Algebra Preliminary Exam Syllabus

In general, the syllabus is determined by the material covered in the course MATH 5125/6, Abstract Algebra, in the year which the student takes the course. This means that the syllabus will depend on the instructor teaching the course that year; however, the following topics are nearly always examined.

Groups

- Fundamental isomorphism theorem
- Groups acting on sets
- Alternating and symmetric groups
- Sylow Theorems
- Finite simple groups
- Finitely generated abelian groups

Rings and Fields

- Polynomial rings
- PID's and UFD's
- Prime and maximal ideals
- Field extensions, normal and Galois extensions, separability
- Computing Galois groups

Modules

- Isomorphism theorems
- Bases, direct sums, free modules, simple modules
- Modules over a PID

Additional topics may include:

- Nilpotent and solvable groups
- Free groups, generators and relations
- Noetherian rings and modules
- Hilbert Basis Theorem
- Hilbert Nullstellensatz

- Power series rings
- Projective and injective modules
- Nakayama's lemma
- Jacobson radical
- Artinian rings and modules
- Semisimple rings and Wedderburn structure theorem
- Jordan and rational canonical forms
- Representation theory of finite groups and character theory
- Tensor products

Books The following have been use for MATH 5125/6, and each cover most of the above material at the right level.

- "Algebra" by Michael Artin, Prentice Hall, 1991, ISBN 0-13-004763-5, MR: 92g:00001
- "Algebra" by Thomas W. Hungerford, Springer-Verlag, 1980, ISBN 0-387-90518-9, MR: 50 #6693
- "Algebra" (third edition) by Serge Lang, Addison Wesley, 1993, ISBN 0-201-55540-9
- "Basic Algebra" volumes I and II (second editions) by Nathan Jacobson, Freeman, 1985 and 1989, ISBN 0-7167-1480-9 and 0-7167-1079-X, MR: 86d:00001 and MR: 90m:00007
- "Algebra, a graduate course" by I. Martin Isaacs, Brooks/Cole, 1994, ISBN 0-534-19002-2, MR: 95k:00003

Other Books The following are good for various sections of MATH 5125/6, but do not cover the whole syllabus.

- "Algebra, a Module Theoretic Approach" by William A. Adkins and Steven H. Weintraub, Graduate Texts in Mathematics no. 136, Springer-Verlag, 1992, ISBN 0-387-97839-9, MR 94a:00001
- "Introduction to Commutative Algebra" by M. F. Atiyah and I. G. Macdonald, Addison-Wesley, 1969, MR: 39 #4129
- "Undergraduate Commutative Algebra" by Miles Reid, London Math. Soc. Student Texts no. 29, Cambridge University Press, 1995, ISBN 0-521-45889-7

- “An Introduction to the Theory of Groups” by Joseph J. Rotman, Graduate Texts in Mathematics no. 148, Springer-Verlag, 1995, ISBN 0-387-94285-8, MR: 95m:20001
- “Rings, Modules and Linear Algebra” by Brian Hartley and Trevor O. Hawkes, Chapman & Hall, 1980, ISBN 0-412-09810-5, MR: 42 #2897
- “Field theory and its classical problems” by Charles Robert Hadlock, Carus Mathematical Monographs no. 19, Mathematical Association of America, 1978, ISBN 0-88385-020-6, MR: 82c:12001
- “Abstract Algebra” by David S. Dummit and Richard M. Foote, Prentice Hall Inc., 1991, ISBN 0-13-004771-6, MR: 92k:00007

Algebra Preliminary Exam, Spring 1980

1. Let A, B and C be finite abelian groups. If $A \oplus C \cong B \oplus C$, prove that $A \cong B$.

2. Show that there exists no simple group of order 56.

3. Let T denote the set of all 5×5 matrices with eigenvalues $4, 4, 17, 17, 17$. Define a relation $\sim$ on T by $M_1 \sim M_2$ if M_1 and M_2 are similar matrices. How many equivalence classes does T have? Justify your answer. (Assume that the matrices are over $\mathbb{C}$.)

4. Give an example of a unique factorization domain (UFD) which is not a principal ideal domain (PID).

5. What is the Galois group of $x^3 - 10$ over $\mathbb{Q}$? Find all normal subfields of the splitting field.

6. Recall: if E is the splitting field of a polynomial f over F, then $\mathrm{Gal}(E/F)$ is called the Galois group of f over F. The Galois group of f over F is said to be transitive if given any two roots r_1 and r_2 of f in E, there exists σ in $\mathrm{Gal}(E/F)$ with $\sigma(r_1) = r_2$.

   (a) Prove that if f is a separable irreducible polynomial, then the Galois group of f is transitive.

   (b) Show that even though the Galois group of f is transitive, not every permutation of the roots need occur. (Hint: consider $x^4 - 2$ over $\mathbb{Q}$.)

7. Let A be a local ring with maximal ideal $\mathfrak{M}$, let k be $A/\mathfrak{M}$, and let M be a finitely generated A-module. Show that if $\mathrm{Hom}_A(M, k) = 0$, then $M = 0$. (Hint: use Nakayama's lemma.)

Algebra Preliminary Exam, Fall 1980

1. Suppose that for each prime integer p dividing the order of a finite group G, there is a subgroup of index p. Prove that G cannot be a nonabelian simple group.

2. A subgroup of a finite group is "p-local" if it is the normalizer of some Sylow p-subgroup. Show that the number of p-local subgroups of a group is congruent to 1 modulo p.

3. Characterize (with proof) all finitely generated $\mathbb{Q}\ x$ -modules with the property that each submodule is a direct summand. ($\mathbb{Q}$ denotes the field of rational numbers.)

4. Let R and S be local Noetherian integral domains with maximal ideals M and N respectively. Assume that $R \subseteq S$ and that S is a finitely generated R-module. If there exists a proper ideal I of R such that $I = IS \quad R$ and the canonical image of R/I in S/IS equals S/IS, then prove that $R = S$.

5. Let R be an integral domain. For $x, y \in R$, define $xR : yR \ = \{r \in R \mid yr \in xR\}$. Let $\{P_\lambda\}_{\lambda \in \Lambda}$ be a set of prime ideals of R with the property that if $x, y \in R$ and $y \notin xR$, then $xr : yR \ = P_\lambda$ for some $\lambda \in \Lambda$. Prove that $R = \bigcap_{\lambda \in \Lambda} R_{P_\lambda}$. ($R_{P_\lambda}$ denotes the localization at P_λ.)

6. If G is a finite group, prove that there exist fields K and L such that L is a Galois extension of K with Galois group isomorphic to G.

7. Show that for each positive integer n, there is a polynomial $d \in \quad x_1, \ldots, x_n$ such that for each $n \times n$ matrix A with complex entries,

$$\det A = d(\mathrm{tr}(A), \mathrm{tr}(A^2), \ldots, \mathrm{tr}(A^n)).$$

(denotes the field of complex numbers and $\mathrm{tr}(A)$ denotes the trace of A.)

Algebra Preliminary Exam, Spring 1981

1. Prove that a group of order p^n, where p is a prime and $n \geq 1$, has a nontrivial center.

2. Let G be a group of order pq, where p and q are primes and $p < q$. Prove that G has only one subgroup of order q.

3. (a) Show that if H is a subgroup of order 12 in a group G of order 36, then H is a normal subgroup of G. (Map G to the automorphisms of the set of right cosets of H.) NOTE: there are examples of groups of order 36 with subgroups of order 12 which are *not* normal.

   (b) Describe *all* abelian groups of order 36, up to isomorphism.

4. Prove that if $p(x)$ is a polynomial irreducible over a field F, and if α and β are roots of $p(x)$ in some extension field of F, then $F(\alpha)$ and $F(\beta)$ are isomorphic. What happens if $p(x)$ is reducible?

5. Let L be a normal extension of a field F, let $G = \mathrm{Gal}(L/F)$, and let H be a subgroup of G. Prove that

   (a) if $K = \{x \in L \mid h(x) = x \text{ for all } h \in H\}$, then K is a subfield of L;

   (b) if K is normal over F, then H is a normal subgroup of G.

6. (a) Let R be a commutative principal ideal domain (PID) with a 1. If $a, b \in R$, show that a and b have a greatest common divisor $d \in R$ (i.e. d divides a and b, and if g divides a and b, then g divides d).

   (b) Show that $\mathbb{Z}[x]$ is not a PID.

7. Let R be a commutative ring with identity, and let I, J be ideals in R. Let $IJ = \{x \in R \mid x = \sum_{i=1}^{n} a_i b_i \text{ for } a_i b_i \in J, n \in \mathbb{Z}^+\}$.

   (a) Show that IJ is an ideal of R.

   (b) Show that if $I + J = R$, where $I + J = \{a + b \mid a \in I, b \in J\}$, then $IJ = I \cap J$ (recall that R has a 1).

   (c) Suppose further that R is a domain, and that $IJ = I \cap J$ for all ideals I, J in R. Prove that R is a field. (Take principal ideals.)

Algebra Preliminary Exam, Fall 1981

Instructions: do all eight problems
Notation: $\mathbb{Z}$ = integers, $\mathbb{Q}$ = rational numbers

1. Let G be a finite group.
   (a) Let A and B be subgroups of G such that $B \lhd G$ and $AB = G$. Prove that $[A : A \cap B] = [G : B]$.
   (b) Let H be a subgroup of G such that $[G : H] = 2$ and let $h \in H$. If m is the number of conjugates of h in G and n is the number of conjugates of h in H, prove that either $n = m$ or $n = m/2$.
2. If the order of G is 105 and H is a subgroup of G of order 35, prove that $H \lhd G$.
3. Let P be a nonnormal p-Sylow subgroup of the finite group G. If $\mathbf{N}_G(P)$ is the normalizer of P in G, prove that $\mathbf{N}_G(P)$ is nonnormal in G.
4. Let F_1 and F_2 be finite fields of orders q_1 and q_2.
   (a) Prove that q_i is a power of a prime, say $q_i = p_i^{\alpha_i}$ for $i = 1, 2$.
   (b) If $F_1 \subseteq F_2$, prove that $p_1 = p_2$ and that α_1 is a divisor of α_2.
5. (a) Prove that $x^4 - 2$ is irreducible over $\mathbb{Q}$.
   (b) Let K be the splitting field of $x^4 - 2$. Prove that the Galois group of K over $\mathbb{Q}$, $\mathrm{Gal}(K/\mathbb{Q})$, is of order 8.
   (c) Exhibit the correspondence (given by the Fundamental Theorem of Galois theory) between the subgroups of $\mathrm{Gal}(K/\mathbb{Q})$ and the intermediate fields between $\mathbb{Q}$ and K.
6. (a) State Nakayama's lemma.
   (b) Let R be a local commutative ring with maximal ideal M. Let X be a finitely generated R-module. Show that if X/MX can be generated by n elements, then so can X.
7. (a) Construct an example of finitely generated nonzero abelian groups A and B so that $\mathrm{Hom}_{\mathbb{Z}}(A,B) = \mathrm{Hom}_{\mathbb{Z}}(B,A) = A \otimes_{\mathbb{Z}} B = 0$.
   (b) If A and B are finitely generated abelian groups such that $\mathrm{Hom}_{\mathbb{Z}}(A,B) \neq 0$ and $\mathrm{Hom}_{\mathbb{Z}}(B,A) = 0$, prove that $B \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ and $A \otimes_{\mathbb{Z}} \mathbb{Q} \neq 0$.
8. Let R be a commutative ring and let A be an ideal of R. Define the radical of A, denoted $\sqrt{A}$, by $\sqrt{A} = \{r \in R \mid r^n \in A \text{ for some positive integer } n\}$. You may assume that $\sqrt{A}$ is an ideal of R and that $A \subseteq \sqrt{A}$.

(a) If P is a prime ideal of R such that $P \supseteq A$, prove that $P \supseteq \sqrt{A}$ and as a consequence show that $\sqrt{P} = P$.

(b) Prove that $\sqrt{A}/A$ is the set of nilpotent elements of R/A. (An element r is nilpotent if $r^n = 0$ for some positive integer n.)

Algebra Preliminary Exam, Fall 1982

Do all problems

1. Prove that a group of order $135 = 3^3 \cdot 5$ has a normal subgroup of order 15.

2. (a) For a positive integer n, show that every ideal in $\mathbb{Z}/(n)$ is principal.

   (b) Explain how one determines the number of ideals of $\mathbb{Z}/(n)$ in terms of n.

3. (a) Calculate the Galois group of $(x^2-2)(x^2+3)$ over $\mathbb{Q}$.

   (b) Explicitly state the correspondence between the subfields of the splitting field K of $(x^2-2)(x^2+3)$ over $\mathbb{Q}$ and the subgroups of $\mathrm{Gal}(K/\mathbb{Q})$.

4. Let V and W be vector spaces over the field k and let W^* be the space of linear functions from W to k. Prove that the map $\phi\colon V \otimes_k W^* \to \mathrm{Hom}_k(W,V)$ defined by $\phi(v \otimes f)(w) = f(w)v$ is

   (a) Well defined.

   (b) Linear.

5. Let K be a field of characteristic $\neq 2$. Suppose $f(x) = p(x)/q(x)$ is a ratio of polynomials in $K[x]$. Prove that if $f(x) = f(-x)$, then there are polynomials $p_0(x^2), q_0(x^2)$ such that $f(x) = p_0(x^2)/q_0(x^2)$. (HINT: look for a field automorphism of $K(x)$ that fixes $f(x)$.)

6. Let G be a solvable group. Prove that if $N \neq \{e\}$ is normal in G and contains no other non trivial subgroups which are normal in G, then N is abelian.

7. Let A, B, C, D be finite abelian groups such that $A \times B \cong C \times D$ and $B \cong D$. Prove that $A \cong C$.

Algebra Preliminary Exam, Spring 1984

Do all problems

1. Find the Galois group of $x^6 - 1$ over $\mathbb{Q}$.

2. Show that a semisimple right Artinian ring without zero divisors is a division ring. (A ring has no zero divisors if $ab = 0$ implies $a = 0$ or $b = 0$.)

3. Show that there are no simple groups of order 300.

4. Let R be a commutative domain with field of fractions F. Prove that F is an injective R-module.

5. State and prove a structure theorem analogous to the Fundamental Theorem for modules over a PID, which describes finitely generated $\mathbb{Z}/n\mathbb{Z}$-modules. (You may assume the Fundamental Theorem for any argument.)

