otimes a) \otimes (v \otimes b) \longmapsto u \otimes v \otimes ab$.

Therefore $H_n(G \times H, k) \cong H_n((P \otimes_{kG} k) \otimes_k (Q \otimes_{kH} k))$.

Note that $P \otimes_{kG} k$ is a complex of projective k-modules [$kG \otimes_{kG} k \cong k$ as k-modules, hence $F \otimes_{kG} k$ is a free k-module for any free kG-module F, thus $P \otimes_{kG} k$ is a projective k-module for any projective kG-module P]. Since k is commutative hereditary, we may apply the Künneth formula to obtain a split exact sequence of k-modules

$$0 \longrightarrow \bigoplus_{r+s=n} H_r(P \otimes_{kG} k) \otimes_k H_s(Q \otimes_{kH} k) \longrightarrow$$

$$H_n((P \otimes_{kG} k) \otimes_k (Q \otimes_{kH} k)) \longrightarrow$$

$$\bigoplus_{r+s=n-1} \operatorname{Tor}_1^k(H_r(P \otimes_{kG} k), H_s(Q \otimes_{kH} k)) \longrightarrow 0.$$

Since $H_r(P \otimes_{kG} k) \cong H_r(G, k)$ and $H_s(Q \otimes_{kH} k) \cong H_s(H, k)$, the above is the required exact sequence.

(6) Let (P, α_0) be a projective resolution of $\mathbb{Z}$ with with $\mathbb{Z}G$-modules. Then

$$H_n(G, M) \cong H_n(P \otimes_{\mathbb{Z}G} M) \quad \text{and}$$

$$H_n(G, M \otimes_{\mathbb{Z}} k) \cong H_n(P \otimes_{\mathbb{Z}G} (M \otimes_{\mathbb{Z}} k)) \qquad (*)$$

$$\cong H_n((P \otimes_{\mathbb{Z}G} M) \otimes_{\mathbb{Z}} k).$$

Note that $P \otimes_{\mathbb{Z}G} M$ is projective as a $\mathbb{Z}$-module ($\mathbb{Z}G \otimes_{\mathbb{Z}G} M \cong M$ as $\mathbb{Z}$-modules, so $\mathbb{Z}G \otimes_{\mathbb{Z}G} M$ is projective as a $\mathbb{Z}$-module, hence so is $P \otimes_{\mathbb{Z}G} M$ for any projective $\mathbb{Z}G$-module P). Since $\mathbb{Z}$ is commutative hereditary, we may apply the Universal Coefficient Theorem to obtain an exact sequence of k-modules which splits

$$0 \to H_n(P \otimes_{\mathbb{Z}G} M) \otimes_{\mathbb{Z}} k \to H_n((P \otimes_{\mathbb{Z}G} M) \otimes_{\mathbb{Z}} k)$$

$$\to \operatorname{Tor}_1^{\mathbb{Z}}(H_{n-1}(P \otimes_{\mathbb{Z}G} M), k) \to 0,$$

and the result follows upon using $(*)$ above.

(7) First do case $G = \mathbb{Z}$. By HW10 prob. 9(i) we have an exact sequence of $\mathbb{Z}G$-modules

$$0 \longrightarrow \mathbb{Z}G \longrightarrow \mathbb{Z}G \longrightarrow \mathbb{Z} \longrightarrow 0,$$

so applying the long exact sequence for $\mathrm{Ext}_{\mathbb{Z}G}$ in the first variable, we obtain $\mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, M) = 0$

$\forall\, n \geq 2$, $\forall$ $\mathbb{Z}G$-modules M. In particular $H^n(G, \mathbb{Z}) = 0 \quad \forall\, n \geq 2$. Since $H^0(G, \mathbb{Z}) \cong \mathbb{Z}$ and $H^1(G, \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(G/G', \mathbb{Z}) \cong \mathbb{Z}$, the case $n = 0$ is established.

In general write $G = H \times K$ where $H \cong \mathbb{Z}^{n-1}$, $K \cong \mathbb{Z}$, and assume the result is true for H. Since H (and K) are of type FP_∞ over $\mathbb{Z}$ (use the Exercise on page 2 of handout 11: it is an immediate consequence of this exercise that any finitely generated <u>abelian</u> group is of type FP_∞).

Therefore we may apply the Künneth formula at the bottom of page 2 Handout 11 to deduce that

$$H^r(G, \mathbb{Z}) \cong \bigoplus_{p+q=r} H^p(H, \mathbb{Z}) \otimes_{\mathbb{Z}} H^q(K, \mathbb{Z})$$

$$\bigoplus_{p+q=r+1} \mathrm{Tor}_1^{\mathbb{Z}}(H^p(H, \mathbb{Z}), H^q(K, \mathbb{Z})).$$

Now the Tor term is zero because $H^q(K,\mathbb{Z})$ is 0 or $\mathbb{Z}$. Therefore

$$H^r(G,\mathbb{Z}) \cong \bigoplus_{p+q=r} H^p(H,\mathbb{Z}) \otimes_{\mathbb{Z}} H^q(K,\mathbb{Z})$$

$$\cong H^r(H,\mathbb{Z}) \oplus H^{r-1}(H,\mathbb{Z})$$

on using the formula for $H^q(K,\mathbb{Z})$ at the start of the question (interpret $H^{-1}(H,\mathbb{Z})=0$). The result follows because $\binom{n}{r-1}+\binom{n}{r}=\binom{n+1}{r}$.

We now apply the Universal Coefficient Theorem (page 3 on handout 11). Since G is of type FP_∞ and $\mathbb{Z}$ is free as a $\mathbb{Z}$-module,

$$H^r(G,\mathbb{Z}\otimes_{\mathbb{Z}} k) \cong H^r(G,\mathbb{Z})\otimes_{\mathbb{Z}} k \oplus \operatorname{Tor}_1^{\mathbb{Z}}(H^{r+1}(G,\mathbb{Z}),k) \text{ as } k\text{-modules.}$$

But $H^{r+1}(G,\mathbb{Z})$ is a free $\mathbb{Z}$-module by the previous part of the question. Therefore

$H^r(G,k) \cong H^r(G,\mathbb{Z})\otimes_{\mathbb{Z}} k$ and the result follows.

(8) From page 2 of handout 11, all finitely generated abelian groups (and all finite groups) are of type FP_∞, so (since $\mathbb{Z}$ is a commutative hereditary ring) there is a split exact of k-modules

$$0 \to \bigoplus_{r+s=n} H^r(G,\mathbb{Z}) \otimes_{\mathbb{Z}} H^s(H,\mathbb{Z}) \to H^n(G\times H,\mathbb{Z})$$
$$\to \bigoplus_{r+s=n+1} \mathrm{Tor}_1^{\mathbb{Z}}(H^r(G,\mathbb{Z}), H^s(H,\mathbb{Z})) \to 0$$

From prob. 7, $H^r(G,\mathbb{Z})$ is a free $\mathbb{Z}$-module $\forall r \in \mathbb{N}$. Therefore the Tor term vanishes and the result follows.

Remark The same result is true if H is any finite group.

(9) (i) Here $P \otimes_k A$ is a right kG-module with action $(p \otimes a)g = pg \otimes ag$, and $A \otimes_k B$ is a left kG-module with action $g(a \otimes b) = ag^{-1} \otimes gb$.

