PCMI 2018 - Oscillations in Harmonic Analysis
Problem Set #6 on 7/10/2018

P1) Suppose f is a Schwartz function supported on an interval $[-M, M]$.

(a) Fix L with $L/2 > M$ and show that $f(x) = \sum a_n(L) e^{2\pi inx/L}$ where

$$a_n(L) = \frac{1}{L} \int_{-L/2}^{L/2} f(x) e^{-2\pi inx/L} dx = \frac{1}{L} \widehat{f}(n/L)$$

Alternatively we may write $f(x) = \delta \sum_{n=-\infty}^{\infty} \widehat{f}(n\delta) e^{2\pi in\delta x}$ with $\delta = 1/L$.
[Hint: Note that the Fourier series of f on $[-L/2, L/2]$ converges absolutely.]

(b) Prove that if F is a Schwartz function then

$$\int_{\infty}^{\infty} F(\xi) d\xi = \lim_{\substack{\delta \to 0 \\ \delta > 0}} \delta \sum_{n=-\infty}^{\infty} F(\delta n)$$

[Hint: Approximate the first integral by $\int_{-N}^{N} F$ and the sum by $\delta \sum_{|n| \leq N/\delta} F(n\delta)$. Then approximate the second integral by Riemann sums.]

(c) Conclude that $f(x) = \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{2\pi ix\xi} d\xi$.

P2) Show the following properties of the Fourier transform for a Schwartz function f.

(i) The Fourier transform of $f(x+h)$ is $e^{2\pi ih\xi} \widehat{f}(\xi)$.

(ii) The Fourier transform of $f(x) e^{-2\pi ixh}$ is $\widehat{f}(\xi + h)$.

(iii) The Fourier transform of $f(\delta x)$ is $\delta^{-1} \widehat{f}(\delta^{-1} \xi)$ when $\delta > 0$.

(iv) The Fourier transform of $f'(x)$ is $2\pi i\xi \widehat{f}(\xi)$.

(v) The Fourier transform of $-2\pi ix f(x)$ is $\frac{d}{d\xi} \widehat{f}(\xi)$.

[Hint: Show that $|\frac{\widehat{f}(\xi+h) - \widehat{f}(\xi)}{h} - \widehat{-2\pi ix} f(\xi)|$ is small. When writing out those integrals, say using the variable x, then use that $f(x)$ decays rapidly if x is large, while if x is bounded then you can make $|\frac{e^{-2\pi ixh} - 1}{h} + 2\pi ix|$ small if h is small enough.]

P3) Consider the sequence of functions

$$S_n(x) = \frac{nx}{1 + n^2 x^2}$$

(a) Show that as $n \to \infty$ we have $S_n(x) \to 0$ for every x.

(b) Show that for every $n > 0$ there exist numbers (possibly depending on n) x_+ and x_- such that $S_n(x_+) = \frac{1}{2}$ and $S_n(x_-) = -\frac{1}{2}$. This is an example of the so-called Gibbs phenomenon.

P4) Let $B_n(t)$ be defined in $-\pi < x < \pi$ to be equal to πn in the interval $-\frac{1}{n} < x < \frac{1}{n}$ and zero elsewhere in the interval, and to be periodic with period 2π.

(a) If f is continuously differentiable and of period 2π show

$$\lim_{n\to\infty} (f * B_n)(x) = f(x)$$

(b) Does the Gibbs phenomenon occur here?

P5) Let D_N denote the Dirichlet kernel

$$D_N(x) = \sum_{k=-N}^{N} e^{ikx} = \frac{\sin((N+1/2)x)}{\sin(x/2)}$$

and define

$$L_N = \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(x)| dx$$

(a) Prove that

$$L_N \geq c\log(N)$$

for some constant $c > 0$.
[Hint: Show that $|D_N(x)| \geq c\frac{\sin((N+1/2)x)}{|x|}$, change variables, and prove that

$$L_N \geq c\int_{\pi}^{N\pi} \frac{|sin(x)|}{|x|} dx + O(1)$$

Write the integral as a sum $\sum_{k=1}^{N-1}\int_{k\pi}^{(k+1)\pi}$. To conclude, use the fact that $\sum_{k=1}^{n}\frac{1}{k} \geq c\log(n)$.]

(b) Prove the following as a consequence: for each $n \geq 1$, there exists a continuous function f_n such that $|f_n| \leq 1$ and $|S_n(f_n)(0)| \geq c^{'}\log(n)$.
[Hint: The function g_n which is equal to 1 when D_n is positive and -1 when D_n is negative has the desired property but is not continuous. Approximate g_n in the integral norm by continuous functions h_k satisfying $|h_k| \leq 1$ which you may take as a given that this is possible.]