6. Assume that K/F is a Galois field extension and α lies in an algebraic closure of K. Prove that $|\mathrm{Gal}(K/F)|$ divides $|\mathrm{Gal}(K(\alpha)/F(\alpha))| \deg \alpha$, where $\deg \alpha$ denotes the degree of α over F.

7. Prove that a finite p-group with a unique subgroup of index p is cyclic. (Hint: first consider abelian p-groups.)

8. Let $f: M \quad N$ be a surjective homomorphism of left R-modules. Show that if P is a projective R-module, then the induced map $\mathrm{Hom}_R(P,M) \overset{f}{-} \quad \mathrm{Hom}_R(P,N)$ is a surjection of abelian groups.

9. Let k be a field and let $R = k \; X_1, \ldots, X_m \;$ be the polynomial ring in m indeterminates. Prove that if M is a simple R-module, then $\dim_k M \quad \infty$.

Algebra Preliminary Exam, Fall 1985

Do ALL problems

1. (a) Define what is meant by a prime ideal in a commutative ring.

   (b) Prove that a nonzero prime ideal in a principal ideal domain is always a maximal ideal.

2. Prove that there is no simple group of order 56.

3. Show that a finite field cannot be algebraically closed.

4. Find all finitely generated abelian groups A with the property that for any subgroups B and C, either $B \subseteq C$ or $C \subseteq B$.

5. Let F be the splitting field of $(x^3-2)(x^2-3)$ over $\mathbb{Q}(i)$. Describe the Galois group in as much detail as possible.

6. Let R be an integral domain and let θ be an element of the quotient field of R. Set $I = \{r \in R \mid r\theta \in R\}$.

   (a) Prove that I is an ideal of R.

   (b) Show that either $\theta \in R$ or there exists a maximal ideal $\mathfrak{M}$ of R such that $\theta \notin R_{\mathfrak{M}}$.

   (c) Conclude that $R = \bigcap R_{\mathfrak{M}}$, where the intersection is taken over all the maximal ideals of R.

7. Prove that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathbb{Q}$ are isomorphic as $\mathbb{Z}$-modules. (Here $\mathbb{Q}$ denotes the rational numbers and $\mathbb{Z}$ denotes the integers.)

8. Let G be a group. Suppose that

   (a) H and K are nilpotent groups,

   (b) there are homomorphisms $\alpha\colon G \to H$ and $\beta\colon G \to K$, and

   (c) $\ker\alpha \cap \ker\beta \subseteq \mathbf{Z}(G)$, the center of G.

   Prove that G is nilpotent.

Algebra Preliminary Exam, Spring 1986

1. Let F be the free group on the set $X = \{x_i \mid i \in I\}$, and let H be the normal subgroup of F generated by $\{x_i x_j x_i^{-1} x_j^{-1} \mid i, j \in I\}$. Prove that F/H is isomorphic to the free abelian group on X.

2. Let G be an abelian group of order p^6, and let $H = \{x \in G \mid x^p = 1\}$. Suppose that $|H| = p^2$. Give all possible such groups G.

3. Prove that S_4 is solvable.

4. In S_5, how many Sylow subgroups of each type are there?

5. (a) Let R be a PID and let S be a multiplicatively closed subset. Show that $S^{-1}R$ is a PID.

   (b) Give an example of a PID with exactly 3 nonassociate irreducible elements.

6. Let $(M_\alpha)_{\alpha \in A}$ be the set of maximal ideals in a commutative ring R with identity. Set $J = \bigcap_{\alpha \in A} M_\alpha$. For $r \in R$, prove that $r \in J$ if and only if $1 + rs$ is a unit for all $s \in R$.

7. Let $\alpha \in \mathbb{C}$ be a root of $x^3 + 4x + 2$.

   (a) Find a basis for $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$. Justify y our answer.

   (b) Express $(\alpha + 1)^{-1}$ in terms of the basis.

   (c) Express $(\alpha^2 + 3\alpha + 5)(2\alpha^2 - 4\alpha + 1)$ in terms of the basis.

8. Let K be a subfield of the field L, and let $\alpha \in L$ such that $[K(\alpha) : K]$ is odd. Prove that $K(\alpha^2) = K(\alpha)$.

Algebra Prelim, Fall 1986

1. (a) Let G be a finite abelian (multiplicative) group. Prove that if G is not cyclic, then there exists a positive integer n such that $n < |G|$ and $g^n = e$ for all $g \in G$.

   (b) Prove that the multiplicative group of a finite field is cyclic.

2. Let G be a group of order $5 \cdot 7^2 \cdot 17$, and let H be a subgroup of order $7^2 \cdot 17$.

   (a) Prove that H is abelian.

   (b) Prove that H is normal in G.

   (c) Prove that G is abelian.

3. Let R be an integral domain. For an element $a \in R$, prove the equivalence of the following two statements.

   (a) There exists an infinite chain $(a_1) \subset (a_2) \subset \cdots$ of principal ideals of R with $a = a_1$.

   (b) There exists an infinite set $\{b_i \mid i = 1, 2, \ldots\}$ of nonunits of R such that $b_1 b_2 \ldots b_n$ divides a for each positive integer n.

4. Let R be a commutative ring with identity and let M be a maximal ideal of R.

   (a) Prove that $R[x]/M[x] \cong (R/M)[x]$.

   (b) Conclude that $M[x]$ is a prime ideal but not a maximal ideal in $R[x]$. Indeed argue that there are infinitely many prime ideals of $R[x]$ which contain $M[x]$.

5. Let $k \subseteq K \subseteq L$ be fields such that K is a splitting field over k. If $\sigma \in \mathrm{Gal}(L/k)$, prove that $\sigma(K) = K$.

6. Let K/k be a finite extension and let $\alpha \in K$ with $f(x) = \mathrm{Irr}(\alpha, k)$. If $n = \deg f(x)$, prove that $n | [K : k]$.

7. Let $f(x) = x^n - 1$ and let K be a splitting field for $f(x)$ over $\mathbb{Q}$. Prove that the Galois group $\mathrm{Gal}(K/\mathbb{Q})$ is abelian.

8. Let M be a module. A submodule S of M is *small* if whenever $S + N = M$ for any submodule N of M, then $N = M$. Suppose S is small in M and there exists an epimorphism $f \colon P \to M/S$ where P is projective. Prove that there exists an epimorphism $\psi \colon P \to M$.

Algebra Preliminary Exam, Spring 1987

Do any 8 problems

1. (a) Let H be a subgroup of the finite group G, and let p be a prime. Prove that two distinct Sylow p-subgroups of H cannot lie in the same Sylow p-subgroup of G.

   (b) Let n be a positive integer and let R be a ring with a 1. Show that R has characteristic n if and only if R has a subring (with the same 1) isomorphic to $\mathbb{Z}/n\mathbb{Z}$ (i.e. the integers modulo n).

2. Let A and B be normal subgroups of the group G.

   (a) Prove that $A \cap B$ is a normal subgroup of G.

   (b) Prove that $G/(A \cap B)$ is isomorphic to a subgroup of $(G/A) \times (G/B)$.

   (c) If G is finite and $G/(A \cap B) \cong (G/A) \times (G/B)$, prove that $AB = G$.

3. (a) Let G be a simple group with a subgroup H of index 6. Prove that there exists a momomorphism $\phi\colon G \to S_6$.

   (b) Prove there are no simple groups of order 300.

4. Let R be the ring $\{a + b\sqrt{29} \mid a, b \in \mathbb{Z}\}$ (so $R = \mathbb{Z}[\sqrt{29}\,]$). Define $N\colon R \to \mathbb{Z}$ by $N(a + b\sqrt{29}) = a^2 - 29b^2$ for $a, b \in \mathbb{Z}$.

   (a) Show that for $\alpha, \beta \in R$, $N(\alpha\beta) = N(\alpha)N(\beta)$.

   (b) Let $\alpha \in R$. Show that α is a unit if and only if $N(\alpha) = \pm 1$.

   (c) Show that $a^2 - 29b^2 = \pm 2$ has no solution with $a, b \in \mathbb{Z}$.

   (d) Show that $2, -2, 5 + \sqrt{29}$ and $5 - \sqrt{29}$ are irreducible elements of R.

   (e) Deduce that R is not a UFD.

5. Let R be a UFD and let S be a multiplicatively closed subset of nonzero elements of R.

   (a) If u is irreducible in R, prove that u is either irreducible or a unit in $S^{-1}R$.

   (b) Prove that $S^{-1}R$ is a UFD.

6. Let V be a vector space over $\mathbb{R}$ and let $T\colon V \to V$ be a linear transformation. Describe how V can be made into an $\mathbb{R}[x]$-module via T.

   Suppose V has basis (e_1, e_2, e_3) and T is the linear transformation defined by $T(e_1) = 2e_1$, $T(e_2) = -4e_2 - 4e_3$, and $T(e_3) = 9e_2 + 8e_3$.

   (a) Express V as a direct sum of two nonzero $\mathbb{R}[T]$-modules.

(b) Calculate $(x^2-4x+4)e_2$.

(c) If $V \cong \bigoplus_{i=1}^n \mathbb{R}[x]/(f_i)$ where $f_1|f_2|\ldots|f_n$ and f_1 is not a unit, what are the possibilities for the ideals (f_i)?

(d) Express V as a direct sum of cyclic modules.

(e) Does there exist $v \in V$ such that $V = \mathbb{R}[x]v$?

7. Let R be a PID, let p be a prime of R, and let M be the R-module $R/Rp^{e_1} \oplus \cdots \oplus R/Rp^{e_n}$ where the e_i and n are positive integers. Define $M(p) = \{m \in M \mid pm = 0\}$ and $pM = \{pm \mid m \in M\}$.

   (a) Prove that $M(p)$ and pM are submodules of M.

   (b) Prove that $M/pM \cong M(p)$.

   (c) In the case $R = \mathbb{Q}[x]$ and $p = x^2+1$, give an example of a finitely generated R-module M such that $M/pM \ncong M(p)$.

8. Let $R = \mathbb{Z} \times \mathbb{Z}$ and $S = \mathbb{Z}^\# \times 0$, where $\mathbb{Z}$ is the ring of integers and $\mathbb{Z}^\# = \mathbb{Z} \backslash 0$. Prove that $S^{-1}R \cong \mathbb{Q}$, where $\mathbb{Q}$ is the field of rational numbers.

9. List without repetition all the abelian groups of order $3^2 2^3$. Which ones are cyclic?

Algebra Preliminary Exam, Fall 1987

Instructions: do all problems

1. Let G be a group with 56 elements. Prove that G is not simple.

2. Let G be a group and let $S \leqslant G$. Prove that $\langle x^{-1}Sx \mid x \in G\rangle \lhd G$. Now suppose that $G = HA$ where H and A are subgroups and A is abelian. Prove that there exists $K \lhd G$ such that $H \cap A \subseteq K \subseteq H$. Deduce that if G is nonabelian simple, then $G = \langle H, x^{-1}Hx\rangle$ for all $x \in G \setminus H$.

3. Let G be the group $\mathbb{C} \setminus 0$ with the operation multiplication. Define $\theta\colon G \to G$ by $\theta(x) = x^2$. Prove that θ is a group homomorphism, $|\ker\theta| = 1$, and $G/\ker\theta \cong G$. Suppose $L = \{x \in G \mid x^{(2^n)} = 1 \text{ for some positive integer } n\}$. Is $G/L \cong G$.

4. If H is a subgroup of the group G, let $\mathbf{N}(H)$ denote the normalizer of H in G. Suppose G is a finite group and P is a Sylow p-subgroup of G. Prove that $\mathbf{N}(\mathbf{N}(P)) = \mathbf{N}(P)$.

5. Let R be a commutative ring. If I and J are ideals of R, define $(I : J) = \{x \in R \mid Jx \subseteq I\}$. Prove that $(I : J)$ is an ideal of R.

   Now suppose $I \subseteq (a)$, $a \notin I$, $K = (I : (a))$ and R/I is a domain. Prove that $K = I$ and $aK = I$. Deduce that $I \subseteq \bigcap_{n=1}^{\infty}(a^n)$. Does the final assertion remain true if the hypothesis $a \notin I$ is dropped? ((a) denotes the ideal generated by a.)

6. Let K be a field. Prove that $K[X]$ has infinitely many irreducible polynomials, no two of which are associates. (Consider $p_1p_2 \dots p_n + 1$). Suppose now $f \in K[X]$, $f \neq 0$. Prove that there exists a homomorphism θ from $K[X]$ to a domain with nonzero kernel such that $\theta(f) \neq 0$.

7. Let R be a ring, let M be an R-module, and let $\theta\colon M \to M$ be an R-module homomorphism. Prove that $\ker\theta$ is a submodule of M.

   Now suppose every submodule of M is finitely generated. Prove there exists an integer n such that $\bigcup_{i=1}^{\infty}\ker\theta^i = \ker\theta^n$. Deduce that if θ is onto, then θ is an isomorphism.

8. Let V be a vector space over $\mathbb{C}$ and let $T\colon V \to V$ be a linear transformation. Describe how V can be made into a $\mathbb{C}[X]$-module via T.

   Now let $\{e_1, e_2, e_3\}$ be a basis for V and suppose $T(e_1) = -e_1 + 2e_2$, $T(e_2) = -2e_1 + 3e_2$, $T(e_3) = -2e_1 + 2e_2 + e_3$. Find the Jordan canonical form for the matrix of T. Hence find the isomorphism type of V (as a $\mathbb{C}[X]$-module) as a direct sum of primary cyclic modules. Does there exist a $\mathbb{C}[X]$-module homomorphism of $\mathbb{C}[X]$ onto V?

Algebra Preliminary Exam, Spring 1988

Do all eight problems

1. Let G be a group and define $\phi\colon G\times G \to G$ by $\phi(g,h) = gh^{-1}$.

   (a) Find necessary and sufficient conditions on G such that ϕ is a homomorphism.

   (b) Under the conditions determined for (a), prove that $\Delta = \{(x,x) \mid x \in G\}$ is a normal subgroup of G and $(G\times G)/\Delta \cong G$.

2. Let $f\colon G \to H$ be a group homomorphism with H an abelian group. Suppose that N is a subgroup of G containing $\ker(f)$. Prove that N is a normal subgroup of G.

3. Let G be a group of order 99.

   (a) Show that G is not a simple group.

   (b) Show that G contains a subgroup of order 33.

4. Prove that a finite abelian group is either cyclic or has at least p elements of order p for some prime p.

5. If S is a simple nonabelian group, prove that $\mathrm{Aut}(S)$ contains a subgroup isomorphic to S. (Hint: consider conjugation.)