Define $\theta : (P \otimes_k A) \times B \to P \otimes_{kG} (A \otimes_k B)$ by $\theta(p \otimes a, b) = p \otimes (a \otimes b)$. Since $\theta((p \otimes a)g, b) = \theta(pg \otimes ag, b) = pg \otimes (ag \otimes b) = p \otimes g(ag \otimes b) = p \otimes (a \otimes gb)$ for $g \in G$ and θ is k-linear, it follows that θ is a kG-balanced map, so it induces a k-map $\hat{\theta} : (P \otimes_k A) \otimes_{kG} B \to P \otimes_{kG} (A \otimes_k B)$.

Define $\phi : P \times (A \otimes_k B) \to (P \otimes_k A) \otimes_{kG} B$ by $\phi(p, a \otimes b) = (p \otimes a) \otimes b$. Since $\phi(pg, a \otimes b) = (pg \otimes a) \otimes b = (p \otimes ag^{-1})g \otimes b = (p \otimes ag^{-1}) \otimes gb = \phi(p, ag^{-1} \otimes gb) = \phi(p, g(a \otimes b))$ for $g \in G$ and ϕ is a k-map, it follows that ϕ is a kG-balanced map, so it induces a k-map

$\hat{\phi}: P \otimes_{kG} (A \otimes_k B) \to (P \otimes_k A) \otimes_{kG} B$.

Now $\hat{\theta}\hat{\phi}(p \otimes (a \otimes b)) = p \otimes (a \otimes b)$ $\forall p \in P$, $a \in A$, $b \in B$, so $\hat{\theta}\hat{\phi} =$ identity.
Also $\hat{\phi}\hat{\theta}((p \otimes a) \otimes b) = (p \otimes a) \otimes b$ $\forall p \in P$, $a \in A$, $b \in B$, so $\hat{\phi}\hat{\theta} =$ identity. It follows that θ and ϕ are isomorphisms (of k-modules), so $(P \otimes_k A) \otimes_{kG} B \cong P \otimes_{kG} (A \otimes_k B)$ as kG-modules.
(This part didn't use the hypothesis A is flat.)

(ii) We need to prove $(P \otimes_k A) \otimes_{kG} -$ is exact.

Now A flat as a k-module $\Rightarrow$ $A \otimes_k -$ exact and P flat as a kG-module $\Rightarrow$ $P \otimes_{kG} -$ exact. Therefore $P \otimes_{kG} (A \otimes_k -)$ is exact. Since $P \otimes_{kG} (A \otimes_k B)$ is naturally isomorphic to $(P \otimes_k A) \otimes_{kG} B$, it follows that $(P \otimes_k A) \otimes_{kG} -$ is exact.

(iii) Let (P, α_0) be a flat resolution of k as kG-modules. Then by (ii), $P \otimes_k A$ is a flat resolution of $k \otimes_k A \cong A$ as kG-modules. Therefore $\mathrm{Tor}_n^{kG}(A, B) \cong H_n((P \otimes_k A) \otimes_{kG} B)$, which using (i) is $\cong H_n(P \otimes_{kG} (A \otimes_k B))$
$\cong H_n(G, A \otimes_k B)$.

TWELFTH HW SOLUTIONS

(1) (i) We have exact sequences $0 \to {}_{\mathcal{I}}\mathcal{I} \to \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0$ (where ε is the augmentation map, so $\varepsilon(g) = 1$ $\forall\, g \in G$) and $0 \to \mathbb{Z} \to \mathbb{Z}G \xrightarrow{\pi} {}_{\mathcal{I}}\mathcal{I} \to 0$ (where $\pi(1) = x - 1$ and $G = \langle x \rangle$; we always have the first sequence, but the second depends on G being cyclic). Then the long exact sequence for Ext in the first variable and naturality yield a commutative diagram with the horizontal maps isomorphisms

$$\begin{array}{ccccc} \mathrm{Ext}^1_{\mathbb{Z}G}(\mathbb{Z}, A) & \xrightarrow{\theta_1} & \mathrm{Ext}^2_{\mathbb{Z}G}({}_{\mathcal{I}}\mathcal{I}, A) & \xrightarrow{\theta_2} & \mathrm{Ext}^3_{\mathbb{Z}G}(\mathbb{Z}, A) \\ \downarrow{\scriptstyle \alpha_{1*}} & & \downarrow{\scriptstyle \alpha_{2*}} & & \downarrow{\scriptstyle \alpha_{3*}} \\ \mathrm{Ext}^1_{\mathbb{Z}G}(\mathbb{Z}, B) & \longrightarrow & \mathrm{Ext}^2_{\mathbb{Z}G}({}_{\mathcal{I}}\mathcal{I}, B) & \longrightarrow & \mathrm{Ext}^3_{\mathbb{Z}G}(\mathbb{Z}, B) \end{array}$$

If $\vartheta = \theta_2\theta_1$, then ϑ is an isomorphism which takes $\ker \alpha_{1*}$ to $\ker \alpha_{3*}$.

(ii) The exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ yields an exact sequence

$$\to H^1(G,A) \xrightarrow{\alpha_{1*}} H^1(G,B) \xrightarrow{\beta_{1*}} H^1(G,C) \xrightarrow{\delta_1}$$

$$H^2(G,A) \xrightarrow{\alpha_{2*}} H^2(G,B) \xrightarrow{\beta_{2*}} H^2(G,C) \xrightarrow{\delta_2} H^3(G,A)$$

Then $\theta^{-1}\delta_2 : H^2(G,C) \to H^1(G,A)$ will complete the hexagon. Perhaps we should check exactness; this is clear everywhere except maybe at $H^2(G,C)$ and

$H^1(G,A)$.

Exactness at $H^2(G,C)$ Since ϑ^{-1} is an isomorphism,

$$\operatorname{im}\beta_{2*} = \ker\delta_2 = \ker\vartheta^{-1}\delta_2.$$

Exactness at $H^1(G,A)$ ϑ^{-1} maps $\operatorname{im}\delta_2$ $(=\ker\alpha_{3*})$ to $\ker\alpha_{1*}$, hence $\operatorname{im}\vartheta^{-1}\delta_2 = \ker\alpha_{1*}$.

(2) (i) Write the exact sequence as $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$. Then $B/\alpha A \cong C$ and $|\alpha A| = |A|$. Therefore $|B|/|A| = |C|$, i.e. $|B| = |A||C|$.

(ii)
$$\begin{array}{ccccc} & & H^1(G,A) & \xrightarrow{\vartheta_1} & H^1(G,B) \\ & \swarrow^{\vartheta_6} & \| & & \| \\ & & M_1 & & M_2 \\ H^2(G,C) & & & & \searrow^{\vartheta_2} \\ = M_6 & & & & H^1(G,C) = M_3 \\ \uparrow & & & & \downarrow \\ \vartheta_5 & & & & \vartheta_3 \\ H^2(G,B) & \xleftarrow{\vartheta_4} & & & H^2(G,A) \\ \| & & & & \| \\ M_5 & & & & M_4 \end{array}$$

For each i, we have $M_i/\ker\vartheta_i \cong \ker\vartheta_{i+1}$ (where we take $6+1=1$), hence $|M_i| = |\ker\vartheta_i||\ker\vartheta_{i+1}|$. It follows that

$$\frac{|M_5|}{|M_2|} = \frac{|\ker\vartheta_5||\ker\vartheta_6|}{|\ker\vartheta_2||\ker\vartheta_3|} = \frac{|M_4||M_6|}{|M_1||M_3|}$$

which is the required result.

