PCMI 2018 - Oscillations in Harmonic Analysis
Problem Set #2 on 7/3/2018

P1) Show that given N points in the plane there exists a point that determines at least $\sim \sqrt{N}$ distinct distances as $N \to \infty$.

[Hint: Out of the N points pick two, call them p and q. Draw s number of circles around p and t number of circles around q. Note that the number of distinct circles around p corresponds to the distinct distances from p. Now note that all the other points are on the intersection of circles. How many intersections could you possibly have?]

P2) Show from the definition of an inner product that

$$\langle \alpha_1 f_1 + \beta_1 g_1, \alpha_2 f_2 + \beta_2 g_2 \rangle = \alpha_1 \overline{\alpha_2} \langle f_1, f_2 \rangle + \alpha_1 \overline{\beta_2} \langle f_1, g_2 \rangle + \beta_1 \overline{\alpha_2} \langle g_1, f_2 \rangle + \beta_1 \overline{\beta_2} \langle g_1, g_2 \rangle$$

P3) Prove the parallelogram equality

$$\|f + g\|^2 + \|f - g\|^2 = 2\|f\|^2 + 2\|g\|^2$$

P4)
a) Complete the proof of the Cauchy-Schwarz inequality. [Hint: One way of doing this is to expand $\|f + \alpha g\|^2$ for some scalar α and then choose it as $\alpha = \beta \langle f, g \rangle$ with β real and realize that you have a quadratic in β. What can you say about the discrimnant of that quadratic?]
b) Prove in detail that if equality holds in the Cauchy-Schwarz inequality one of the vectors is a scalar multiple of the other.

P5) Let f and g be elements of $C[0, 1]$ defined by $f(x) = 1$ and $g(x) = x$. Find the projection of f in the direction of g.

P6) Using the Gram-Schmidt process, find an orthonormal basis for the four-dimensional subspace of $C[-1, 1]$ spanned by $1, x, x^2$.

P7) Let W_n be a subspace of a vector space V with an orthonormal basis $\{\phi_1, \ldots, \phi_n\}$. Show that $\text{proj}_{W_n}(f) = f$ if and only if $f \in W_n$.