6. Let R be a commutative ring with identity. A *simple* R-module S is a module whose only submodules are 0 and S.

   (a) Prove that an R-module S is simple if and only if there is a maximal ideal $\mathfrak{M}$ such that S is isomorphic to $R/\mathfrak{M}$.

   (b) Let R be a commutative ring with identity. Show that simple R-modules exist.

7. Let R be a ring with identity and let I be a (two-sided) ideal in R. Let M and N be R-modules.

   (a) Show that $R/I \otimes_R M$ is isomorphic to M/IM as left R/I-modules.

   (b) Show that $(M\oplus N)/I(M\oplus N)$ is isomorphic to $(M/IM)\oplus(N/IN)$. You may use any results about tensor products you know.

8. Let E be an extension field of F with $[E:F] = 11$. Prove that if $x, y \in E$ with neither in F and if θ is an F-automorphism of E, then

$$\theta(x) \neq x \text{ implies } \theta(y) \neq y.$$

Algebra Preliminary Exam, Fall 1988

Instructions: do all problems

1. Prove that there are no simple groups of order 600.

2. Let R be a principal ideal domain and assume that A, B, and C are finitely generated R-modules. Suppose that $A \oplus B$ is isomorphic to $A \oplus C$. Prove that B is isomorphic to C.

3. Prove that the Galois group of a splitting field K of an irreducible polynomial p over the rational numbers $\mathbb{Q}$ acts transitively on the roots of p. Show by examples that this theorem does not necessarily hold if either

   (a) K is not a splitting field, or

   (b) p is reducible over $\mathbb{Q}$.

4. Prove that a group of order 255 is cyclic.

5. Define what is meant by a solvable group. Prove that if $H \lhd G$, and H and G/H are solvable, then G is solvable.

6. Let $T\colon V \to V$ be a linear map where V is a finite dimensional vector space over an algebraically closed field. Prove that if 0 is the only eigenvalue of T, then $T^n = 0$ where $n = \dim(V)$.

7. Prove that if $f\colon S \to T$ is a homomorphism between simple R-modules S and T, then either f is an isomorphism or f is the zero homomorphism. (Recall that a nonzero R-module is simple if 0 and the module itself are the only submodules.)

8. Let R be a commutative ring with a 1.

   (a) Prove that if M is a cyclic R-module, then M is isomorphic to R/I for some ideal I of R.

   (b) Prove that if M is a cyclic R-module and N is an arbitrary R-module, then $M \otimes_R N$ is isomorphic to N/IN for some ideal I of R.

Algebra Prelim, Spring 1989

Do all problems

1. Show that a group of order 540 cannot be simple.

2. Compute the Galois group of $5x^5 + 3x^4 + 15$ over

$$\text{(i) } \mathbb{Z}/2\mathbb{Z} \qquad \text{(ii) } \mathbb{Q}$$

3. Let R and S be domains and let $\theta\colon R \to S$ be an epimorphism. Which of the following statements are true? (Prove or give a counterexample.)

   (a) If R is a PID, then S is a PID.

   (b) If R is a UFD, then S is a UFD.

   (c) If $\ker\theta \neq 0$ and R is a PID, then S is a field.

4. Let K be a field and let $f \in K[x]$ be a polynomial.

   (a) Let $\alpha_1, \ldots, \alpha_r$ be distinct zeros of f in K. Prove that there exists $g \in K[x]$ such that

$$f = (x - \alpha_1)\ldots(x - \alpha_r)g.$$

   (b) Let p be a prime number and let $K = \mathbb{Z}/p\mathbb{Z}$ be the finite field with p elements. For each integer m, let $\overline{m}$ denote its residue class in K. Prove that as polynomials in $K[x]$, we have

$$x^{p-1} - \overline{1} = \prod_{m=1}^{p-1}(x - \overline{m}).$$

   Deduce that p divides $(p-1)! + 1$.

5. Let G be a group of finite order and let F be the intersection of all maximal subgroups of G.

   (a) Prove that $F \lhd G$.

   (b) If $H \leqslant G$ and $FH = G$, prove that $H = G$.

   (c) If S is a Sylow subgroup of F and $x \in G$, prove that $xSx^{-1} = fSf^{-1}$ for some $f \in F$. Deduce that $G = F\,\mathbf{N}_G(S)$.

6. Which of the following statements are true? (Prove or give counterexample.)

(a) If K/F and E/K are finite Galois extensions, then E/F is a finite Galois extension.

(b) Let $f, g \in \mathbb{Q}[x]$ be irreducible, let α be a root of f, and let β be a root of g. If $\mathbb{Q}(\alpha) = \mathbb{Q}(\beta)$, then $\mathrm{Gal}(f/\mathbb{Q}) \cong \mathrm{Gal}(g/\mathbb{Q})$.

7. Let R be a commutative ring. Define what is meant by saying that an R-module is Noetherian.

   Suppose R has the property that the R-modules R^n are Noetherian for all $n \in \mathbb{N}$ (where R^n denotes the direct sum of n copies of R, and $\mathbb{N} = \{0, 1, 2, \dots\}$). Let M be a finitely generated R-module.

   (a) Show that $M \cong R^n/N$ for some $n \in \mathbb{N}$ and some R-submodule N of R^n.

   (b) Deduce that if L is a submodule of M, then $L \cong K/N$ where K is a submodule of R^n containing N.

   (c) Conclude that all finitely generated R-modules are Noetherian.

8. Determine the matrices in $\mathrm{M}_3(\mathbb{Q})$ commuting with $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$.

Algebra Qualifying Exam, Fall 1989

Do six problems

1. Compute the Galois group of $3x^2+7x+21$ over

$$\text{(a) } \mathbb{Z}/2\mathbb{Z} \qquad \text{(b) } \mathbb{Q}$$

2. By an "N-group", we mean a finite group with the property that every nonidentity homomorphic image has a nonidentity center. Prove that maximal subgroups of N-groups are always normal.

3. Let R be an integral domain and let K be its field of fractions. Assume that if $0 \neq x \in K$, then either $x \in R$ or $x^{-1} \in R$. Prove that

   (a) R is a local ring.

   (b) R is integrally closed in K.

4. Let R be a PID and let A, M be nonzero finitely generated R-modules.

   (a) Show that if A is torsion free, then $A \otimes_R M \neq 0$.

   (b) Provide a counterexample to the conclusion of 4a in the case A is not torsion free.

5. Assume that p and q are distinct primes. Show that a group of order p^2q cannot be simple.

6. Let k be a field. If $f \in k[X_1, X_2, \ldots, X_n]$, define

$$V(f) = \{(a_1, \ldots, a_n) \in k^n \mid f(a_1, \ldots, a_n) = 0\}.$$

   Prove that if $f_1, f_2, \ldots$ is a countable list of polynomials in $k[X_1, \ldots, X_n]$, then there is a positive integer T such that

$$\bigcap_{j=1}^{\infty} V(f_j) = V(f_1) \cap V(f_2) \cap \cdots \cap V(f_T).$$

7. Let k be a field. Prove that if A and B are two $n \times n$-matrices with entries in k, both of which have minimal polynomial X^{n-1}, then A and B are similar.

Algebra Prelim, Spring 1990

Answer all questions

1. (a) Let M be a left R-module, and let A and B be Artinian submodules. Show that $A+B$ is an Artinian R-submodule.

   (b) If R is also left Noetherian and M is finitely generated, show that M has a unique maximum Artinian submodule $A(M)$ and that $A(M/A(M)) = 0$.

2. Let A be an abelian group with no elements of infinite order. Suppose that every element of prime order is of order 3. Show that the order of every element is a power of 3. (Hint: do finitely generated abelian groups first.)

3. Let G be a simple group of order 144.

   (a) Prove that a group of order 18 has exactly one Sylow 3-subgroup.

   (b) If H is a proper subgroup of G, show that $|H| \leq 26$.

   (c) If P and Q are distinct Sylow 3-subgroups of G, show that $|P \cap Q| = 1$. (If $|P \cap Q| > 1$, consider $N_G(P \cap Q)$).

4. Prove that a group of order 765 is abelian.

5. Let $f(x)$ in $\mathbb{Q}[x]$ be an irreducible polynomial of degree 5. Suppose a and b are distinct roots and that $\mathbb{Q}(a) = \mathbb{Q}(b)$. Show that $\mathbb{Q}(a)$ is a normal extension of $\mathbb{Q}$.

6. Let $S \subseteq \mathbb{Z}[x_1, x_2, \ldots, x_n]$. Prove that there is a smallest principal ideal containing S. If this ideal is generated by α, show that $\alpha\mathbb{Q}[x_1, x_2, \ldots, x_n]$ is the smallest principal ideal in $\mathbb{Q}[x_1, x_2, \ldots, x_n]$ containing S.

7. Let V be a vector space over R, and let $T: V \to V$ be a linear transformation. Describe how V can be made into an $R[x]$-module.

   Now suppose there are $v_1, \ldots, v_n$ in V such that $\{T^i(v_j) \mid i = 0, 1, \ldots,\ j = 1, 2, \ldots, n\}$ span V as a vector space.

   (a) Prove that V is a finitely generated $R[x]$-module.

   (b) If T is onto, show that V cannot have a summand isomorphic to $R[x]$.

   (c) If T is onto, show that V is finite dimensional.

8. (a) Let $A = \{\iota, (12)(34)\}$ and $V = \{\iota, (12)(34), (13)(24), (14)(23)\}$ in S_4. Show that A is normal in V and V is normal in S_4, but A is not normal in S_4.

   (b) Give an example of fields $F \subseteq K \subseteq L$ such that K a normal extension of F and L a normal extension of K, but L is not a normal extension of F.

Algebra Prelim, Fall 1990

Answer all questions

1. Give a complete list of all non-isomorphic abelian groups of order $200 = 2^3 \cdot 5^2$.

2. Show that a group of order $216 = 2^3 \cdot 3^3$ cannot be simple.

3. Let G be a finite p-group with $|G| > p^2$. Prove that G contains a normal abelian subgroup of order p^2.

4. (a) Show that if G is a subgroup of S_n, then either $G \subseteq A_n$ or $[G : G \cap A_n] = 2$.

   (b) Show that if $n \geq 5$ and G is a normal subgroup of S_n, then $G = 1$, A_n or S_n.

   (c) Show that if $n \geq 5$, then S_n has no subgroups of index 3.

5. Let G be an abelian group with 54 elements. Suppose that G cannot be generated by one element, but can be generated by two elements. Prove that G is isomorphic to $\mathbb{Z}/18\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

6. Let K be an extension field of F with $[K : F] = 14$. Let $f(x) \in F[x]$ be a polynomial of degree 5. Suppose $f(x)$ has no roots in F but has a root in K. What can you say about the factorization of $f(x)$ into irreducibles in $F[x]$ and $K[x]$?

7. Let $f(x)$ be irreducible over $\mathbb{Q}$ with splitting field E, and let α and β be roots of f in E. If $E/\mathbb{Q}$ has an abelian Galois group, prove that $\mathbb{Q}(\alpha) = \mathbb{Q}(\beta)$.

8. Let R be a commutative ring with identity, and let $(M_\alpha)_{\alpha\in\Gamma}$ be the set of maximal ideals of R. Let A be an ideal of the polynomial ring $R[x]$ such that $A \subseteq \bigcup_\alpha M_\alpha[x]$. Show that $A \subseteq M_\beta[x]$ for some $\beta \in \Gamma$. (Hint: consider the set $B = \{r \in R \mid r \text{ is a coefficient of some polynomial in } A\}$.)

Qualifying Exam Algebra Spring 1991

1. Suppose that A, H are normal subgroups of a group G such that G/A is a simple group of order n.

   (a) Prove that $H \cap A$ is a normal subgroup of H.

   (b) Prove that either $H \subseteq A$ or $H/(H \cap A)$ is a simple group of order n. (Hint: use an isomorphim theorem.)

2. (a) Prove that a group of order 100 cannot be simple.

   (b) Describe all abelian groups of order 100 up to isomorphism.

   (c) Either show that every group of order 100 is abelian, or exhibit a nonabelian example.

3. Let G be a group and let $f\colon G \to H$ be a group homomorphism. Prove that if H is a solvable group and if $\ker(f)$ is abelian, then G is a solvable.

4. Let R be a PID.

   (a) Prove that the intersection of two nonzero maximal ideals cannot be zero.

   (b) Assume that R contains an infinite number of maximal ideals. Show that the intersection of all the nonzero maximal ideals of R equals zero.

5. Let $R \subseteq S$ be rings with a 1 such that S/R is a free left R-module. Prove that if L is a left ideal of R, then $LS \cap R = L$. (Hint: write S as a direct sum of R-modules.)

6. Let R be a ring with 1. A nonzero left R-module S is simple if 0 and S are the only submodules of S. Let

   $$0 \longrightarrow S \xrightarrow{\alpha} M \xrightarrow{\pi} S \longrightarrow 0$$

   be a short exact sequence of R-modules which is *not* split, and such that S is a simple R-module. Show that the only nonzero submodules of M are $\alpha(S)$ and M. (Hint: if $0 \neq N \subseteq M$ and $\alpha(S) \cap N = 0$, show that there is an isomorphism $\sigma\colon S \to N$ such that $\pi\sigma = 1_S$.)

7. Suppose that F is a Galois extension of $\mathbb{Q}$ with $[F : \mathbb{Q}] = 25$. What possible groups can occur as the Galois group of F over $\mathbb{Q}$? In all cases, describe the intermediate fields between F and $\mathbb{Q}$ is in terms of inclusion and dimension over $\mathbb{Q}$. Which intermediate fields are Galois over $\mathbb{Q}$?

8. Recall that a group G of permutations of a set S is called *transitive* if given $s, t \in S$, then there exists $\sigma \in G$ such that $\sigma(s) = t$. Let $f(x)$ be a separable polynomial in $K[x]$ and let F be a splitting field of $f(x)$ over K. Prove that $f(x)$ is irreducible over K if and only if the Galois group of F over K is a transitive subgroup when viewed as permutations of the roots of $f(x)$.

Algebra Prelim, Spring 1992

Answer all questions

1. (a) Let G be a finite group of order m, and let p be the smallest prime which divides m. Prove that if H is a subgroup of index p, then H is a normal subgroup of G.

   (b) Prove that any group of order p^2q is solvable, where $p \neq q$ are primes. (Hint: consider separately the cases $p < q$ and $p > q$).