(3) (i) The exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ yields an exact sequence

$$\to H^1(G,\mathbb{Q}) \to H^1(G,\mathbb{Q}/\mathbb{Z}) \to H^2(G,\mathbb{Z}) \to H^2(G,\mathbb{Q}) \to \cdots$$

Since G is finite, $n\,H^r(G,\mathbb{Q}) = 0 \;\forall\, r \geq 1$ where $n = |G|$. But multiplication by n on $\mathbb{Q}$ is an automorphism, hence multiplication by n on $H^r(G,\mathbb{Q})$ is also an automorphism and we deduce that $H^r(G,\mathbb{Q}) = 0 \;\forall\, r \geq 1$. Therefore the above exact sequence shows

$$H^1(G,\mathbb{Q}/\mathbb{Z}) \cong H^2(G,\mathbb{Z}).$$

The result follows because $H^1(G,\mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}(G,\mathbb{Q}/\mathbb{Z})$.

(ii) Using (i), $H^2(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}) \cong \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z},\mathbb{Q}/\mathbb{Z})$

$$\cong \{x \in \mathbb{Q}/\mathbb{Z} \mid nx = 0\}$$

$$\cong \mathbb{Z}/n\mathbb{Z}.$$

(iii) $\mathbb{Z}/n\mathbb{Z}$ is finite, so certainly of type FP_∞, hence we can apply the Universal Coefficient Theorem for group cohomology (as stated on handout II) to deduce that

$$H^2(\mathbb{Z}/n\mathbb{Z},k) \cong H^2(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}) \otimes_{\mathbb{Z}} k \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H^3(\mathbb{Z}/n\mathbb{Z},k))$$

i.e. $H^2(\mathbb{Z}/n\mathbb{Z},k) \cong \mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} k \cong k/nk.$

(4) (i) Since G is finite, we may apply prob. 3(i) to deduce that $H^2(G,\mathbb{Z}) \cong \mathrm{Hom}(G/G', \mathbb{Q}/\mathbb{Z})$. Now G/G' is a finite abelian group, so we may write $G/G' = A_1 \times \cdots \times A_t$ where the A_i are finite cyclic groups. As in prob. 3(ii), $\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$ $\forall n \in \mathbb{P}$, hence $\mathrm{Hom}(A_i, \mathbb{Q}/\mathbb{Z}) \cong A_i$ $\forall i$. It follows that

$$\begin{aligned}\mathrm{Hom}(G/G', \mathbb{Q}/\mathbb{Z}) &\cong \mathrm{Hom}(A_1 \times \cdots \times A_t, \mathbb{Q}/\mathbb{Z}) \\ &\cong \mathrm{Hom}(A_1, \mathbb{Q}/\mathbb{Z}) \times \cdots \times \mathrm{Hom}(A_t, \mathbb{Q}/\mathbb{Z}) \\ &\cong A_1 \times \cdots \times A_t \cong G/G',\end{aligned}$$

and the result follows.

(ii) The exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ yields an exact sequence

$$H^r(G,\mathbb{Q}) \to H^r(G,\mathbb{Q}/\mathbb{Z}) \to H^{r+1}(G,\mathbb{Z}) \to H^{r+1}(G,\mathbb{Z}).$$

Since G is finite, $H^r(G,\mathbb{Q}) = 0$ $\forall r \in \mathbb{P}$ (as proved in prob. 3(i)), hence $H^r(G,\mathbb{Q}/\mathbb{Z}) \cong H^{r+1}(G,\mathbb{Z})$.

(iii) We may write $\mathbb{Q}/\mathbb{Z} = \varinjlim \mathbb{Z}/n\mathbb{Z}$. Since G is finite, $\mathbb{Z}G$ is Noetherian, hence we may apply HW5 prob. 5 to deduce that

$$H^r(G,\mathbb{Q}/\mathbb{Z}) \cong \varinjlim H^r(G, \mathbb{Z}/n\mathbb{Z}).$$

Now use (ii).

(5) (i) Define $\theta : M \times kG \longrightarrow M \otimes_k kG$ by

$$\theta\left(m, \sum_{g\in G} a_g g\right) = \sum_{g\in G} mg^{-1} \otimes a_g g$$

(Where $a_g \in k$ $\forall g$; G acts diagonally on LHS, by right multiplication on kG on RHS.)

It is easily checked that θ is k-balanced, hence it induces a k-map

$$\hat{\theta} : M \otimes_{kG} kG \longrightarrow M \otimes_k kG.$$

We must verify that $\hat{\theta}$ respects the G-action, for then we will have shown that $\hat{\theta}$ is a kG-map. By k-bilinearity, it will be sufficient to do this on elements of the form $m \otimes g$ (since they generate M as a k-module). Let $h \in G$. Then

$$\hat{\theta}[(m\otimes g)h] = \hat{\theta}[mh \otimes gh] = mhh^{-1}g^{-1} \otimes gh = mg^{-1} \otimes gh$$

$$[\hat{\theta}(m\otimes g)]h = [mg^{-1} \otimes g]h = mg^{-1} \otimes gh.$$

Similarly we define $\phi : M \times kG \longrightarrow M \otimes_k kG$ by $\phi\left(m, \sum_{g\in G} a_g g\right) = \sum_{g\in G} mg \otimes a_g g$.

This induces a k-map $\hat{\phi} : M \otimes_{kG} kG \longrightarrow M \otimes_k kG$; this time, G is acting diagonally on the RHS ($\hat{\phi}$ is, of course, going to be the inverse for $\hat{\theta}$).

Let $h \in G$.

Then

$\hat{\phi}[(m\otimes g)h] = \hat{\phi}(m\otimes gh) = mgh\otimes gh$

and $[\hat{\phi}(m\otimes g)]h = (mg\otimes g)h = mgh\otimes gh$,

so $\hat{\phi}$ is indeed a kG-map.

$\hat{\theta}\hat{\phi}(m\otimes g) = \hat{\theta}(mg\otimes g) = mgg^{-1}\otimes g = m\otimes g$

and

$\hat{\phi}\hat{\theta}(m\otimes g) = \hat{\phi}(mg^{-1}\otimes g) = mg^{-1}g\otimes g = m\otimes g$.

Thus $\hat{\theta}\hat{\phi}$ and $\hat{\phi}\hat{\theta}$ are the identity maps and the result follows.

(ii) If $M = k$ and $F = kG$, then

$k\otimes_k kG$ (diagonal action)

$\cong$ $k\otimes_k kG$ (kG acting by right multiplication on kG)

$\cong$ kG, so $M\otimes_k F$ free in this case.

Since tensor product commute with arbitrary direct sums, the result follows.

(iii) Map a free kG-module E onto M, so we have an exact sequence of kG-modules

$$0 \to M' \to E \to M \to 0.$$

This will split as k-modules (but not as kG-modules), because M is projective as a k-module, so $M\oplus M'$ ($\cong E$ as k-modules) is a kG-module which is free as a k-module.

Now choose a kG-module P' such that $P \oplus P'$ is a free kG-module. Then

$(M \oplus M') \otimes_k (P \oplus P')$ is a free kG-module by (ii), and is $\cong$

$M \otimes_k P \oplus M \otimes_k P' \oplus M' \otimes_k P \oplus M' \otimes_k P'$.

This shows that $M \otimes_k P$ is a direct summand of a free kG-module, and the result follows.

(6) Since we have a natural epimorphism $G \to G/H \cong P$, the inflation map $\inf_{P,G}$ is define, so we have maps

$$H^n(P,k) \xrightarrow{\inf_{P,G}} H^n(G,k) \xrightarrow{res_{G,P}} H^n(P,k)$$

$$\xrightarrow{tr_{P,G}} H^n(G,k) \quad \text{where } tr \text{ is the transfer map,}$$

$res_{G,P} \inf_{P,G}$ is the identity, so $res_{G,P}$ is onto. $tr_{P,G} res_{G,P}$ is multiplication by $[G:P]$, and multiplication by $[G:P]$ on k is an automorphism ($\because p \nmid [G:P]$). Therefore $tr_{P,G} res_{G,P}$ is an automorphism, hence $res_{G,P}$ is onto. Therefore $res_{G,P}$ is an isomorphism.

