PCMI 2018 - Oscillations in Harmonic Analysis
Problem Set #3 on 7/5/2018

P1) Consider the vector space $\ell^p(\mathbb{Z})$ of integer valued sequences $(a_k)_{k=1}^{\infty}$ with $\sum_{k=1}^{\infty} |a_k|^p < \infty$.

(a) Verify that the function defined for a sequence $\mathbf{a} = (a_k)_{k=1}^{\infty}$ by

$$\|\mathbf{a}\|_p = \left(\sum_{k=1}^{\infty} |a_k|^p \right)^{1/p}$$

is a norm on $\ell^p(\mathbb{Z})$ when $p \geq 1$. [Hint: Look up Minkowski's inequality for sums.]

(b) Show that the norm $\|\cdot\|_p$ from (a) comes from an inner product if and only if $p = 2$. [Hint: If it comes from an inner product then the parallelogram equality holds true by problem P3 on Problem Set #2. Plug some simple sequences into it and see what happens.]

P2) Let H_n denote the linear class of functions spanned by

$$1, \cos(x), \cos(2x), \ldots, \cos(nx), \sin(x), \sin(2x), \ldots, \sin(nx).$$

a) What is the dimension of H_n?

b) Is H_n a subspace of H_{n+1}?

c) For what values of n is $\sin^2(x)$ an element of H_n?

P3) In this exercise we show how the symmetries of a function imply certain properties of its Fourier coefficients. Let f be a 2π-periodic integrable function defined on $\mathbb{R}$.

(a) Show that the Fourier series of the function f can be written as

$$\widehat{f}(0) + \sum_{n \geq 1} [\widehat{f}(n) + \widehat{f}(-n)] \cos(nx) + i[\widehat{f}(n) - \widehat{f}(-n)] \sin(nx)$$

(b) Prove that if f is even, then $\widehat{f}(n) = \widehat{f}(-n)$, and we get a cosine series.

(c) Prove that if f is odd, then $\widehat{f}(n) = -\widehat{f}(-n)$, and we get a sine series.

(d) Suppose that $f(x + \pi) = f(x)$ for all $x \in \mathbb{R}$. Show that $\widehat{f}(n) = 0$ for all odd n.

(e) Show that a continuous function f is real-valued if and only if $\overline{\widehat{f}(n)} = \widehat{f}(-n)$ for all n.