2. List all groups of order 6 (up to isomorphism), and prove that they are the only ones.

3. If A and B are finitely generated abelian groups with $A \oplus A$ isomorphic to $B \oplus B$, prove that A and B are isomorphic.

4. Suppose $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is a short exact sequence of modules over a ring R. Prove that if the sequence splits, i.e. there is an R-module homomorphism $h\colon C \to B$ such that $gh = 1_C$, then $B \cong A \oplus C$.

5. Let R be a commutative ring with a 1. If S is a multiplicative set (i.e. $x, y \in S \Rightarrow xy \in S$) containing 1, but not 0, prove there exists a prime ideal of P of R with $P \cap S = \emptyset$.

6. Let A be an abelian group and let $m \geq 1$ be an integer. Prove that $A \otimes \mathbb{Z}/m\mathbb{Z} \cong A/mA$.

7. Let K/k be a Galois extension of fields and let $f(x) \in k[x]$ be an irreducible polynomial which has a root in K. Prove that $f(x)$ splits into linear factors in $K[x]$.

8. Let K be a finite Galois extension of $\mathbb{Q}$ with Galois group isomorphic to A_4. For each divisor d of 12, how many subfields L of K have $[K : L] = d$? In each case give the isomorphism class of $\mathrm{Gal}(K/L)$, and state whether or not $L/\mathbb{Q}$ is a Galois extension. (Recall A_4 is a counter-example to the converse of Lagrange's theorem.)

Algebra Prelim, Fall 1992

Answer all questions

1. Prove that any group of order 765 is abelian.

2. Let G be a finite group and let N be a normal subgroup of G. Prove that G is solvable if and only if both N and G/N are solvable.

3. (a) Prove that if G is a subgroup of S_n, then either $G \subseteq A_n$ or $[G : G \cap A_n] = 2$.

   (b) Prove that if $n \geq 5$ and G is a normal subgroup of S_n, then $G = \{e\}$, A_n or S_n.

   (c) Prove that if $n \geq 5$, then S_n has no subgroup of index 3.

4. Let $f(x)$ be an irreducible polynomial over $\mathbb{Q}$ with splitting field K. If the Galois group of $K/\mathbb{Q}$ is abelian, prove that for any roots α, β of $f(x)$ in K, we have $\mathbb{Q}(\alpha) = \mathbb{Q}(\beta) = K$.

5. Let K be a field and $f(x) \in K[x]$ be a separable irreducible polynomial of degree 4, and let E be a splitting field for $f(x)$ over K. If $\alpha \in E$ is a root of $f(x)$ and $L = K(\alpha)$, prove that there exists a subfield F of L with $[F : K] = 2$ if and only if the Galois group of E/K is not isomorphic to either A_4 or S_4.

6. (a) If G, H and K are finitely generated abelian groups with $G \oplus H \cong G \oplus K$, prove that $H \cong K$.

   (b) Give an example to show that part (a) is false if G is not finitely generated.

7. Let R be an integral domain. A nonzero element π of R is a *prime* if $\pi | ab$ implies that either $\pi | a$ or $\pi | b$. A nonzero element π is *irreducible* if $\pi = ab$ implies that either a or b is a unit.

   (a) Prove that every prime is irreducible.

   (b) If R is a UFD, prove that every irreducible is prime.

8. Let R be a commutative ring with a 1, and let M be a cyclic R-module.

   (a) Prove that M is isomophic to R/I for some ideal I of R.

   (b) If N is any R-module, prove that $M \otimes_R N$ is isomorphic to N/IN for some ideal I of R.

Algebra Preliminary Exam, Spring 1993

Do six problems

1. Let R be a principal ideal domain. Assume that M is a nonzero finitely generated R-module with the property that the intersection of any two nonzero submodules is nonzero. Prove that $M = R/Rt$ where t is either zero or some power of an irreducible element in R.

2. Let G be a group which acts transitively on a finite set X. Assume that there is an element $x_0 \in X$ whose stabilizer has no element of finite order other than 1.

   (a) Show that if $f \in G$ has finite order larger than 1, then f has no fixed points.

   (b) Show that if the order of $f \in G$ is a prime q, then $|X| \quad 0 \mod q$.

3. Let S be a commutative ring with prime ideals $P_1, P_2, \ldots, P_t$. Show that if $S/P_1 \quad P_2 \quad \cdots \quad P_t$ is a finite set, then each of the P_i is a maximal ideal.

4. Let p be a prime. Prove that if every nontrivial finite field extension of the field F has degree divisible by p, then every finite field extension of F has degree a power of p. (You may assume that $\mathrm{char} F = 0$.)

5. Let R be a commutative ring with a 1. If M and N are R-modules, then $\mathrm{Hom}(M,N)$ denotes the set of all R-module homomorphisms from M to N. If $f : N \quad N$ is an R-module homomorphism, then $f \ : \ \mathrm{Hom}(M,N) \quad \mathrm{Hom}(M,N\ )$ is defined by $f \ (g) = f \quad g$, the composition of g followed by f. Prove that M is a projective R-module if and only if for all surjective $f : N \quad N$, the function f is surjective.

6. Let D be a finite dihedral group, and let V be a finite dimensional complex vector space which is a D-module. (You may regard D as a group of linear transformations from V to itself.) Prove that if the only D-invariant subspaces of V are 0 and V itself (i.e. V is a simple or irreducible D-module), then $\dim_{\mathbb{C}} V \quad 2$.

7. Determine the Galois group of (the splitting field for) the polynomial $X^{10} - 1$ over the rational numbers.

Algebra Preliminary Exam, Fall 1993

Do all problems

1. Assume that R is a ring and $e \in R$ has the property that $e^2 = e$. Prove that Re is a projective left R-module.

2. Let F be a field of characteristic zero and let K be a finite field extension of F.

   (a) Explain why there is a polynomial $p(X) \in F[X]$ such that $K \cong F[X]/(p)$.

   (b) Prove that if $c \in K[X]/(p)$ and $c^2 = 0$, then $c = 0$.

3. Let $\mathrm{M}_2(\mathbb{Q})$ denote the group of 2×2 matrices with rational entries under addition, and let $\mathrm{GL}_2(\mathbb{Q})$ denote the group of invertible 2×2 matrices with rational entries under multiplication.

   (a) Prove that if $\mathrm{M}_2(\mathbb{Q})$ acts on a set, then all orbits are either infinite or singletons.

   (b) Show that $\mathrm{GL}_2(\mathbb{Q})$ acts on $\mathbb{Q}$ via $g * \lambda = \dfrac{\det(g)}{|\det(g)|}\lambda$, and that there exists a finite orbit which is not a singleton.

4. Let G be the direct product of the dihedral group of order 34 and the cyclic group of order 9. Suppose that L is a field and G is a group of automorphisms of L. Prove that there is a unique field K such that $L^G \subseteq K$ and $\dim_K L = 17$. (You may assume that $\operatorname{char} L = 0$.)

5. Let S be a commutative integral domain. Prove that if every prime ideal of $S[X]$ is principal, then S is a field.

6. Let A be an abelian group.

   (a) Show that the collection H of all homomorphisms from A to $\mathbb{Z}$ is a group under addition of functions.

   (b) Prove that if $f_1, \ldots, f_m \in H$, then the subgroup generated by $f_1, \ldots, f_m$ is free (i.e. free as a $\mathbb{Z}$-module).

7. Let p be a prime, and let G be the group of invertible 2×2 matrices under multiplication with entries in the field of integers modulo p. Let H be the subgroup consisting of all matrices of the form $\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$.

   (a) Show that $|G| = (p^2 - 1)(p^2 - p)$.

   (b) Find all values of p such that the number of conjugates of H in G is congruent to 8 mod p.

Algebra Prelim Spring 1994
Answer All Problems

(1) Let G be a group with exactly three elements of order two. Prove that G is not simple.

(2) Let $G = \langle x, y \mid x^6 = e, y^4 = e, yxy^{-1} = x^{-1} \rangle$. Prove that G has a homomorphic image isomorphic to S_3 (the symmetric group of degree 3), but is *not* isomorphic to S_3.

(3) Let R be a PID which is *not* a field, and let M be a finitely generated R-module which is *not* a torsion module.

(i) Prove that the R-module R is isomorphic to a proper submodule of itself.
(ii) Prove that M is isomorphic to a proper submodule of itself.

(4) Let R be a ring, and let A, B, C be R-modules.

(i) Prove that $\mathrm{Hom}_R(A, B \oplus C) \cong \mathrm{Hom}_R(A, B) \oplus \mathrm{Hom}_R(A, C)$ as abelian groups.
(ii) Prove that $\mathrm{Hom}_R(A, A \oplus A)$ is *not* isomorphic to $\mathbb{Z}$.

(5) Let R be a commutative Noetherian ring.

(i) If S is a multiplicative subset of R, prove that $S^{-1}R$ is Noetherian.
(ii) Prove that $R[[X^{-1}, X]]$ (the Laurent Series ring in X) is a Noetherian ring (you may assume that the power series ring $R[[X]]$ is Noetherian).

(6) (i) Prove that $X^4 + X^3 + X^2 + X + 1$ is irreducible in $\mathbb{Q}[X]$. (Set $Y = X - 1$.)
(ii) Let $A \in \mathrm{M}_n(\mathbb{Q})$, and suppose no eigenvalue of A is equal to 1. Prove that if $A^5 = I$, then $4 | n$ (where I denotes the identity matrix).

Algebra Prelim Fall 1994 Answer All Problems

(1) Let p be a prime, let G be a finite p-group, let Z be the center of G, and let $1 = H \lhd G$.

(i) Let $x \in H \setminus Z$, and let $\mathfrak{C}(x)$ denote the conjugacy class containing x. Prove that $\mathfrak{C}(x) \subseteq H$ and p divides $|\mathfrak{C}(x)|$.
(ii) Prove that $Z \cap H = 1$.
(iii) Let A be a maximal normal abelian subgroup of G. Prove that A is also a maximal abelian subgroup of G. (Apply (ii) with $\bar{G} = G/A$ and H the centralizer of A in G.)

(2) Let G be a simple group of order 180.

(i) Prove that the number of 5-Sylow subgroups of G is 36.
(ii) Prove that the normalizer of a 3-Sylow subgroup of G has order 18.
(iii) Prove that the 3-Sylow subgroup of a group of order 18 is normal in that group.
(iv) If A and B are distinct 3-Sylow subgroups of G, prove that $A \cap B = 1$ (consider the centralizer in G of $A \cap B$).
(v) Prove that there is no simple group of order 180.

(3) Let R be a PID (principal ideal domain), and let M be a cyclic left R-module. Suppose $M = A \oplus B$ where A and B are nonzero left R-modules. Prove that there exists $r \in R \setminus 0$ such that $rM = 0$. Prove further that for such an r, there exist distinct primes $p, q \in R$ such that pq divides r.

(4) Let R be the ring $\mathbb{Z}_{(2)}[[X]]/(X-2)$, where $\mathbb{Z}_{(2)}[[X]]$ denotes the power series ring in X over $\mathbb{Z}_{(2)}$, the localization of $\mathbb{Z}$ at the prime 2.

(i) If q is an odd integer, prove that q is invertible in $\mathbb{Z}[[X]]/(X-2)$.
(ii) Define $\theta\colon \mathbb{Z}[[X]] \to \mathbb{Z}_{(2)}[[X]]$ by $\theta \sum a_i X^i = \sum a_i X^i$, and let $\pi\colon \mathbb{Z}_{(2)}[[X]] \to R$ be the natural epimorphism. Prove that $\pi\theta$ is surjective and deduce that $R = \mathbb{Z}[[X]]/(X-2)$.
(iii) Prove that $R \ncong \mathbb{Z}[X]/(X-2)$.

(5) Let p be a prime, let $K \subseteq L$ be fields of characteristic p, let $\alpha, \beta \in L$, and let d be a positive integer. Suppose $[K(\alpha) : K] = d$, $[K(\beta) : K] = p$, α is separable over K, and β is not separable over K.

(i) Prove that $K(\alpha) = K(\alpha^p)$ and $\beta^p \in K$.
(ii) Prove that $K(\alpha) \subseteq K(\alpha + \beta)$.
(iii) Prove that $[K(\alpha + \beta) : K] = pd$.

(6) Let p be a prime, and let $K = \mathbb{F}_p(t)$, the quotient field (field of fractions) of the polynomial ring $\mathbb{F}_p[t]$.

(i) Prove that $X^p - t$ is irreducible in $K[X]$.
(ii) Let L be the splitting field of $X^p - t$ over K. Determine the Galois group of L over K.

Qualifying Exam Algebra Spring 1995

1. Determine, up to isomorphism, all groups of order $1127 = 7^2 \cdot 23$.

2. Let G be a noncyclic nilpotent group. Show that there is a normal subgroup N of G such that G/N is a noncyclic abelian group.

3. Describe all finitely generated abelian groups G such that if A and B are subgroups of G, then either $A \subseteq B$ or $B \subseteq A$.

4. Let R be a commutative ring with a 1. Recall that if I is an ideal, then $\operatorname{rad} I = \{x \mid x^m \in I \text{ for some } m \in \quad\}$ is also an ideal. Let $P_1, \ldots, P_n$ be distinct prime ideals in R.

   (a) Show that $R/\operatorname{rad} I$ has no nonzero nilpotent elements.

   (b) Prove that $\operatorname{rad}(P_1 \cdots P_n) = P_1 \quad P_2 \quad \cdots \quad P_n$.

   (c) Prove that $R/\operatorname{rad}(P_1 \cdots P_n)$ is an integral domain if and only if there exists an i such that P_i is contained in every P_j for $j = 1, \ldots, n$.

5. Suppose that K is a subfield of a field L. Assume that $K = \mathbb{Q}(\omega)$, where $\omega = e^{\frac{2\pi i}{3}} = -1/2 \qquad \overline{3}/2 \cdot i$. Note that $\omega^3 = 1$. Let $\alpha \in L$ such that $K(\alpha) : K \ = 2^3$. Prove that $K(\alpha^3) = K(\alpha)$.

6. Let R be a commutative ring with 1 and let $a \in R$ be non-nilpotent. Let $S = \{a^i \mid i \geq 0\}$.

   (a) Prove that there is a prime ideal P not containing a.

   (b) Let K be the quotient field of R/P. Prove that there exists a ring homomorphism $\phi \colon S^{-1}R \qquad K$.