(7) Since G is finite, it is of type FP_∞, so we may apply the Universal Coefficient Theorem for group cohomology (handout 11) to deduce that

$$H^n(G, M \otimes_{\mathbb{Z}} k) \cong H^n(G,M) \otimes_{\mathbb{Z}} k \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H^{n+1}(G,M), k).$$

Thus $\mathrm{Tor}_1^{\mathbb{Z}}(H^{n+1}(G,M), k) = 0$.
Since $k = \mathbb{Z}/p\mathbb{Z} \oplus ?$ as abelian groups,

$$\mathrm{Tor}_1^{\mathbb{Z}}(H^{n+1}(G,M), \mathbb{Z}/p\mathbb{Z}) = 0.$$

Therefore $\{ x \in H^{n+1}(G,M) \mid px = 0 \} = 0$ by the Proposition on handout 3. Also $p^r H^{n+1}(G,M) = 0$ where $p^r = |G|$. The result follows.

(8) If G is finite, then $L^p(G) = \mathbb{C}G$ and $\sum_{g \in G} g \in \mathbb{C}G \setminus 0$. (because the sum is a finite sum).

Conversely suppose $\sum_{g \in G} a_g g \in L^p(G)^G$ $(a_g \in \mathbb{C})$. Then $\left(\sum_{g \in G} a_g g\right) x = \sum_{g \in G} a_g g x \quad \forall x \in G$, hence

$\sum_{g \in G} a_{gx^{-1}} g = \sum_{g \in G} a_g g$, hence (equating the coefficient of g) $a_{gx^{-1}} = a_g \quad \forall g, x \in G$.

Thus if $a_g \neq 0$ for some $g \in G$, say $a_g = a \neq 0$, then $a_g = a \ \forall \ g \in G$. Thus for

$\sum_{g \in G} |a_g|^p$ to be $< \infty$, we must have G finite.

(9) (i) HW10 prob. 9(i) shows $\mathcal{I} \cong \mathbb{Z}H$, so the augmentation sequence $0 \to \mathcal{I} \to \mathbb{Z}H \to \mathbb{Z} \to 0$ becomes $0 \to \mathbb{Z}H \to \mathbb{Z}H \to \mathbb{Z} \to 0$. It follows immediately from the definition of $\mathrm{Ext}^r_{\mathbb{Z}H}(\mathbb{Z}, M)$ that this is zero $\forall\ r > 1$, hence $H^r(H, M) = 0\ \forall r > 1$.

(ii) $\mathrm{tr}_{H,G}\ \mathrm{res}_{G,H} : H^r(G, M) \to H^r(G, M)$ is

multiplication by $[G : H] = n$. Also $\mathrm{res}_{G,H} = 0$ because $H^r(H, M) = 0$ from (i). Therefore $n\, H^r(G, M) = 0\ \forall r > 1$.

Thirteenth Homework Solutions

(1) (i) The endomorphism $m \mapsto m(1-g)$ of M is certainly a $\mathbb{Z}G$-map because $\mathbb{Z}G$ is commutative. Now $H^r(G,M) \cong \mathrm{Ext}^r_{\mathbb{Z}G}(\mathbb{Z},M)$, and the endomorphism multiplication by $1-g$ on M induces the same endomorphism on $\mathrm{Ext}^r_{\mathbb{Z}G}(\mathbb{Z},M)$ as multiplication by $1-g$ on $\mathbb{Z}$. But this latter map is zero (because the action of G on $\mathbb{Z}$ is trivial) as required.

(ii) Let μ denote multiplication by $1-g$ on M. Then $u \in \ker \mu = 0 \iff u(1-g)=0 \iff u = ug \iff ug^i = u \ \forall i \in \mathbb{Z} \iff uh = u \ \forall h \in G \iff u \in M^G$. By hypothesis $M^G = 0$, hence μ is a monomorphism. Since M is finite, μ is also onto and the result follows.

(iii) Let ν denote the endomorphism of $H^r(G,M)$ induced by multiplication by $1-g$ on M.

By (i), $\nu = 0$.

By (ii), ν is an automorphism.

It follows that $H^r(G,M) = 0 \ \forall r \in \mathbb{N}$ as required.

(2) Since G is finite, $\mathbb{Z}G$ is Noetherian, so there exists a resolution (P, α_0) consisting of finitely generated free $\mathbb{Z}G$-modules. Since G is finite, $\mathbb{Z}G$ is finitely generated as a $\mathbb{Z}$-module and it follows P_r is finitely generated as a $\mathbb{Z}$-module $\forall\, r \in \mathbb{N}$. Also M is finitely generated as a $\mathbb{Z}G$-module ($\because$ G is finite and M is a finitely as a $\mathbb{Z}G$-module; specifically if M is generated as a $\mathbb{Z}G$-module by the finite subset X, then M is generated as a $\mathbb{Z}$-module by $\{xg \mid x \in X, g \in G\}$). Therefore $\mathrm{Hom}_{\mathbb{Z}}(P_r, M)$ is finitely generated as a $\mathbb{Z}$-module. Since $\mathrm{Hom}_{\mathbb{Z}G}(P_r, M) \subseteq \mathrm{Hom}_{\mathbb{Z}}(P_r, M)$, we see that $\mathrm{Hom}_{\mathbb{Z}G}(P_r, M)$ is also a finitely generated abelian group. But $H^r(G, M)$ is a subquotient of $\mathrm{Hom}_{\mathbb{Z}G}(P_r, M)$, hence $H^r(G, M)$ is a finitely generated $\mathbb{Z}$-module as required.

If $r \in \mathbb{P}$ and $n = |G|$, then $n\, H^r(G, M) = 0$ (see handout 12, proposition at the top of page 1). This combined with the result from the previous paragraph shows that $H^r(G, M)$ is finite.

(3) As in the trace map (see Handout 12, page 1), we define $\theta : M \otimes_{\mathbb{Z}H} \mathbb{Z}G \longrightarrow \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M)$ by $(\theta(m\otimes\alpha))\beta = m\,\mathrm{tr}_H(\alpha\beta)$ for $\alpha, \beta \in \mathbb{Z}G$, $m \in M$; it is shown there that θ is a $\mathbb{Z}G$-map, so all we need to do is to prove that $\ker\theta = 0$.

Let T be a right transversal for H in G. Then as $\mathbb{Z}$-modules, $M \otimes_{\mathbb{Z}H} \mathbb{Z}G = \bigoplus_{t\in T} M \otimes_{\mathbb{Z}H} \mathbb{Z}Ht$ and every element of $M \otimes_{\mathbb{Z}H} \mathbb{Z}Ht$ can be written in the form $M \otimes t$. It follows that every element of $M \otimes_{\mathbb{Z}H} \mathbb{Z}G$ can be written in the form $\sum_{t\in T} m_t \otimes t$ where $m_t \in M$. Suppose

$\theta(\sum_{t\in T} m_t \otimes t) = 0$. Then $\sum_{t\in T} m_t \mathrm{tr}_H(ts^{-1}) = 0$ $\forall\, s \in T$. But $ts^{-1} \in H \Leftrightarrow HtS^{-1} = H \Leftrightarrow Ht = Hs$, hence $s = t$. Therefore $\mathrm{tr}_H ts^{-1} \neq 0 \Leftrightarrow s = t$ and we deduce that $m_s = m_s \mathrm{tr}_H 1 = 0$. This shows that $\sum m_t \otimes t = 0$, as required.

