PCMI 2018 - Oscillations in Harmonic Analysis
Problem Set #4 on 7/6/2018

P1) Let f be the function defined on $[-\pi, \pi]$ by $f(x) = |x|$.

(a) Draw the graph of f.

(b) Calculate the Fourier coefficients of f, and show that

$$\widehat{f}(n) = \begin{cases} \frac{\pi}{2} & \text{if } n = 0 \\ \frac{-1+(-1)^n}{\pi n^2} & \text{if } n \neq 0 \end{cases}$$

(c) What is the Fourier series of f in terms of sines and cosines?

(d) Taking $x = 0$, prove that $\sum_{n \text{ odd } \geq 1} \frac{1}{n^2} = \frac{\pi^2}{8}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

P2) Show that if the function f is integrable and bounded on $[-\pi, \pi]$ with $\widehat{f}(n) = 0$ for all $n \in \mathbb{Z}$ then $f(x_0) = 0$ whenever f is continuous at x_0.

[Hint: Start by assuming that f is real valued. For contradiction assume $f(x_0) \neq 0$. Without loss of generality you may then assume $x_0 = 0$ and $f(0) > 0$. By continuity at 0 you can choose $0 < \delta \leq \frac{\pi}{2}$ such that $f(x) > \frac{f(0)}{2}$ when $|x| < \delta$. Next introduce a new function $p(x) = \epsilon + \cos(x)$ where $\epsilon > 0$ is chosen so small that $|p(x)| < 1 - \frac{\epsilon}{2}$ whenever $\delta \leq |x| \leq \pi$. Then choose $0 < \eta < \delta$ such that $p(x) \geq 1 + \frac{\epsilon}{2}$ for $|x| < \eta$. Define $p_k(x) = (p(x))^k$. Sketch a little cartoon of what p_k looks like as k gets large. Why does $\widehat{f}(n) = 0$ imply that $\int_{-\pi}^{\pi} f(x)\cos(nx)dx = 0$ for all $n \in \mathbb{Z}$ and similarly with $\sin(nx)$? How does that imply $\int_{-\pi}^{\pi} f(x)p_k(x)dx = 0$ for all k? In order to reach a contradiction break the integral $\int_{-\pi}^{\pi} f(x)p_k(x)dx = 0$ up into the three parts $|x| < \eta$, $\eta \leq |x| < \delta$ and $\delta \leq |x| \leq \pi$. Show that the integral on the first region tends to infinity, is non-negative on the second region and tends to 0 on the third region. To complete the proof note that if f is complex valued you can write it as $f(x) = u(x) + iv(x)$ where u, v real valued and note $u(x) = \frac{f(x)+\overline{f}(x)}{2}$ and something similar for v. Show that $\widehat{f}(n) = 0$ implies $\widehat{u}(n) = 0$ and $\widehat{v}(n) = 0$ and use the previous result on the real and imaginary parts.]

P3) Given 2π periodic, integrable functions f and g on $\mathbb{R}$ show the following properties for their convolution.

(a) $f * g = g * f$

(b) $\widehat{f * g}(n) = \widehat{f}(n)\widehat{g}(n)$