7. Let F be a finite Galois extension of K and suppose that $\mathrm{Gal}(F/K) = S_4$.

   (a) Show that there are at least 9 different proper intermediate fields between F and K.

   (b) Show that there is a proper Galois extension E of K (in F) and describe the Galois group of E over K.

8. Find the Jordan canonical form of $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$ viewed as matrices over the complex numbers $\quad$. Find all 3×3-matrices with entries in $\quad$ that commute with this canonical form.

Qualifying Examination Algebra Fall 1995

#1. Suppose that G is a group of order $(35)^3$. Show that G has normal Sylow 5- and 7-subgroups. Also show that G has a normal subgroup of order 25.

#2. Suppose that G is an abelian group isomorphic to $\mathbb{Z}/(36) \times \mathbb{Z}/(45)$. Let H be a subgroup of G of order 27. Up to isomorphism, describe H.

#3. Let $f\colon G \to H$ be a group epimorphism with G finite, let $g \in G$, and let C denote the centralizer of $f(g)$ in H.

a). Prove that if D is a conjugacy class in $f^{-1}(C)$, then $f(D)$ is a conjugacy class in C.

b). Prove that the order of the conjugacy class of g in $f^{-1}(C)$ is at most $|\ker f|$.

c). Prove that the order of the centralizer of g in G is at least $|C|$.

#4. Suppose that R is a unique factorization domain which is NOT a principal ideal domain.

a). Show that R must have at least two (nonassociate) prime elements.

b). Show that R must have a nonprincipal maximal ideal.

#5. Let R be a ring with a 1 and suppose that X is an R-module and N is a submodule of an R-module M. Let $i\colon N \to M$ denote the inclusion map and let $\sigma\colon N \to N \oplus X$ be the R-module homomorphism $\sigma(n) = (n, 0)$. Prove that if

$$\begin{array}{ccc} N & \xrightarrow{i} & M \\ || & & \downarrow f \\ N & \xrightarrow{\sigma} & N \oplus X \end{array}$$

is a commutative diagram for some R-homomorphism f, then M is isomorphic to $N \oplus M/N$.

#6. Suppose that E is a Galois field extension of F with $[E:F] = p^n$ for some prime p and positive integer n. Show that there are intermediate fields $F = K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_n = E$ so that $[K_i : K_{i-1}] = p$ and K_i is Galois over F for $i = 1, \ldots, n$.

#7. Prove that there is no finite field which is algebraically closed.

Qualifying Examination Algebra January 1996

(1) Let $f\colon G \to H$ be a group epimorphism between the finite abelian groups G and H. Suppose that G has order $2^3 \cdot 3^3 \cdot 5^2 \cdot 11^2$ and H has order $2^3 \cdot 5^2$. What is the order of $\ker f$? Describe, up to isomorphism, the possible groups of that order.

(2) Suppose that G is a group of order p^4q^5 where p and q are distinct primes. Suppose further that both a Sylow p-subgroup and a Sylow q-subgroup are normal in G.

(i) Prove that $G \cong A \times B$, where A and B are subgroups of orders p^4 and q^5 respectively.

(ii) Prove that G has a normal subgroup of order pq.

(3) Prove that a group of order $2^2 \cdot 3 \cdot 11^2$ is not simple.

(4) Find the degree and a $\mathbb{Q}$-basis of $\mathbb{Q}[\sqrt{2}+\sqrt{3}]$ over $\mathbb{Q}$ where $\mathbb{Q}$ is the rational numbers. Justify your answer.

(5) Prove that in a principal ideal domain D, every nonzero prime ideal is a maximal ideal. Deduce that if K is an integral domain and $f\colon D \to K$ is a ring epimorphism with $\ker f \neq 0$, then K is a field.

(6) Let R be a commutative ring. Prove that R has no nonzero nilpotent elements if and only if $R_{\mathfrak{P}}$ has no nonzero nilpotent elements for all prime ideals $\mathfrak{P}$ of R (where $R_{\mathfrak{P}}$ denotes the localization of R at the prime ideal $\mathfrak{P}$). Is it true that R is a domain if and only if $R_{\mathfrak{P}}$ is a domain for all prime ideals $\mathfrak{P}$ of R?

(7) Suppose that M is an R-module with submodules A and B such that $A \cap B = 0$. Prove that the submodule of M generated by A and B is isomorphic to $A \oplus B$ (the direct sum of A and B).

(8) Suppose that the Galois group of a Galois extension E over F is S_6.

(i) Show that there are at least 35 proper subfields between E and F.

(ii) Show that there is a subfield L between E and F such that L is Galois over F, but there is no subfield between E and L which is Galois over L.

(iii) What is the dimension of L over F?

Algebra Prelim, August 1997

Answer all questions

1. Let R be a commutative ring with unity and let I, J be ideals of R.

   (a) Prove that the product $IJ = \{x \in R \mid x = \sum_{i=1}^{n} a_i b_i \text{ with } a_i \in I \text{ and } b_i \in J\}$ is an ideal of R.

   (b) Prove that $IJ \subseteq I \quad J$.

   (c) If $I \quad J = R$, prove that $IJ = I \quad J$.

   (d) If $IJ = I \quad J$ for all ideals of R and R is an integral domain, prove that R is a field. (Hint: let $I = Ra$ where $a = 0$ be a principal ideal or R.)

2. Let F be a finite Galois extension of the field K with $\mathrm{Gal}(F/K) = S_5$.

   (a) Show that there are more than 40 fields strictly between F and K.

   (b) Show that there is a unique proper subfield E of F with $E = K$ such that E/K is a Galois extension. Determine $E : K$ and describe $\mathrm{Gal}(E/K)$ up to isomorphism.

3. Let G be a group of order 455.

   (a) Prove that G is not simple.

   (b) Prove that G is cyclic.

4. Let R be a PID, let n be a positive integer, and let A and B be finitely generated R-modules. If $A^n = B^n$, prove that $A = B$. (A^n denotes the direct sum of n copies of A.)

5. Let P be a finitely generated projective $\mathbb{Z}$-module. If P is also injective, prove that $P = 0$.

6. Let A, B be abelian groups, and let m be a positive integer. Prove that $A \otimes (B/mB) = (A \otimes B)/m(A \otimes B)$.

7. Prove that a group of order 588 is solvable.

8. Let $K = \mathbb{Q}(\ \overline{2} \quad \overline{3},\ \overline{5})$.

   (a) Determine $K : \mathbb{Q}$.

   (b) Compute $\mathrm{Gal}(K/\mathbb{Q})$.

Algebra Prelim, January 1998

Answer all questions

1. Let F and K be fields of characteristic 0 with K an extension of F of degree 21. Let $f(x)$ be a polynomial in $F[x]$ of degree 6 which has no roots in F and exactly two roots in K.

   (a) Describe the factorization of $f(x)$ into irreducible polynomials in $F[x]$.

   (b) Describe the factorization of $f(x)$ into irreducible polynomials in $K[x]$.

2. Let G be a group of order $1947 = 3 \cdot 11 \cdot 59$. Prove that G is cyclic.

3. Let G be a group of order p^n with $n \geq 2$ and p prime. Prove that G has a normal abelian subgroup of order p^2.

4. Let $K = \mathbb{Q}(\sqrt[3]{2}, \omega)$ where $\omega = \cos(2\pi/3) + i\sin(2\pi/3)$ is a primitive cube root of unity.

   (a) What is $[K : \mathbb{Q}]$?

   (b) Prove that $K/\mathbb{Q}$ is a Galois extension.

   (c) Describe the Galois group of $K/\mathbb{Q}$.

5. Let R be a PID with field of fractions (quotient field) F, let S be subring of F which contains R, and let A be an ideal of S.

   (a) Prove that $A \cap R$ is an ideal of R.

   (b) If $A \cap R = Rd$, prove that $A = Sd$. (Hint: if $a/b \in S$ with $(a,b) = 1$, prove that $1/b \in S$.)

6. Let p be a prime, let a, k be positive integers such that p does not divide k, and let G be a group of order $p^a k$. Let M be a normal subgroup of G and let P be a Sylow p-subgroup of G.

   (a) Prove that PM/M is a Sylow p-subgroup of G/M.

   (b) Let H/M be the normalizer of PM/M in G/M and let N be the normalizer of P in G. Prove that $N \subseteq H$.

   (c) Prove that the number of Sylow p-subgroups of G/M is a divisor of the number of Sylow p-subgroups of G.

7. (a) Give an example of a group of order $3540 = 59 \cdot 60 = 59 \cdot 5 \cdot 3 \cdot 4$ which is not solvable.

   (b) Give an example of a group of order 3540 which is solvable but not cyclic.

8. Let G and H be finitely generated abelian groups such that $G \times G = H \times H$. Prove that $G = H$.

Algebra Prelim, January 1999

Answer all questions

1. Let R be a commutative ring with a $1 = 0$. If every ideal of R except the ideal R is a prime ideal, prove that R is a field.

2. Let p and q be distinct primes and let G be a group of order p^3q^3. If G has a normal p-Sylow subgroup, prove that G has a normal subgroup H of order p^3q.

3. Let R be a commutative ring with a 1. Prove that R is isomorphic to a proper R-submodule of R if and only if there exists an element in R which is neither a zero divisor nor a unit. (A zero divisor is an element r such that there exists $s \in R \quad 0$ such that $rs = 0$. A unit in R is an element r such that there exists $s \in R$ such that $rs = 1$.)

4. Let K be a subfield of the field L and let $\alpha \in L$. If $\ K(\alpha) : K\ $ is odd, prove that $K(\alpha^2) = K(\alpha)$.

5. Let p be a prime and let G be an abelian group of order p^6. Suppose the set $\{x \in G \mid x^p = 1\}$ has order p^2. Describe all possible groups G (up to isomorphism). Justify your answer.

6. Let $k \subseteq K \subseteq L$ be fields such that K is a splitting field over k, and let $\sigma \in \mathrm{Gal}(L/k)$. Prove that $\sigma(K) = K$.

7. Prove that there is no simple group of order 280.

8. Let n be a positive integer, let E be a field of characteristic zero, and let F be a subfield of E such that $\ E : F\ = n$. Prove that there are at most $2^{n!}$ fields between F and E.

Algebra Prelim, August 1999

Answer all questions

1. Let G be a simple group of order 480 with an abelian Sylow 2-subgroup.

   (a) If P and Q are distinct Sylow 2-subgroups of G, by considering $\mathbf{C}_G(P \cap Q)$, prove that $P \cap Q = 1$.

   (b) Prove that there is no such group G.

2. Let R be a UFD, let S be a multiplicatively closed subset of R such that $0 \notin S$, and let p be a prime in R. Prove that $p/1$ is either a prime or a unit in $S^{-1}R$.

3. Let k be the field $\mathbb{Z}/2\mathbb{Z}$. Classify the finitely generated projective $k[X]/(X^3 + X)$-modules up to isomorphism.

4. Let R be a ring, let M be a Noetherian R-module, and let J denote the Jacobson radical of R. Prove that either $MJ^n = 0$ for some positive integer n, or $MJ^{n+1} \subsetneq MJ^n$ (strict inequality) for all positive integers n.

5. Let R be a nonzero right Artinian ring (with a 1) with no nonzero nilpotent ideals and no nontrivial ($\neq 0, 1$) idempotents. Prove that R is a division ring.

6. Compute the character table of S_4.

7. Let K be a splitting field of the polynomial $X^4 - 2$ over $\mathbb{Q}$. Determine the order of $\mathrm{Gal}(K/\mathbb{Q})$. Use this to show that K contains a subfield L such that $[L : \mathbb{Q}] = 4$ and L is normal over $\mathbb{Q}$.

ALGEBRA PRELIMINARY EXAMINATION: Fall 2000

Do all problems

1. Let

$$1 \longrightarrow A \longrightarrow G \longrightarrow P \longrightarrow 1$$

   be a short exact sequence of groups such that A is abelian, $|P| = 81$, and $|A| = 332$. Show that G has a nontrivial center.

2. Let F be a finite field of odd characteristic. Prove that the rings $F[X]/(X^2 - \alpha)$ as α ranges over all nonzero elements of F fall into exactly two isomorphism classes.

3. Let R be a finite-dimensional simple algebra and let M be a finite-dimensional left R-module. Prove that there is a positive integer d such that

$$\underbrace{M \oplus M \oplus \cdots \oplus M}_{d \text{ copies}}$$

   is a free module.

4. Let B be a square matrix with rational entries. Show that if there is a monic polynomial $f \in \mathbb{Z}[T]$ such that $f(B) = 0$ then the trace of B is an integer.

5. Let k be a field. Compute the dimension over k of

$$k[X]/(X^m) \otimes_{k[X]} k[X]/(X^n)$$

   and prove your assertion.

6. In this problem X, Y, Z are indeterminates. Define $\sigma \colon \mathbb{C}(X, Y, Z) \to \mathbb{C}(X, Y, Z)$ by $\sigma(h(X, Y, Z)) = h(Y, Z, X)$ for every rational function h in three variables. Prove or disprove: every member of $\mathbb{C}(X, Y, Z)$ which is left unchanged by σ is a rational function of $X + Y + Z$, $XY + YZ + XZ$ and XYZ.

ALGEBRA PRELIMINARY EXAMINATION:
Winter 2001

Do all problems. All rings should be assumed to have a 1.

1. Let R, A and B be commutative rings with $R \subseteq A$ and $R \subseteq B$. Prove that if A is an integral extension of R and B is an integral extension of R, then the ring $A \otimes_R B$ is also an integral extension of R.

2. For this problem all fields have characteristic 0. Let K/L be a Galois extension with Galois group G and let H be a subgroup of G. Prove that there exists some $\beta \in K$ such that H coincides with
$$\{\sigma \in G \mid \sigma(\beta) = \beta\}.$$

3. Let S be a semisimple ring. Prove that $S \times S$ is semisimple.

4. Let G be a finite group and assume that p is a fixed prime divisor of its order. Set $K = \bigcap \mathrm{N}_G(P)$ where the intersection is taken over all Sylow p-subgroups P of G and $\mathrm{N}_G(\_)$ denotes the normalizer. Show that
   (a) $K \lhd G$.
   (b) G and G/K have the same number of Sylow p-subgroups.

5. Suppose A is an abelian group (written additively) of order p^M for some prime p. Prove that if n is a positive integer such that $p^n A = 0$, then
$$|\{a \in A \mid pa = 0\}| \geq p^{M/n}.$$

6. Let G be a finite group. Prove that if H and K are normal nilpotent subgroups of G, then so is HK.

7. Prove or disprove: let $\mathbb{Z}_6$ denote the ring of integers modulo 6. Then every projective $\mathbb{Z}_6$-module is free.