(4) (i) Clearly the map $P_r \times M \to P_r \otimes_{\mathbb{Z}H} M$ defined by $(p,m) \mapsto p \otimes m$ is $\mathbb{Z}$-balanced, so it induces a well defined map $\theta : P_r \otimes_{\mathbb{Z}} M \to P_r \otimes_{\mathbb{Z}H} M$. It does not depend on the choice of transversal because if we replace x_i with hx_i, then $px_i^{-1} \otimes x_i$ becomes $p(hx_i)^{-1} \otimes (hx_i)m = px_i^{-1}h^{-1} \otimes hx_i m = px_i^{-1} \otimes x_i m$.

(ii) $P_r \otimes_{\mathbb{Z}G} M$ is $P_r \otimes_{\mathbb{Z}} M$ modded out by elements of the form $pg \otimes m - p \otimes gm$ where $p \in P_r$, $g \in G$, $m \in M$. We must show that θ sends such elements to 0.

$\theta(p \otimes gm) = \sum_{i'} px_{i'}^{-1} \otimes x_i g m$

Write $x_i g = h_i x_{i'}$ where $h_i \in H$ and $i' \in \{1, \dots, n\}$. Then $i' = j' \Rightarrow (h_i x_{i'})(h_j x_{j'})^{-1} \in H \Rightarrow (x_i g)(x_j g)^{-1} \in H \Rightarrow Hx_i = Hx_j \Rightarrow i = j$, so $i \mapsto i'$ is a permutation of $\{1, \dots, n\}$. Also $x_{i'}^{-1} h_i = g x_{i'}^{-1}$. Therefore

$\theta(p \otimes gm) = \sum_{i'} px_{i'}^{-1} \otimes h_i x_{i'} m =$

$\sum_{i'} px_{i'}^{-1} h_i \otimes x_{i'} m = \sum_{i'} pg x_{i'}^{-1} \otimes x_{i'} m =$

$\theta(pg \otimes m)$ as required.

(iii) We need to show that θ_r commutes with the boundary maps α_r; specifically

$$(\alpha_r \otimes 1)\theta_r = \theta_{r-1}(\alpha_r \otimes 1).$$

It will be sufficient to do this on generators. Now

$$(\alpha_r \otimes 1)\theta_r(p \otimes m) = \alpha_r \otimes 1 \sum_i p x_i^{-1} \otimes x_i m$$

$$= \sum_i \alpha_r(p x_i^{-1}) \otimes x_i m$$

whereas

$$\theta_{r-1}(\alpha_r \otimes 1)(p \otimes m) = \theta_{r-1}(\alpha_r p \otimes m) =$$

$$\sum_i (\alpha_r p) x_i^{-1} \otimes x_i m.$$

These two expressions are equal because

$$(\alpha_r p) x_i^{-1} = \alpha_r(p x_i^{-1}).$$

(5) (i) Let A be generated by $X = \{x_1, \dots, x_\ell\}$. Write $x_i = \sum_j x_{ij}$ where the x_{ij} are homogeneous elements of A, and $Y = \{x_{ij} \mid i,j \text{ arbitrary}\}$, a finite subset of A consisting of homogeneous elements. Then any product of the x_i can be written as a sum of products of the x_{ij}. Thus any k-linear sum of products of the x_i can be written as a k-linear sum of products of the elements of Y, which proves (i).

(ii) Write $Y = \{y_1, \dots, y_m, y_{m+1}, \dots, y_n\}$ where $y_1, \dots, y_m$ have even degree and $y_{m+1}, \dots, y_n$ have odd degree. Define $Z = \{y_1, \dots, y_m, y_i y_j \mid m+1 \leq i \leq j \leq n\}$. Any element of B is a k-linear sum of products of the elements of Y in which the total number of $y_{m+1}, \dots, y_n$ occurring in any product is even (we simply discard the products which involve an odd number of the $y_{m+1}, \dots, y_n$). Moreover using $y_j y_i = (-1)^{\deg y_i \deg y_j} y_i y_j$, we may assume that any element of B is a k-linear sum of products of the form $y_{i_1} \cdots y_{i_t}$ where $i_1 \leq i_2 \leq \cdots \leq i_t$. Such an element is a product of elements of Z.

Write $Z = \{z_1, \dots, z_p\}$. Then the z_i commute (because they have even degree). Thus $B = k[z_1, \dots, z_p]$ is a homomorphic image of the polynomial ring $k[X_1, \dots, X_p]$. The result now follows from Hilbert's Basis theorem.

(6) We shall continue to use the notation of the previous problem.

(i) Any element of M is a k-linear sum of products of the elements of Y in which the total number of $y_{m+1}, \dots, y_n$ occurring in any product is odd. Moreover using $y_i y_j = (-1)^{\deg y_i \deg y_j} y_j y_i$,

we may assume that a y_i with $m+1 \le i \le n$ occur first, and then such a product is of the form $y_i \beta$ with $\beta \in B$. It follows that $M = y_{m+1} B + \dots + y_n B$, so M is indeed a finitely generated B-submodule of A. Since B is Noetherian by prob. 5, it follows that M is Noetherian.

(ii) $A = B + M$, so A is a Noetherian B-module. Therefore A is a Noetherian A-module, hence A is right Noetherian. Similarly A is left Noetherian.

(7) (i) We must show that if $\alpha \in A$ and $\beta \in B$, then $\alpha\beta = \beta\alpha$. It will be sufficient to do this in the case α, β homogeneous. But then $\alpha\beta = (-1)^{\deg\alpha \deg\beta}\beta\alpha = \beta\alpha$ because $\deg\beta$ is even.

(ii) $xx = (-1)^{\deg x \deg x} xx$, so $x^2 + x^2 = 0$. Therefore $x^2 = 0$ because char $k \neq 2$.

(iii) Since any element is a sum of homogeneous elements, we may assume that all the a_i are homogeneous. Since at least $n+1$ of the a_i are in X and $|X| = n$, at least two of the a_i are equal. Moreover using $a_i a_j = (-1)^{\deg a_i \deg a_j} a_j a_i$, we can bring these two a_i together. The result now follows from (ii).

(iv) Let N be generated by the elements $x_1, \ldots, x_l$ where $x_i \in M \; \forall i$. We may write each x_i as a sum of homogeneous elements from M, say each x_i is a sum of element from Y, where Y is a finite subset of homogeneous elements of M. Let $\bar{N} = \sum_{y \in Y} yA$. Then $\bar{N} \supseteq N$, so it will be sufficient to show that $\bar{N}$ is nilpotent. Let $|Y| = t-1$. Then any element of $\bar{N}^t$ is a sum of products, where in each product at least t of the elements come from Y. The result now follows from (iii).

(8) (i) Let $\beta = \alpha g - \alpha$, write $\alpha = \sum_{x \in G} \alpha_x x$ $(\alpha_x \in \mathbb{C})$ and $\beta = \sum_{x \in G} \beta_x x$ $(\beta_x \in \mathbb{C})$. Let $X = \{x \in G \mid \alpha_x \neq 0\}$ and $Y = \{x \in G \mid \beta_x \neq 0\}$. Then $|Y| < \infty$ because $\beta \in \mathbb{C}G$. If $x \in G \setminus Y$, then $(\alpha g - \alpha)_x = 0$, i.e. $\alpha_{xg^{-1}} = \alpha_x$. Setting $\bar{Y} = Y \cup Yg^{-1}$, we can put this in more symmetrical form:

if $x \in G \setminus \bar{Y}$, then $\alpha_{xg^{-1}} = \alpha_x = \alpha_{xg}$ (1)

Let $H = \langle g \rangle$. First show $X \subseteq \bigcup_{y \in \bar{Y}} yH$.

