PCMI 2018 - Oscillations in Harmonic Analysis
Problem Set #8 on 7/13/2018

P1) Consider

$$z^2 - x^2 + 2xyz - y^2 = 0.$$

Can z be solved as a function of x and y near the point $(1, 0, 1)$?

[Hint: Use the Implicit Function Theorem]

P2) **Inverse Function Theorem** Let $A \subset \mathbb{R}^n$ be an open set and let $f : A \to \mathbb{R}^n$ be of class C^1. Let $x_0 \in A$ and suppose $|Df(x_0)| \neq 0$. Then there is a neighborhood U of x_0 in A and an open neighborhood W of $f(x_0)$ such that $f(U) = W$ and f has a C^1 inverse $f^{-1} : W \to U$. Moreover for $y \in W$, $x = f^{-1}(y)$ we have

$$Df^{-1}(y) = [Df(x)]^{-1}.$$

If f is of class C^p, $p \geq 1$, then so is f^{-1}.

The goal of this problem is for you to explore the proof of this fact in small steps.

(a) Recall the proof of the theorem in in $\mathbb{R}$ which you may have seen in a calculus or real analysis class. Show in particular

$$(f^{-1})'(f(a)) = \frac{1}{f'(a)}.$$

If you can't recall the proof then look it up online.

(b) In the proof above did you get an inverse on an entire open neighborhood (as opposed to just getting the formula)? Make sure you understand that.

(c) Can you show that if the original function was of class C^p then the inverse you found is also of the same class?

(d) Can you generalize your proof to $\mathbb{R}^n$?