Algebra Qualifying Exam, Summer 2001

Instructions: do all problems.

1. Let G be a group.

   (a) Show that if A and B are normal subgroups of G, then $A \cap B$ is also a normal subgroup of G.

   (b) Suppose that N is a proper nontrivial normal simple subgroup of G and that G/N is also a simple group. Prove that either N is the only nontrivial proper normal subgroup of G or that G is isomorphic to $N \times G/N$.

2. Let A be a finite noncyclic abelian group with two generators. Let p be a prime. Assume that for all primes q with $q \neq p$, there are no nonzero group homomorphisms from $\mathbb{Z}/(q)$ to A. Describe the structure of A and prove that there is no nonzero group homomorphism from A to $\mathbb{Z}/(q)$ for all primes q with $q \neq p$.

3. Let R be a ring with a 1 and let $0 \to P \to M \to Q \to 0$ be a short exact sequence of R-modules. Show that if P and Q are projective R-modules, then M is a projective R-module.

4. Prove that every group of order 441 has a quotient which is isomorphic to $\mathbb{Z}/(3)$.

5. Let R be a commutative ring with a $1 \neq 0$. Suppose that every ideal of R different from R is a prime ideal.

   (a) Prove that R is an integral domain.

   (b) Prove that R is a field.

6. (a) Prove that $\mathbb{Q}(\sqrt[4]{2})$ and $\mathbb{Q}(i\sqrt[4]{2})$ are isomorphic fields.

   (b) Prove that $\mathbb{Q}(i, \sqrt[4]{2})$ is a Galois extension of $\mathbb{Q}$.

   (c) Find the Galois group $\mathrm{Gal}(\mathbb{Q}(i, \sqrt[4]{2})/\mathbb{Q})$ and prove your claim.

7. Recall that a group G of permutations of a set S is called *transitive* if given $s, t \in S$, then there exists $\sigma \in G$ such that $\sigma(s) = t$. Let K be a field. Let $f(x)$ be a separable polynomial in $K[x]$ and let F be a splitting field of f over K. Prove that $f(x)$ is irreducible over K if and only if the Galois group of F over K is a transitive group when viewed as a group of permutations of the roots of $f(x)$.

8. An ideal in a commutative ring R with a 1 is called *primary* if $I \neq R$ and if $ab \in I$ and $a \notin I$, then $b^n \in I$ for some positive integer n.

   (a) Prove that prime ideals are primary.

   (b) Prove that if R is a PID, then I is primary if and only if $I = P^n$ for some prime ideal P of R.

Algebra Prelim, Summer 2002

Instructions: do all problems.

1. Let G be a finite group.

   (a) Let H and Q be subgroups of G. Note that H acts on the set of conjugates of Q via conjugation. Let O_Q denote the orbit containing Q with respect to this action. Prove that if $H \cap \mathrm{N}_G(Q) = 1$, then the orbit O_Q has $|H|$ subgroups in it.

   (b) Now suppose that $|G| = p^m q$ where p and q are distinct primes and m is a positive integer. Let Q be a Sylow q-subgroup of G and suppose that $\mathrm{N}_G(Q) = Q$. Prove that G has a normal Sylow p-subgroup.

2. Let K be a finite Galois extension of the rationals $\mathbb{Q}$. Suppose that $\sqrt{2}$ and $\sqrt{3}$ are both elements of K. Show that $\mathrm{Gal}(K/\mathbb{Q})$ has a normal subgroup N of index 4. Show further that if $|N|$ is odd, then $\sqrt[8]{2} \notin K$.

3. (a) Let R be a ring. Let A and B be right R-modules and let C be a left R-module. Prove that $(A \oplus B) \otimes_R C \cong (A \otimes_R C) \oplus (B \otimes_R C)$.

   (b) Let M be a finitely generated $\mathbb{Z}$-module. Prove that if $M \otimes_{\mathbb{Z}} M = 0$, then $M = 0$.

4. Let R be a commutative Noetherian ring with unity and let M be a nonzero R-module. Given $m \in M$, set $\mathrm{Ann}(m) = \{r \in R \mid rm = 0\}$. Show there exists some $w \in M$ such that $\mathrm{Ann}(w)$ is a prime ideal of R.

5. Let R be an integral domain. Prove that R is a field if and only if every R-module is projective.

6. Denote the center of a group by $Z(\cdot)$. Let G be a finite group with identity element e. Define a sequence of subgroups of G inductively by $Z_0 = \{e\}$ and

   $$Z_{j+1} \text{ is the preimage in } G \text{ of } Z(G/Z_j).$$

   Since $Z_0 \subseteq Z_1 \subseteq Z_2 \subseteq \cdots$, there is a positive integer N such that $Z_N = Z_{N+1} = Z_{N+2} \ldots$. Prove that Z_N is equal to the intersection of all normal subgroups K in G such that $Z(G/K)$ is the trivial group.

7. Explicitly find a simple (i.e. minimal) left ideal of the following ring of 2×2 matrices: $\mathrm{M}_2(\mathbb{Q}[x]/(x^2 - 1))$.

Algebra Prelim, Winter 2003

Instructions: do all problems.

1. Let K denote the splitting field over the rational numbers $\mathbb{Q}$ of the polynomial $f(x) = x^5 + x^4 + 3x + 3$.

   (a) What is $[K : \mathbb{Q}]$?

   (b) Determine the Galois group $\mathrm{Gal}(K/\mathbb{Q})$.

2. Prove or disprove: If x and y are elements of a finite abelian group G with the same order, then there is an automorphism θ of G such that $\theta(x) = y$.

3. Let H be a group of order 2002. Prove that the number of elements in the set $\{h \in H \mid h^2 = e\}$ is even (where e is the identity of H).

4. Let G be a group with the following property: for each $g \in G - 1$, there exists a normal subgroup K of G such that G/K is abelian and $g \notin K$ (where 1 is the trivial subgroup). Prove that G is abelian.

5. Prove that if A and B are commutative Noetherian rings, then so is the cartesian product $A \times B$.

6. Let R be a commutative ring, let P be a projective R-module and let I be an ideal in R. Prove P/IP is a projective R/I-module.

7. Let k be a field and let I be an ideal of $k[x]$ (where $k[x]$ is the polynomial ring over k in the variable x).

   (a) Show that $k + I$ is a subring of $k[x]$.

   (b) Prove that if $I \neq 0$ then $k[x] \otimes_{k+I} k[x]/I$ is finite dimensional over k.

Algebra Prelim, Fall 2003

Do all problems

1. Prove that a group of order $2256 = 47 * 48$ cannot be simple.

2. Let $G = \langle x, y \mid x^7 = y^3 = 1, yxy^{-1} = x^2 \rangle$.

   (i) Prove that every element of G can be written in the form $x^i y^j$ where i, j are non-negative integers.

   (ii) Prove that G has order at most 21.

   (iii) Prove that there is a homomorphism $\theta \colon G \to S_7$ such that

   $$\theta x = (1\ 2\ 3\ 4\ 5\ 6\ 7),\ \theta y = (2\ 3\ 5)(4\ 7\ 6).$$

   (iv) Prove that G has order 21.

3. Let R be a UFD. Suppose that for every coprime $p, q \in R$, the ideal $pR + qR$ is principal. Prove that for every $a, b \in R$, the ideal $aR + bR$ is principal. (Coprime means that the greatest common divisor of p, q is 1.)

4. Let R be the ring $\mathbb{Z}/4\mathbb{Z}$ and let M be the ideal $2\mathbb{Z}/4\mathbb{Z}$. Prove that $M \otimes_R M \cong M$ as R-modules.

5. Let R be a PID and let M, N be R-modules. Suppose M is finitely generated and $M \oplus M \cong N \oplus N$. Prove that $M \cong N$.

6. Let K be a field of characteristic zero, let $f \in K[x]$ be an irreducible polynomial, let L be a splitting field for f over K, and let $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in L$ be the roots of f. Suppose $[L : K] = 24$ (i.e. $\dim_K L = 24$).

   (i) If $1 \leq i, j \leq 4$ are integers, prove that $L \neq K(\alpha_i, \alpha_j)$.

   (ii) Prove that $L = K[\alpha_1 + 2\alpha_2 + 3\alpha_3]$.

7. Let k be an algebraically closed field, let n be a positive integer, and let U, V be affine algebraic sets in k^n (so U is the zero set of a collection of polynomials in $k[x_1, \ldots, x_n]$). Suppose $U \cap V = \emptyset$. Prove that $I(U) + I(V) = k[x_1, \ldots, x_n]$ (where $I(U)$ is the set of all polynomials in $k[x_1, \ldots, x_n]$ which vanish on U).

Algebra Prelim, Fall 2004

Do all problems

1. An automorphism of a group is an isomorphism of the group with itself. The set of automorphisms $\mathrm{Aut}(G)$ of a group G is itself a group under composition of functions. Find the group of automorphisms of the cyclic group C_{2p} of order $2p$ where p is an odd prime.

2. Let G be a simple group of order $4032 = 8!/10$. Prove that G is not isomorphic to a subgroup of the alternating group A_8. Deduce that G has at least 216 elements of order 7.

3. An ideal I in a commutative ring R with unit is called *primary* if $I \neq R$ and whenever $ab \in I$ and $a \notin I$, then $b^n \in I$ for some positive integer n. Prove that if R is a PID, then I is primary if and only if $I = P^n$ for some prime ideal P of R and some positive integer n.

4. Let k be a field and let $k[x^2, x^3]$ denote the subring of the polynomial ring $k[x]$ generated by k and $\{x^2, x^3\}$. Prove that every ideal of R can be generated by two elements. Hint: if the ideal is nonzero, we may choose one of the generators to be a polynomial of least degree.

5. Let k be a field, let $f \in k[x]$ be a polynomial of positive degree and let M be a finitely generated $k[x]$-module. Suppose every element of M can be written in the form fm where $m \in M$. Prove that M has finite dimension as a vector space over k.

6. Let R be an integral domain (commutative ring with $1 \neq 0$ and without nontrivial zero divisors) and suppose R when viewed as a left R-module is injective. Prove that R is a field.

7. Let K be a splitting field over the rational numbers $\mathbb{Q}$ of the polynomial $x^4 + 16$. Determine the Galois group of $K/\mathbb{Q}$.

Algebra Prelim, January 2005

Do all problems

1. If p and q are distinct primes and G is a finite group of order p^2q, prove that G has a nontrivial normal Sylow subgroup.

2. Find the Galois group of K over the rationals $\mathbb{Q}$ where K is the splitting field of the polynomial $x^4 + 4x^2 + 2$.

3. Show that $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module.

4. Let R be a commutative Noetherian ring with a 1 and let M be a finitely generated R-module. Show that if $f : M \to M$ is a surjective R-module homomorphism, then it must also be injective. (Hint: consider the kernels of f^n.)

5. Suppose R is a principal ideal domain that is not a field, and that M is a finitely generated R-module. Suppose further that for every irreducible element $p \in R$, the R/pR-module M/pM is cyclic (has a single generator). Show that M is cyclic.

6. Let G be a finite group with a composition series of length 2. Prove that if M and N are distinct nonidentity proper normal subgroups of G, then $G = M \times N$.

7. Let R be the ring $\mathbb{Q} + x^2\mathbb{Q}[x]$, the collection of all polynomials with rational coefficients that have no x term.

   (a) Show that if $0 \neq f \in R$, then R/fR is a finite dimensional vector space over $\mathbb{Q}$.

   (b) Use part (a) to prove that every nonzero prime ideal of R is maximal.

Algebra Prelim, Fall 2005

Do all problems

1. Show that there are exactly 5 nonisomorphic groups of order 18.

2. Let A be a commutative ring and set $B = A[X,Y]/(X^2 - Y^2)$. Prove that A is a Noetherian ring if and only if B is a Noetherian ring.

3. Let F be a field with more than 2 elements and let $\mathrm{GL}_2(F)$ denote the group of 2×2 invertible matrices with entries in F. Consider the action of $\mathrm{GL}_2(F)$ on one-dimensional subspaces of F^2. Show that the stabilizer of a one-dimensional subspace is never simple.

4. Let R be the ring $\mathbb{Z}[X]$ and set $M = 2R + XR$. Prove or disprove: M is a free R-module.

5. Let F be a field of characteristic zero. Suppose that K/F is finite Galois extension with Galois group G. Prove that if $a \in K$ and $\sigma(a) - a \in F$ for all $\sigma \in G$, then $a \in F$.

6. Let S be a simple algebra of finite dimension n over $\mathbb{C}$. Prove that there are $\sqrt{n}$ maximal left ideals of S whose intersection is zero.

7. Recall that if F is a field, then the tensor product of two F-algebras (over F) is another F-algebra. Let L be a finite field extension of F and let $\overline{F}$ be the algebraic closure of F. Show that if $\overline{F} \otimes_F L$ is a field, then $F = L$.

Algebra Prelim, May 2006

Do all problems

1. Prove that there are no simple groups of order 1755.

2. Let P be a finite p-group. Prove that every subgroup of P appears in some composition series for P.

3. Let R be a principal ideal domain. Let A be a finitely generated R-module and let B be an R-submodule of A. Assume that there exist nonzero elements r and s of R such that $\gcd(r,s) = 1$, $rB = 0$, and $s(A/B)$ is a torsion-free R-module. Prove that $A \cong B \oplus A/B$ as R-modules.

4. Let $F = \mathbb{Q}(i)$ and let K be the splitting field of $x^6 - 7$ over F.

   (a) Determine $[K : F]$ and write down a basis for K over F.

   (b) Show that $\mathrm{Gal}(K/F)$ is a dihedral group.

5. Let R be a ring with unity 1. Let P be a projective R-module and let M be an R-submodule of P. Prove: if P/M is a projective R-module, then M is a projective R-module.

6. Let R be a commutative Noetherian ring with unity 1. Let M be a nonzero R-module. Given $m \in M$, set $\operatorname{Ann} m = \{a \in R \mid am = 0\}$ and note that $\operatorname{Ann} m$ is an ideal of R. Prove that there exists $s \in M$ such that $\operatorname{Ann} s$ is a prime ideal in R. (Remember: R itself is not a prime ideal in R.)

7. Let $A = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$.

   (a) Prove that there is a well-defined multiplication on A that satisfies the distributive property such that

   $$(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$$

   for all complex numbers a_1, a_2, b_1, b_2.