Indeed if $x \in X \setminus \bigcup_{y \in \bar{Y}} yH$, then

$xH \cap (\bigcup_{y \in \bar{Y}} yH) = \emptyset$, and then repeated application of (1) shows that $\alpha_{xg^i} = \alpha_x$

$\forall i \in \mathbb{N}$. But this is not possible because the coefficients of α are p-summable.

Thus if $|X| = \infty$, then $|X \cap yH| = \infty$ for some $y \in \bar{Y}$. Without loss of generality, we may

assume that $|X \cap \{yg^i \mid i \geq 0\}| = \infty$.

Since $|\bar{Y}| < \infty$, $\exists\, N \in \mathbb{P}$ such that $\bar{Y} \cap \{yg^i \mid i \geq N\} = \emptyset$. But then repeated application of (1) shows that $\alpha_{yg^N} = \alpha_{yg^{N+i}}$ $\forall\, i \in \mathbb{N}$. Since $\alpha_{yg^j} \neq 0$ for some $j \geq N$, this contradicts the fact that the coefficients α_g are p-summable.

Now consider the long exact sequence for Ext in the first variable applied to $0 \to \sigma J \to \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0$ where $\varepsilon(g) = 1\ \forall g \in G$. This yields a commutative diagram with exact rows

$$\begin{array}{ccccccc}
\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, \mathbb{C}G) & \xrightarrow{\alpha_1} & \mathrm{Hom}_{\mathbb{Z}G}(\sigma J, \mathbb{C}G) & \xrightarrow{\alpha_2} & \mathrm{Ext}^1_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{C}G) & \to & 0 \\
\downarrow \vartheta_1 & & \downarrow \vartheta_2 & & \downarrow \vartheta_3 & & \\
\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, L^p(G)) & \xrightarrow{\beta_1} & \mathrm{Hom}_{\mathbb{Z}G}(\sigma J, L^p(G)) & \xrightarrow{\beta_2} & \mathrm{Ext}^1_{\mathbb{Z}G}(\mathbb{Z}, L^p(G)) & \to & 0
\end{array}$$

because $\mathrm{Ext}^1_{\mathbb{Z}G}(\mathbb{Z}G, \_) = 0$. Suppose $u \in \ker \vartheta_3$. Then $u = \alpha_2 v$ for some $v \in \mathrm{Hom}_{\mathbb{Z}G}(\sigma J, \mathbb{C}G)$. Then $\beta_2 \vartheta_2 v = \vartheta_3 \alpha_2 v = 0$, hence $\vartheta_2 v = \beta_1 w$ for some $w \in \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, L^p(G))$. Set $\gamma = w1 \in L^p(G)$. If $g \in G$, then $\gamma(g-1) = (w1)(g-1) =$

$$w(g-1) = \beta_1 w(g-1)$$
(because $\beta_1 w$ is the restriction w to ${}^0\mathcal{J}$)
$$= \theta_2 v(g-1) \in \mathbb{C}G$$

(because $v(g-1) \in \mathbb{C}G$ and $\theta_2 v$ is the composition of v and the natural inclusion $\mathbb{C}G \longrightarrow L^p(G)$). Thus $\gamma(g-1) \in \mathbb{C}G$, so by the first part $\gamma \in \mathbb{C}G$. Since $wx = (w1)x = \gamma x$ $\forall\, x \in \mathbb{Z}G$, it follows that $\beta_1 w x = \gamma x$ $\forall\, x \in {}^0\mathcal{J}$.

Define $y \in \mathrm{Hom}_{\mathbb{Z}G}({}^0\mathcal{J}, \mathbb{Z}G)$ by $yx = \gamma x$ $\forall\, x \in {}^0\mathcal{J}$.

Then $\theta_2 y x = yx = \gamma x = \beta_1 w x = \theta_2 v x$ $\forall\, x \in {}^0\mathcal{J}$, hence $\theta_2 y = \theta_2 v$ and we deduce that $y = v$. Therefore $vx = \gamma x$ $\forall\, x \in {}^0\mathcal{J}$. Define $z \in \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, \mathbb{C}G)$ by $zx = \gamma x$ $\forall\, x \in {}^0\mathcal{J}$. Then $(\alpha_1 z)x = \gamma x = vx$ $\forall\, x \in {}^0\mathcal{J}$, hence $v = \alpha_1 z$, so

$$u = \alpha_2 v = \alpha_2 \alpha_1 z = 0.$$

Therefore $\ker \theta_3 = 0$ as required.

(ii) For each $i \in \mathbb{P}$, set $n_i = |G_i|$ and $\sigma_i = \sum_{g \in G_i} g$. Then clearly $\alpha := \sum_{i=1}^{\infty} \frac{\sigma_i}{2^i n_i} \in L^p(G)$

for all $p \geq 1$. If $g \in G$, then $g \in G_r$

$\forall\, r \geq N$ for some $N \in \mathbb{P}$ and we have $\sigma_r g = \sigma_r$ $\forall\, r \geq N$. It follows that $\alpha g - \alpha \in \mathbb{C}G_N$.

Now consider the commutative diagram

$$\begin{array}{ccccccc} \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, \mathbb{C}G) & \xrightarrow{\alpha_1} & \mathrm{Hom}_{\mathbb{Z}G}(\mathcal{I}, \mathbb{C}G) & \xrightarrow{\alpha_2} & \mathrm{Ext}^1_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{C}G) & \longrightarrow & 0 \\ \downarrow \theta_1 & & \downarrow \theta_2 & & \downarrow \theta_3 & & \\ \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, L^p(G)) & \xrightarrow{\beta_1} & \mathrm{Hom}_{\mathbb{Z}G}(\mathcal{I}, \mathbb{C}G) & \xrightarrow{\beta_2} & \mathrm{Ext}^1_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{C}G) & \longrightarrow & 0 \end{array}$$

again. Define $u \in \mathrm{Hom}_{\mathbb{Z}G}(\mathcal{I}, \mathbb{C}G)$ by $ux = \alpha x$ $\forall\, x \in \mathcal{I}$, and $v \in \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, L^p(G))$ by $vx = \alpha x$ $\forall\, x \in \mathbb{Z}G$. Then $\theta_2 u = \beta_1 v$, so

$$\theta_3 \alpha_2 u = \beta_2 \theta_2 u = \beta_2 \beta_1 v = 0,$$

hence $\alpha_2 u \in \ker \theta$. We now want to show $\alpha_2 u \neq 0$, equivalently $u \notin \mathrm{im}\, \alpha_1$. Suppose on the contrary we can write $u = \alpha_1 w$ with $w \in \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, \mathbb{C}G)$. Then $\beta_1 \theta_1 w = \theta_2 \alpha_1 w = \theta_2 u = \beta_1 v$. Now $\ker \beta_1 = \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, L^p(G)) = (L^p(G))^G$ by HW12, prob. 8, this is zero because G is infinite. Therefore $\theta_1 w = v$, hence $\theta_1 w 1 = v1 = \alpha$. Thus $\theta_1 w 1 \notin \mathbb{C}G$, which is a contradiction because $\theta_1 w 1 = w 1 \in \mathbb{C}G$.

