PCMI 2018 - Oscillations in Harmonic Analysis
Problem Set #11 on 7/19/2018

P1) Consider $\phi(x) = 2e_n \cdot x$ for $x \in S^{n-1}$. Verify that $\nabla\phi = 0$ only at $\{\pm e_n\}$.

[Hint: One way to work this out is to see that only there the gradient is normal to the sphere. Why does that do the trick? Alternatively you could look at this in a special case such as in $\mathbb{R}^3$, write it out in spherical coordinates and see what happens.]

P2) Consider $\phi(x_1, \ldots, x_{n-1}) = 2\sqrt{1 - x_1^2 - \ldots - x_{n-1}^2}$. Show that the Hessian of this function at $(0, \ldots, 0)$ is $-2I$.

P3) In this problem you will construct a smooth dyadic decomposition of the real line.

(a) Suppose $a < b$ and f is the function such that $f(x) = 0$ if $x \le a$ or $x \ge b$ and

$$f(x) = e^{-1/(x-a)}e^{-1/(b-x)} \quad \text{if } a < x < b.$$

Show that f is indefinitely differentiable on $\mathbb{R}$.

(b) Prove that there exists an indefinitely differentiable function F on $\mathbb{R}$ such that $F(x) = 0$ if $x \le a$, $F(x) = 1$ if $x \ge b$ and F is strictly increasing on $[a, b]$.
[Hint: Consider $F(x) = c\int_{-\infty}^{x} f(t)dt$ where c is a constant you get to pick.]

(c) Let $\delta > 0$ be so small that $a+\delta < b-\delta$. Show that there exists an indefinitely differentiable function g such that g is 0 if $x \le a$ or $x \ge b$, g is 1 on $[a+\delta, b-\delta]$ and g is strictly monotonic on $[a, a+\delta]$ and $[b-\delta, b]$.

(d) By part (c) we can construct an indefinitely differentiable function ϕ such that ϕ is 0 if $x \le -2$ or $x \ge 2$, ϕ is 1 on $[-1, 1]$ and ϕ is strictly monotonic on $[-2, -1]$ and $[1, 2]$. [In some contexts this function ϕ might be called a mother wavelet.]

(i) Sketch what ϕ looks like.

(ii) Consider the function $\psi(x) = \frac{\phi(\frac{x}{2}) - \phi(2x)}{2}$. Sketch it.

(iii) Define $\psi_j(x) = \psi(2^{-j}x)$. Sketch it.

(iv) For any $2^k < y < 2^{k+1}$ (and similarly for negative y) show that

$$\sum_{j=-\infty}^{\infty} \psi_j(y) = 1$$

This shows that $\{\phi_j\}_{j=-\infty}^{\infty}$ form a smooth partition of unity on the real line (except for perhaps not catching the origin).

P4) Revisit the fact that $|\widehat{\sigma}(\xi)| \le C|\xi|^{-(\frac{n-1}{2})}$. Dig into the proof and explicitly write out what the leading term should look like according to the stationary phase theorem. Namely, at the north pole we have the critical point $x = 0 \in \mathbb{R}^{n-1}$ in local coordinates and a phase function

$2\sqrt{1-|x|^2}$ with $\phi(0) = 2$, $\Delta = 1$ and signature $-(n-1)$ and an amplitude $a_1(x)(1-|x|^2)^{-1/2}$ which is 1 at the critical point and similar for the south pole. Verify all of this and show

$$\widehat{\sigma}(\lambda e_n) = 2\lambda^{-\frac{n-1}{2}} \cos\left(2\pi(\lambda - \frac{n-1}{8})\right) + O(\lambda^{-\frac{n+1}{2}})$$

P5) Show that the Hausdorff dimension of the middle third Cantor set is $\frac{\ln(2)}{\ln(3)}$.

[Hint: When constructing the middle third Cantor set you start at step 0 with the interval $[0,1]$. Then at step 1 you cut out the middle third and note that you are left with 2 intervals of length $\frac{1}{3}$ each. At the second step you cut out the middle thirds of the two intervals that are left and so forth. The set that is left in the limit is called the Cantor middle third set. Note that at step n you have 2^n intervals of length $\frac{1}{3^n}$. Use this to cover the set and then explore in the limit as $n \to \infty$ what happens.]