   (b) Now assume that this multiplication makes A into a ring. Prove that A is not an integral domain.

Algebra Prelim, August 2007

Do all problems

1. How many elements of order 7 must there be in a simple group of order 168?

2. Let ρ be a primitive 4th root of 1 over $\mathbb{Q}$.

   (a) Compute the Galois group of $(x^4-2)(x^2-3)$ over $\mathbb{Q}$ and $\mathbb{Q}(\rho)$.

   (b) Is $\mathbb{Q}(\rho)$ Galois over $\mathbb{Q}$? (Explain your answer.)

   (c) Are there any proper subfields of the splitting field of $(x^4-2)(x^2-3)$ over $\mathbb{Q}(\rho)$ that are Galois over $\mathbb{Q}(\rho)$? (Explain your answer.)

3. Suppose that $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a split exact sequence of left R-modules, where R is a ring with a 1. If D is a right R-module, prove that $1 \otimes f : D \otimes_R A \to D \otimes_R B$ is a monomorphism.

4. Let R be a PID and let S be a multiplicatively closed subset of R. Assume that S is nonempty and that S does not contain 0. Prove that $S^{-1}R$ is a PID.

5. Prove that a group of order $2^4 \cdot 11^2$ is solvable.

6. (a) Prove that $f(x) = x^4 + 9x - 30$ is an irreducible polynomial over $\mathbb{Q}$.

   (b) Let $g(x) = x^2 + 2$ and let I be the ideal in $\mathbb{Q}[x]$ generated by the product $f(x)g(x)$. Show that $\mathbb{Q}[x]/I$ is the product of two fields. What is the dimension over $\mathbb{Q}$ of these fields?

7. Let R be an integral domain. If X is an R-module, then let $t(X)$ denote the subset $\{x \in X \mid rx = 0 \text{ for some nonzero } r \in R\}$.

   (a) Prove that $t(X)$ is a submodule of X.

   (b) Prove that $t(X/t(X)) = 0$.

   (c) Prove that if $X/t(X)$ is a nonzero cyclic R-module, then X is isomorphic to $t(X) \oplus R$.

Algebra Prelim, December 2007

Do all problems

1. Let p and q be distinct prime integers.

   (a) List all nonisomorphic abelian groups of order p^3q, listing only one for each isomorphism class.

   (b) Show that if G is an abelian group of order p^3q such that G cannot be generated by one element but G can be generated by two elements, then $G \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_{pq}$.

2. Let K be a finite field extension of k, let $\alpha \in K$, and let $f(x)$ be the irreducible polynomial of f over k. Prove that if $n \mid \deg f(x)$, then $n \mid [K:k]$.

3. Let R be a UFD with quotient field Q and let $f(x)$ be an irreducible polynomial of degree ≥ 1 in $R[x]$. Let I denote the ideal in $Q[x]$ generated by $f(x)$. Prove that $Q[x]/I$ is a field.

4. Prove that S_4 is solvable.

5. Let R be a ring with a 1 and let P and Q be projective R-modules. Prove that if $f\colon P \to Q$ is a surjective R-module homomorphism, then $\ker f$ is a projective R-module.

6. Let R be an integral domain with quotient field Q. Show that if V is a finite dimensional vector space over Q, then $(Q \otimes_R Q) \otimes_Q V \cong V$ as vector spaces over Q.

7. Let K be a Galois extension of a field F of order 11^4. Prove that there are intermediate fields $F = K_0 \subseteq K_1 \subseteq K_2 \subseteq K_3 \subseteq K_4 = K$ such that $[K_i : K_{i-1}] = 11$ and K_i is a Galois extension of F, for $i = 1, 2, 3, 4$.

8. Let R be a local commutative ring with 1 and with maximal ideal M. Suppose I is an ideal such that $0 \subsetneq I \subseteq M$. Prove that R/I is not a projective R-module.

Algebra Prelim, August 2009

Do all problems

1. Let p be a prime, let n be a non-negative integer, and let S be a set of order p^n. Suppose G is a finite group that acts transitively by permutations on S (so if $s,t \in S$, then there exists $g \in G$ such that $gs = t$) and let P be a Sylow p-subgroup of G. Prove that P acts transitively on S.

2. Prove that there is no simple group of order $448 = 7 * 64$.

3. (a) Prove that $x^2 + 1$ is irreducible in $\mathbb{Z}/3\mathbb{Z}[x]$.

   (b) Prove that $x^3 + 3x^2 - 9x + 12$ is irreducible in $\mathbb{Z}[i][x]$.

4. Let R be a PID, let M be a finitely generated right R-module, and let N be an R-submodule of M. Prove that there exists an R-submodule L of M and $0 \neq r \in R$ such that $L \cap N = 0$ and $Mr \subseteq L + N$.

5. Let R be a ring and suppose we are given a commutative diagram of R-modules and homomorphisms,

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ C & \xrightarrow{j} & D \end{array}$$

   where f is onto and j is one-to-one. Prove that there exists a unique R-module homomorphism $k\colon B \to C$ such that the resulting diagram commutes (so $kf = g$ and $jk = h$).

6. Compute the isomorphism type of the Galois group of $x^4 - 2x^2 + 9$ over $\mathbb{Q}$.

7. Let R be a commutative ring with a 1 and let I, J be ideals of R. Prove that $R/I \otimes_R R/J \cong R/(I+J)$ as R-modules.

Algebra Prelim, August 2010

Do all problems

1. Let G be a group of order 105 with a normal Sylow 3-subgroup. Show that G is abelian.

2. Find the Galois group of $f(x) = x^4 - 2x^2 - 2$ over $\mathbb{Q}$, describing its generators explicitly as permutations of the roots of f.

3. Let p be a prime. Prove that the extension $\mathbb{F}_{p^n} \supset \mathbb{F}_p$ has Galois group generated by the Frobenius automorphism $\sigma\colon \mathbb{F}_{p^n} \to \mathbb{F}_{p^n}$ given by $\sigma(a) = a^p$ for all $a \in \mathbb{F}_{p^n}$.

4. Show that there is no 3 by 3 matrix A with entries in $\mathbb{Q}$, such that $A^8 = I$ but $A^4 \neq I$.

5. Let R be an integral domain. A nonzero nonunit element $p \in R$ is *prime* if $p \mid ab$ implies $p \mid a$ or $p \mid b$. A nonzero nonunit element $p \in R$ is *irreducible* if $p = ab$ implies a or b is a unit. Show that

   (a) Every prime is irreducible.

   (b) If R is a UFD, then every irreducible is prime.

6. Let R be the ring $\mathbb{Z}/6\mathbb{Z}$ and let I be the ideal $3\mathbb{Z}/6\mathbb{Z}$. Prove that $I \otimes_R I \cong I$ as R-modules.

7. Let S be a multiplicatively closed nonempty subset of the commutative ring R with a 1. Assume that $0 \notin S$.

   (a) Show that if R is a PID, then $S^{-1}R$ is a PID.

   (b) Show that if R is a UFD, then $S^{-1}R$ is a UFD.

8. Let R be a commutative ring with a 1.

   (a) Show that if $x \in R$ is nilpotent and $y \in R$ is a unit in R, then $x + y$ is a unit in R.

   (b) Let $f = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \in R[x]$. Show that f is a unit in $R[x]$ if an only if a_0 is a unit in R and a_i is nilpotent for $i > 0$.

Algebra Prelim, August 2011

Do all problems

1. Prove that there is no simple group of order 380.

2. Let k be a field with $|k| = 7$, let n be a positive integer, and let f, g be coprime polynomials in $k[x_1, \dots, x_n]$. If $f^3 - g^3 = h^3$ for some nonzero polynomial $h \in k[x_1, \dots, x_n]$, prove that there exists $p \in k[x_1, \dots, x_n]$ and $u \in k$ such that $f - g = up^3$. Hint: factor $f^3 - g^3$ as a product of three polynomials, and note that these polynomials are pairwise coprime.

3. Let R be a PID, let p be a prime in R, and let M, N be finitely generated left R-modules such that $pM \cong pN$. Assume that if $0 \neq m \in M$ or N and $pm = 0$, then Rm is not a direct summand of M or N respectively (i.e. there is no submodule X such that $Rm \oplus X = M$ or N). Prove that $M \cong N$.

4. Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of degree 9, let K be a splitting field for f over $\mathbb{Q}$, and let $\alpha \in K$ be a root of f. Suppose that $[K : \mathbb{Q}] = 27$. Prove that $\mathbb{Q}(\alpha)$ contains a field of degree 3 over $\mathbb{Q}$.

5. Let p be a prime and let A denote all p^n-th roots of unity in $\mathbb{C}$. Thus A is the abelian subgroup of the nonzero complex numbers under multiplication defined by $\{e^{2\pi i m/p^n} \mid m, n \in \mathbb{N}\}$, in particular A is a $\mathbb{Z}$-module. Determine $A \otimes_{\mathbb{Z}} A$.

6. Let k be a field, let $A \in \mathrm{M}_3(k)$, the 3 by 3 matrices with entries in k, and suppose the characteristic polynomial of A is x^3. Prove that A has a square root, that is a matrix $B \in \mathrm{M}_3(k)$ such that $B^2 = A$, if and only if the minimal polynomial of A is x or x^2.

7. Let R be an integral domain, let n be a positive integer, let S be a subset of the polynomial ring in n variables $R[x_1, \dots, x_n]$, and define $Z(S) = \{(r_1, \dots, r_n) \in R^n \mid f(r_1, \dots, r_n) = 0 \text{ for all } f \in S\}$, the zero set of S. Prove that there exists a finite subset T of S such that $Z(S) = Z(T)$.

Algebra Prelim, January 2012

Do all problems

1. Recall that a proper subgroup of the group G is a subgroup H of G with $G \neq H$. Now suppose G is a finite cyclic group. Prove that G is not a union of proper subgroups.

2. Prove that a group of order $6435 = 9 \cdot 5 \cdot 11 \cdot 13$ cannot be simple.

3. Prove that $\mathrm{M}_2(\mathbb{Q}) \otimes_{\mathrm{M}_2(\mathbb{Z})} \mathrm{M}_2(\mathbb{Q}) \cong \mathrm{M}_2(\mathbb{Q})$ as $(\mathrm{M}_2(\mathbb{Q}), \mathrm{M}_2(\mathbb{Q}))$-bimodules ($\mathrm{M}_2(\mathbb{Q})$ indicates the ring of 2 by 2 matrices with entries in $\mathbb{Q}$).

4. Let R be a PID which is not a field and let M be a finitely generated injective R-module. Prove that $M = 0$.

5. Let p be an odd prime and for a positive integer n, let $\zeta_n = e^{2\pi i/n}$, a primitive nth root of 1.

   (a) Prove that $\mathbb{Q}(\zeta_p) = \mathbb{Q}(\zeta_{2p})$.

   (b) Prove that $1 + x^2 + x^4 + \cdots + x^{2p-2}$ is the product of two irreducible polynomials in $\mathbb{Q}[x]$.

6. Determine the isomorphism class of the Galois group of the polynomial $x^5 - 5x - 1$ over $\mathbb{Q}$.

7. For n a positive integer, let $\mathbb{A}^n$ denote affine n-space over $\mathbb{Q}$.

   (a) Prove that every element of $\mathbb{Q}[x,y]/(x^3 - y^2)$ can be written in the form $(x^3 - y^2) + f(x) + yg(x)$ where $f(x), g(x) \in \mathbb{Q}[x]$.

   (b) Prove that $\mathbb{Q}[x,y]/(x^3 - y^2) \cong \mathbb{Q}[t^2, t^3]$, the subring of the polynomial ring $\mathbb{Q}[t]$ generated by t^2, t^3.

   (c) Prove that $\mathbb{Q}[t^2, t^3]$ is not a UFD.

   (d) Let V denote the affine algebraic set $\mathscr{Z}(x^3 - y^2)$, the zero set of $x^3 - y^2$ in $\mathbb{A}^2$. Determine the coordinate ring of V.

   (e) Is V isomorphic to $\mathbb{A}^1$ as affine algebraic sets? Justify your answer.

Algebra Prelim, August 2013

Do all problems

1. Let p be a prime, let $H \lhd G$ be finite groups, and let P be a subgroup of G. Prove that P is a Sylow p-subgroup of G if and only if $P \cap H$ and PH/H are Sylow p-subgroups of H and G/H respectively.

2. Prove that there is no simple group of order $576 = 9 \cdot 64$.

3. Let R be a UFD with exactly two primes, p and q (i.e. p and q are nonassociate primes, and any prime is an associate of either p or q). Given positive integers m, n, prove that $(p^m, q^n) = R$ (consider $p^m + q^n$). Deduce that R is a PID.

4. Let k be a field, let M be a finitely generated $k[x]$-module, and let C be a cyclic $k[x]$-module. Suppose M has a proper submodule N (so $M \neq N$) such that $M \cong N$. Prove that there exists a $k[x]$-module epimorphism $M \twoheadrightarrow C$.

5. Let K and L be finite fields, let K^+ indicate the abelian group K under addition, and let $L^\times$ indicate the abelian group of nonzero elements of L under multiplication. Determine the order of $K^+ \otimes_{\mathbb{Z}} L^\times$ in terms of $|K|$ and $|L|$. (You will need to consider two cases, namely whether or not $\operatorname{ch} K$ divides $|L^\times|$.)

6. Let K and L be finite Galois extensions of $\mathbb{Q}$. Prove that $K \cap L$ is also a finite Galois extension of $\mathbb{Q}$.

7. Let G be a group and let $0 \to \mathbb{Z} \to P \to Q \to \mathbb{Z} \to 0$ be an exact sequence of $\mathbb{Z}G$-modules (here $\mathbb{Z}$ is the trivial G-module), where P and Q are projective $\mathbb{Z}G$-modules. Prove that $H^1(G, X) \cong H^3(G, X)$ for all $\mathbb{Z}G$-modules X.

Algebra Prelim, August 2014

Do all problems

1. Let p be a prime, let G be a finite group, let P be a Sylow p-subgroup of G, and let X denote all elements of G with order a power of p (including 1).

   (a) Show that P acts by conjugation on X (so for $g \in P$ and $x \in X$, we have $g \cdot x = gxg^{-1}$).

   (b) Show that $\{z\}$ is an orbit of size 1 if and only if z is in the center of P.

   (c) If p divides $|G|$, prove that p divides $|X|$.