Fourteenth Homework Solutions

(1) (i) Let $n = |G|$. Using the formula for the cohomology of a cyclic group from handout 11, we see that

$H^1(G,M) \cong \{ x \in M \mid xn = 0 \}$ and

$H^2(G,M) \cong M/Mn$.

However we have an exact sequence

$$0 \to \ker \mu \to M \xrightarrow{\mu} Mn \to 0$$

where μ is multiplication by n. Then (see HW12, prob. 2(i))

$|M|/|Mn| = |\ker \mu|$ and $\ker \mu = \{x \in M \mid xn = 0\}$.

Therefore $|H^2(G,M)| = |H^1(G,M)|$ and the result follows.

(ii) By HW13 prob. 1(iii), $H^r(G,M) = 0 \quad \forall r > 0$.

Therefore $h(M) = 1$.

② (i) Define an increasing sequence (M_i) of submodules of M by $M_{i+1}/M_i = (M/M_i)^G$, $M_1 = 0$. Since M is finite, this sequence must become constant, so we have

$$M_1 \subset M_2 \subset \cdots \subset M_N = M_{N+1} =$$

for some N. Since $(M_{i+1}/M_i)^G = M_{i+1}/M_i$, $h(M_{i+1}/M_i) = 0$ by (i) of previous problem. Since $(M/M_N)^G = 0$, $h(M/M_N) = 0$ by (ii) of previous problem. By HW13 prob. 2, all cohomology groups are finite and so the Herbrand quotient is well defined. Now HW12 prob. 2(ii) shows that if $0 \to A \to B \to C \to 0$ is exact, then $h(B) = h(A)\,h(C)$. Repeated use of this to the chain of submodules M_i yields $h(M) = 1$.

(ii) The exact sequence $0 \to B \to A \to A/B \to 0$ yields $h(A) = h(B)\,h(A/B)$ (HW12 prob. 2(ii)). Since A/B is finite, (i) shows that $h(A/B) = 1$. The result follows.

(3) The additive structure of $H^*(G,k)$ was determined in HW11 prob 7. The answer was

$$H^0(\mathbb{Z},k) \cong k \cong H^1(\mathbb{Z},k),\ H^r(\mathbb{Z},k)=0 \text{ for } r\geq 2.$$

Thus we may write $H^*(\mathbb{Z},k) = k1 \oplus kx$ where 1 is the identity and has degree 0, and x has degree 1. Since x^2 has degree 2, $x^2=0$, hence the multiplication is determined by the formulae

$$11=1,\quad 1x=x=x1,\text{ and } x^2=0,$$

i.e. $(a+bx)(c+dx) = ac+(ad+bc)x$ for $a,b,c,d \in k$.

The Künneth formula (can be applied because G is of type FP_∞)

$$0 \to \bigoplus_{r+s=n} H^r(\mathbb{Z},k) \otimes_k H^s(\mathbb{Z},k) \xrightarrow{\pi} H^{r+s}(\mathbb{Z}\times\mathbb{Z},k)$$

$$\to \bigoplus_{r+s=n+1} \operatorname{Tor}_1^k(H^r(\mathbb{Z},k), H^s(\mathbb{Z},k)) \to 0.$$

Since $H^r(\mathbb{Z},k) = 0$ or k, it is free as a k-module and therefore the Tor term vanishes. It follows that π is onto and hence it induces an isomorphism

$H^*(\mathbb{Z}, k) \otimes_k H^*(\mathbb{Z}, k) \cong H^*(\mathbb{Z} \times \mathbb{Z}, k)$.

Thus $H^*(\mathbb{Z} \times \mathbb{Z}, k) \cong (k1 \oplus kx) \otimes_k (k1 \oplus ky)$

where $11 = 1$, $1y = y = y1$, $y^2 = 1$. Thus

$H^*(\mathbb{Z} \times \mathbb{Z}, k)$ is the k-algebra which as a k-module is free on $1, x, y, xy$, and multiplication is determined by

$11 = 1$, $1x = x = x1$, $1y = y = y1$, $x^2 = y^2 = 0$, $xy = -yx$.

Another way of describing $H^*(\mathbb{Z} \times \mathbb{Z}, k)$ is that it is isomorphic (as graded anticommutative k-algebras) to the exterior algebra on two generators $E_k[x, y]$.

(4) From the previous problem, we may write $H^*(\mathbb{Z}, k) = k1 \oplus kx$ where $11 = 1$, $1x = x1 = x$, $x^2 = 0$. Now β_0 is always the zero map, and $\beta_1 x = 0$ because it has degree two and $H^*(\mathbb{Z}, k)$ has no nonzero elements of degree two. It follows that $\beta_r = 0 \ \forall r$ (on $H^*(\mathbb{Z}, k)$).

From the previous problem, $H^*(\mathbb{Z} \times \mathbb{Z}, k) \cong (k1 \oplus kx) \otimes_k (k1 \otimes ky)$. To show that $\beta_r = 0 \ \forall \ r \in \mathbb{N}$, it will be sufficient to show that $\beta_r = 0$ on the k-basis $\{1 \otimes 1, x \otimes 1, 1 \otimes y, x \otimes y\}$. However if $\beta_r u = 0 = \beta_s v$, then

$$\beta_{r+s}(uv) = (\beta_r u)v + (-1)^r u(\beta_s v)$$

(by the part (iii) of the theorem on p. 4 of the previous handout) $= 0$ as required.

Remark One can prove this by looking at the cohomology sequence for $0 \to k \to \mathbb{Z}/4\mathbb{Z} \to k \to 0$ and applying the results of HW 11 prob. 7.

(5) (i) Let $n = [G:H]$. From handout 11, $tr_{H,G}\, res_{G,H}$ is multiplication by n on $H^*(H,k)$.

Since k is a field of characteristic p and $p \nmid n$, we see that $\frac{1}{n} \in k$ and so

$\frac{1}{n} tr_{H,G}\, res_{G,H}$ is the identity on $H^*(G,k)$.

[well known fact: if $\alpha : A \to B$, $\beta : B \to A$ are homomorphisms such that $\beta\alpha = 1_A$, then $B = \operatorname{im}\alpha \oplus \ker\beta$]

Therefore $H^*(H,k) = \operatorname{im}(res_{G,H}) \oplus \ker\left(\frac{1}{n} tr_{H,G}\right)$.

But $\ker\left(\frac{1}{n} tr_{H,G}\right) = \ker(tr_{H,G})$, hence

$H^*(H,k) = R \oplus T$.

(ii) If $x, y \in H^*(G,k)$, then $res_{G,H}\, x\, res_{G,H}\, y = res_{G,H}\, xy$ which shows that $RR \subseteq R$.

(iii) From handout 14, if $u \in H^*(G,k)$ and $x \in H^*(H,k)$, then

$tr_{H,G}((res_{G,H}\, u)x) = u\, tr_{H,G}\, x$, so if

$x \in T$, we see that $tr_{H,G}((res_{G,H}\, u)) = 0$.

Therefore $RT \subseteq T$.