2. Let p be a prime and let G be a finite p-group. Prove that if H is a maximal subgroup of G, then $H \lhd G$ and $|G/H| = p$. (Hint: use induction on $|G|$, so the result is true for proper quotients of G and consider HZ. Maximal means H has largest possible order with $H \neq G$.)

3. Let R be a noetherian UFD with the property whenever $x_1, \dots, x_n \in R$ such that no prime divides all x_i, then $x_1R + \cdots + x_nR = R$. Prove that R is a PID. (Hint: consider gcd.)

4. Let M be an injective $\mathbb{Z}$-module and let q be a positive integer. Prove that $M \otimes_{\mathbb{Z}} \mathbb{Z}/q\mathbb{Z} = 0$.

5. Let M be a finitely generated $\mathbb{C}[x]$-module. Suppose there exists a submodule N of M such that $N \cong M$ and $N \neq M$. Prove that there exists $c \in \mathbb{C}$ such that $(x-c)M \neq M$ and $(x-c)M \cong M$.

6. Let $f(x) = (x^5+x^3+1)(x^4+x+1) \in \mathbb{F}_2[x]$, and let K be a splitting field for f over $\mathbb{F}_2$. ($\mathbb{F}_2$ denotes the field with two elements.)

   (a) Show that x^2+x+1 is the only irreducible polynomial of degree 2 in $\mathbb{F}_2[x]$.

   (b) Let K be a splitting field for f over $\mathbb{F}_2$, let $\alpha \in K$ be a root of x^5+x^3+1, and let $\beta \in K$ be a root of x^4+x+1. Determine $[\mathbb{F}_2(\alpha, \beta) : \mathbb{F}_2]$

   (c) Determine $\mathrm{Gal}(K/\mathbb{F}_2)$. (Galois group)

7. Compute the character table of $S_3 \times \mathbb{Z}/3\mathbb{Z}$.

Algebra Prelim, August 2015

Do all problems

1. Let G be the group of upper triangular invertible 3×3 matrices over the field $\mathbb{F}_5$ of 5 elements.

   (a) Show that G has a unique Sylow 5-subgroup P_5.

   (b) Construct explicitly a composition series for P_5.

   (c) Show that G is isomorphic to the semidirect product $P_2 \ltimes P_5$.

2. Let S be the ring of $n \times n$-matrices with entries in a field F.

   (a) Show that the S-module $V = \mathbb{F}^n$ of column vectors is a simple left S-module (simple means it is nonzero and has no submodules other than 0 and itself).

   (b) Show that every left ideal of S is a projective left S-module.

3. Find the Galois group of the polynomial $f(x) = (x^{12} - 1)(x^2 - 2x + 2)$ over $\mathbb{Q}$.

4. Classify the conjugacy classes of 5×5 matrices of order 3

   (a) with coefficients in $\mathbb{Q}$.

   (b) with coefficients in $\mathbb{C}$.

5. Let M be a module over the integral domain R.

   (a) Prove directly that $M = 0$ if and only if $M_P = 0$ for all prime ideals P, where M_P is the localization of M at P.

   (b) Prove that an R-module homomorphism $f\colon M \to N$ is surjective if and only if, for all prime ideals P, the maps $f_P\colon M_P \to N_P$ are surjective, where by definition $f_P(m/d) = f(m)/d$ for all $m \in M$ and $d \in R \setminus P$.

6. Let G be a group of order $2^a p$, where $1 \leq a \leq 3$ and $p \geq 3$ is a prime. Prove that G cannot be simple.

7. Let $F = (F_1, \ldots, F_m)$ be a system of m polynomial equations, where each $F_i \in \mathbb{Z}[x_1, \ldots, x_r]$. Consider the following statements:

(a) The system F has solutions in $\mathbb{Z}$.

(b) The system F has solutions in $\mathbb{Z}/n\mathbb{Z}$ for any $n \geq 1$.

(c) The system F has solutions in $\mathbb{Z}/p^s\mathbb{Z}$ where p is any prime and $s \geq 1$.

(d) The system F has solutions in $\mathbb{Z}/p\mathbb{Z}$ for any prime p.

Prove that each statement implies the next. Prove that (c) implies (b) and give counterexamples to the other five backward implications (d) implies (c), (d) implies (b), etc.

Algebra Prelim, August 2016

Do all problems

1. Let G be a simple group of order $240 = 2^4 \cdot 3 \cdot 5$.

   (a) Prove that G has a subgroup of order 15, and that all groups of order 15 are cyclic.

   (b) Prove that G has exactly 32 elements of order 3. (Hint: show 15 divides $|\mathrm{N}_G(P)|$ where P is a Sylow 3-subgroup.)

2. Let R be a UFD with the property that any ideal that can be generated by two elements is principal.

   (a) If $I_1 \subseteq I_2 \subseteq \cdots$ is an ascending chain of principal ideals in R, prove that there exists $N \in \mathbb{N}$ such that $I_n = I_N$ for all $n > N$.

   (b) Prove that R is a PID.

3. Let R be a ring with a 1 and let S be a subring of R with the same 1. Prove or give a counterexample to the following statements.

   (a) If P is a projective left S-module, then $R \otimes_S P$ is a projective left R-module.

   (b) If P is an injective left S-module, then $R \otimes_S P$ is an injective left R-module.

4. Let R be a PID and let M and N be finitely generated R-modules. Suppose that $M^3 \cong N^2$. Prove that there exists an R-module P such that $P^2 \cong M$.

5. Let k be an algebraically closed field of characteristic 2 and let A be a square matrix over k such that A is similar to A^2.

   (a) Prove that A is similar to A^{2^n} for all positive integers n.

   (b) Prove that A is similar to a diagonal matrix over k.

6. Explicitly construct a subfield K of $\mathbb{C}$ such that $[K : \mathbb{Q}] = 3$ and K is Galois over $\mathbb{Q}$. For such a field K, prove that $K(\sqrt{2})$ is a Galois extension of $\mathbb{Q}$, and determine the Galois group $\mathrm{Gal}(K(\sqrt{2})/\mathbb{Q})$.

7. Let H be a central subgroup of the finite group G and let χ be a character of H. Assume that $H \neq G$. Prove that $\mathrm{Ind}_H^G(\chi)$ (the induced character) is not an irreducible character of G (all characters are assumed to be over $\mathbb{C}$).

Algebra Prelim, August 2018

Do all problems

1. Let G be a group order $495 = 9 \cdot 5 \cdot 11$. Prove that G has a nontrivial normal Sylow subgroup. Deduce that G has a normal subgroup of order 9.

2. Let $0 \neq I \lhd \mathbb{Z}[x]$ and let n be the lowest degree of a nonzero polynomial in I. Suppose I contains a monic polynomial of degree n. Prove that I is a principal ideal.

3. Let I be an injective $\mathbb{Z}$-module. Prove that $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} I = 0$.

4. Let R be a PID, let M be a finitely generated R-module, and let C be a cyclic submodule of M. Prove that there exists an R-epimorphism $M \twoheadrightarrow C$.

5. Let $K/\mathbb{Q}$ (where $K \subset \mathbb{C}$) be a finite Galois extension with Galois group A_5 and let $R = K \cap \mathbb{R}$. Suppose $R \neq K$.

   (a) Prove that K is the splitting field of an irreducible polynomial $f \in \mathbb{Q}[x]$ of degree 6 (you may assume that A_5 has a subgroup of order 10).

   (b) Prove that $[R : \mathbb{Q}] = 30$.

   (c) How many real roots does f have?

   (d) Prove that f has roots a, b such that $\mathbb{Q}(a, b) = R$.

6. Determine the possible rational canonical forms for a 3×3 matrix over $\mathbb{F}_2$ (the field with two elements) which has trace and determinant 1.

7. Compute the character table of A_4. (You may assume that A_4 has 4 conjugacy classes with representatives (1), (1 2 3), (1 3 2), (1 2)(3 4), and that A_4 has a normal subgroup of order 4.)

Algebra Prelim, January 2019

Do all problems

1. Prove that a group of order $992 = 31 \cdot 32$ has either a normal subgroup of order 32 or a normal subgroup of order 62.

2. Let R be a commutative ring with a $1 \neq 0$. Prove that the set of prime ideals of R has a minimal element with respect to inclusion.

3. Let R be a commutative ring with a $1 \neq 0$ and suppose every irreducible R-module is free. Prove that R is a field. Hint: show that if $R \cong R \oplus M$, then $M = 0$.

4. Let k be a field and let M be a finitely generated $k[x]$-module. Prove that there exists a submodule N of M with $N \neq M$ and $N \cong M$ if and only if $\dim_k M = \infty$.

5. Let p be a prime, let $f \in \mathbb{Q}[x]$ be an irreducible polynomial, and let α, β be distinct roots of f. Suppose that $\mathbb{Q}(\alpha) = \mathbb{Q}(\beta)$ and that $[\mathbb{Q}(\alpha) : \mathbb{Q}] = p$. Prove that $\mathbb{Q}(\alpha)$ is a Galois extension of $\mathbb{Q}$.

6. Let p be a prime, let $n \in \mathbb{N}$, let $k = \mathbb{F}_p$, and let $K = \mathbb{F}_{p^n}$. Define $\theta \colon K \to K$ by $\theta(a) = a^p$ ($\mathbb{F}_{p^n}$ denotes the field with p^n elements).

   (a) Prove that θ is a k-linear map.

   (b) Prove that the minimal polynomial of θ (as a k-linear map) divides $X^n - 1$.

   (c) Prove that if n divides $p - 1$, then θ is diagonalizable over k (i.e. there is a k-basis of K such that the matrix of θ with respect to this basis is a diagonal matrix).

7. Compute the character table of $S_3 \times \mathbb{Z}/2\mathbb{Z}$.

Algebra Prelim, August 2019

Do all problems

1. Prove that there is no simple group of order $4860 = 4 \cdot 3^5 \cdot 5$.

2. Prove that $2x^3 + 19x^2 - 54x + 3$ is irreducible in $\mathbb{Q}[x]$.

3. Let R be a subring of the (not necessarily commutative) nonzero ring S. Assume that R and S have the same 1. Prove that $S \otimes_R S \neq 0$.

4. Let R be a PID which is not a field, and let I be a finitely generated injective R-module. Prove that $I = 0$.

5. Determine the number of conjugacy classes of matrices in $\mathrm{GL}_8(\mathbb{Q})$ that consist of elements of order 7.

6. Let $f \in \mathbb{Q}[x]$ be a polynomial over $\mathbb{Q}$, let K be a subfield of $\mathbb{C}$ which is a splitting field for f over $\mathbb{Q}$, let p be an odd prime, and let $\gamma \colon \mathbb{C} \to \mathbb{C}$ denote complex conjugation. Assume that $\deg f = 5$, $\mathrm{Gal}(K/\mathbb{Q}) \cong S_5$, and that $e^{2\pi i/p} \in K$.

   (a) Prove that f is irreducible and has exactly 5 distinct roots.

   (b) Prove that γ permutes the roots of f and that $p = 3$.

   (c) How many roots of f does γ fix? Prove that f has exactly two complex roots and three real roots.

7. Let H be a normal subgroup of index 3 in the finite group G, let $x \in G \setminus H$, and let χ be an irreducible complex character of H. Suppose that $\chi(xhx^{-1}) = \chi(h)$ for all $h \in H$. Prove that $\mathrm{Ind}_H^G \chi$ (the induced character) is the sum of 3 distinct irreducible characters of G.

Groups of order 36

Here we construct a group of order 36 which has a *nonnormal* subgroup of order 12. Let S_3 denote the symmetric group of degree 3. Then $G := S_3 \times S_3$ is a group of order 36 which has a normal subgroup K such that $G/K \cong S_3$ (e.g., we could let $K = \{(x,1) \mid x \in S_3\}$). Now S_3, and hence also G/K, have a nonnormal subgroup of order 2. Using the subgroup correspondence theorem, we deduce that G has a nonnormal subgroup of order $2 \cdot |K| = 12$ (which contains K), as required.

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Algebra Prelim, August 2020

Do all problems

1. Classify groups of order 63 up to isomorphism.

2. Prove that any invertible matrix A with complex entries has a square root (i.e., there exists a matrix B such that $A = B^2$).

   (*Hint*: Use the Jordan canonical form.)

3. Let R be a PID. Determine, up to isomorphism, all finitely generated R-modules A and B whose tensor product $A \otimes_R B$ is a free R-module (possibly 0).

4. Suppose A is a finite abelian group and p is a prime dividing its order. Let p^n be the largest power of p dividing $|A|$. Prove that $\mathrm{Hom}(\mathbb{Z}/p^n\mathbb{Z}, A)$ is isomorphic to the (unique) Sylow p-subgroup of A.

   (*Note*: $\mathrm{Hom}(\mathbb{Z}/p^n\mathbb{Z}, A)$ is the group of all homomorphisms $f\colon \mathbb{Z}/p^n\mathbb{Z} \to A$.)

5. Let E be the splitting field of $x^3 - 4$ over $\mathbb{Q}$. Find, with proof:

   (a) the Galois group and all intermediate fields of $E/\mathbb{Q}$;

   (b) $\theta \in E$ such that $E = \mathbb{Q}(\theta)$.

6. Let R be a commutative ring with $1 \neq 0$. Suppose $a \in R$ is an element such that $a^n \neq 0$ for all $n > 0$.

   (a) Let $\mathscr{S}$ be the set of all ideals I of R such that $a^n \notin I$ for all $n \geq 0$. Show that $\mathscr{S}$ satisfies the conditions of Zorn's lemma with respect to inclusion.

   (b) Use (a) to prove that there exists a prime ideal P of R such that $a \notin P$.

7. Let G be a finite group with subgroups $H, K < G$. Consider the left action of the direct product $H \times K$ on G given by $(h,k) \cdot x = hxk^{-1}$.

   (a) Show that the $(H \times K)$-orbit of $x \in G$ has $|H \times K|/|H \cap xKx^{-1}|$ elements.

   (b) Consider the special case of the above where $G = GL_2(\mathbb{F}_q)$, the group of invertible 2×2 matrices over a finite field $\mathbb{F}_q$ (with $|\mathbb{F}_q| = q$), and

   $$H = K = \left\{ \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} : a, b, c \in \mathbb{F}_q, a, b \neq 0 \right\} < G.$$

   Show that the $(H \times H)$-action on G has two orbits, and compute the size of each orbit. (You may use without proof that $|G| = (q^2 - 1)(q^2 - q)$.)