It remains to show $RT = TR$. Observe

(i) If $\rho \in R$, then $\rho = \operatorname{res}_{G,H} \alpha$ for some $\alpha \in H^*(G, k)$. Writing $\alpha = \sum \alpha_i$ where $\alpha_i \in H^i(G, k)$, we see that $\rho = \sum \operatorname{res}_{G,H} \alpha_i$. Since $\operatorname{res}_{G,H} \alpha_i \in H^i(H, k)$ and we deduce that ρ is a sum of <u>homogeneous</u> elements of R

(ii) If $\tau \in T$, then $\tau = \sum \tau_i$ where $\tau_i \in H^i(H, k)$. Then $\operatorname{tr}_{H,G} \tau = \sum \operatorname{tr}_{H,G} \tau_i$ and since

$\operatorname{tr}_{H,G} \tau_i \in H^i(G, k)$, we see that $\operatorname{tr}_{H,G} \tau = 0$ $\Rightarrow \operatorname{tr}_{H,G} \tau_i = 0 \ \forall i$. We deduce that τ is a sum of <u>homogeneous</u> elements of T.

We now prove $RT \subseteq TR$. Let $\rho \in R$, $\tau \in T$: we need to prove $\rho\tau \in TR$. By (i) and (ii) above we may assume that ρ and τ are homogeneous. But then $\rho\tau = (-1)^{\deg \rho \deg \tau} \tau\rho$ which show that $\rho\tau \in TR$. Therefore $RT \subseteq TR$ and similarly $TR \subseteq RT$. We conclude that $RT = TR$.

(6) Continue to use the notation of (5), so $R = \mathrm{res}_{G,H} H^*(G,k)$ and $T = \ker \mathrm{tr}_{H,G}$. Then

$$(\mathrm{res}_{G,H} I) H^*(H,k)$$
$$= (\mathrm{res}_{G,H} I)(R \oplus T) \qquad \text{by 5(i)}$$
$$= (\mathrm{res}_{G,H} I) R \oplus (\mathrm{res}_{G,H} I) T \quad \text{by 5(ii) and (iii)}.$$

Now $(\mathrm{res}_{G,H} I) R = \mathrm{res}_{G,H} I \, \mathrm{res}_{G,H} H^*(G,k) =$
$\mathrm{res}_{G,H}(I H^*(G,k)) = \mathrm{res}_{G,H} H^*(G,k)$.

Therefore
$$(\mathrm{res}_{G,H} I) H^*(H,k) \cap R = \mathrm{res}_{G,H} I.$$
Similarly
$$(\mathrm{res}_{G,H} J) H^*(H,k) \cap R = \mathrm{res}_{G,H} J$$
and we deduce that $\mathrm{res}_{G,H} I = \mathrm{res}_{G,H} J$.

We have seen before that $\mathrm{res}_{G,H}$ is injective (reason: $\mathrm{tr}_{H,G} \mathrm{res}_{G,H}$ is multiplication by $[G:H]$ and $p \nmid [G:H]$). We deduce that $I = J$.

If $I_1 \subset I_2 \subset I_3 \subset \cdots$ is a strictly ascending chain of right ideals in $H^*(G,k)$, then by the above $I_1 H^*(H,k) \subset I_2 H^*(H,k) \subset \cdots$ is a strictly ascending chain of right ideals in $H^*(G,k)$. Thus if $H^*(H,k)$ is right Noetherian, so is $H^*(G,k)$.

(7) (i) Define a $\mathbb{Z}$-map $\phi : T \to \mathcal{I}$ by $\phi 1 = g-1$; thus $\phi a = (g-1)a \quad \forall a \in \mathbb{Z}$. We must check that ϕ respects the G-action.

$(\phi a)g = (g-1)ag = (g^2-g)a = (1-g)a$ (because $g^2 = 1$) $= -(g-1)a = (g-1)(-a) = \phi(-a)$

$= \phi(ag)$ as required.

(ii) $(g+1)g = g^2+g = 1+g = g+1$.

This shows $(g+1)\mathbb{Z}G = (g+1)\mathbb{Z}$. Define a $\mathbb{Z}$-map $\psi : \mathbb{Z} \to (g+1)\mathbb{Z}G$ by $\psi 1 = (g+1)$, so $\psi a = (g+1)a \quad \forall a \in \mathbb{Z}$. This is clearly a $\mathbb{Z}$-isomorphism. We need to check that ψ respects the G-action. $(\psi a)g = (g+1)ag = (g+1)a = \psi a = \psi(ag)$ as required.

(iii) $\theta(a+bg) = 0 \Rightarrow \theta a + (\theta b)g = 0 \Rightarrow a-b = 0 \Rightarrow a = b$, so $a+bg = a(1+g)$. Therefore $(1+g)\mathbb{Z} \supseteq \ker\theta$. Since $\theta(1+g) = 0$, it follows that $\ker\theta = (1+g)\mathbb{Z}G$. Therefore we have an exact sequence $0 \to (1+g)\mathbb{Z}G \to \mathbb{Z}G \to T \to 0$. Since $(1+g)\mathbb{Z}G \cong \mathbb{Z}$ by (ii), we deduce that we have an exact sequence $0 \to \mathbb{Z} \to \mathbb{Z}G \to T \to 0$ as required.

(8) (i) The cohomology exact sequence applied to $0 \to \mathbb{Z} \to \mathbb{Z}G \to T \to 0$ yields an exact sequence

$$H^r(G, \mathbb{Z}G) \to H^r(G, T) \to H^{r+1}(G, \mathbb{Z}) \to H^{r+1}(G, \mathbb{Z}G)$$

G finite $\Rightarrow H^r(G, \mathbb{Z}G) = 0 \quad \forall\, r \in \mathbb{P}$

Therefore $H^r(G, T) \cong H^{r+1}(G, \mathbb{Z}) \quad \forall\, r \in \mathbb{P}$.

Using the results for $H^{r+1}(G, \mathbb{Z})$ from handout II, we deduce that $H^r(G, T) \cong \mathbb{Z}/2\mathbb{Z}$ if r is odd, and $H^r(G, T) = 0$ if r even, $r \neq 0$. Also $T^G = 0$, and the result is proven.

(ii) We use the Universal Coefficient thm (note that $K \cong T \otimes_{\mathbb{Z}} \mathbb{Z}/4\mathbb{Z}$ as $\mathbb{Z}G$-modules)

$$H^r(G, K) \cong H^r(G, T) \otimes_{\mathbb{Z}} \mathbb{Z}/4\mathbb{Z} \oplus \operatorname{Tor}_1^{\mathbb{Z}}(H^{r+1}(G, T), \mathbb{Z}/4\mathbb{Z}).$$

Since $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$ and $\operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, the result follows.

<u>Remark</u> It would also be possible to get this from the short exact sequence $0 \to T \xrightarrow{\text{mult}^n \text{ by } 4} T \to K \to 0$

(9) Let $L = \{ a \in K \mid 2a = 0 \}$. Then L is a $\mathbb{Z}G$-submodule of K, and L, $K/L \cong \mathbb{Z}/2\mathbb{Z}$ as $\mathbb{Z}$-modules But G acts trivially on any module of order two, because g must send the nonzero element to itself. Therefore $L \cong \mathbb{Z}/2\mathbb{Z} \cong K/L$ as $\mathbb{Z}G$-modules and the result follows.

The long exact cohomology sequence applied to $0 \to \mathbb{Z}/2\mathbb{Z} \to K \to \mathbb{Z}/2\mathbb{Z} \to 0$ yields

$$0 \to H^0(G, \mathbb{Z}/2\mathbb{Z}) \to H^0(G, K) \to H^0(G, \mathbb{Z}/2\mathbb{Z})$$

$$\xrightarrow{\beta_0'} H^1(G, \mathbb{Z}/2\mathbb{Z}) \to \cdots$$

By problem 8(ii) and the results for the cohomology of a cyclic group from handout 11, all the cohomology groups in the above sequence are isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Therefore the maps in the sequence are alternately positive and negative, and the result follows.