PCMI 2018 - Oscillations in Harmonic Analysis
Problem Set #10 on 7/17/2018

P1) Consider

$$I(\lambda) = \int_{\mathbb{R}^3} e^{-\pi i\lambda(x^2+y^2+z^2-1)} a(x, y, z)dxdydz$$

where $a \in C_c^\infty(\mathbb{R}^3)$ supported in a small enough neighborhood of 0. What can you say about the decay of $I(\lambda)$ in terms of λ where you think of λ as being large? [Only use the first power of λ to describe the decay if there is any, not a whole series.]

P2) **Lemma:** Suppose ϕ is a smooth, $\nabla\phi(p) = 0$ and G is a smooth diffeomorphism with $G(0) = p$. Then

$$H_{\phi\circ G}(0) = DG(0)^T H_\phi(p)DG(0).$$

Thus $H_\phi(p)$ and $H_{\phi\circ G}(0)$ have the same signature and

$$\det(H_{\phi\circ G}(0)) = \det(DG(0))^2 \det(H_\phi(p)).$$

(a) Prove the Lemma.

[Hint: Use the chain rule.]

(b) This Lemma is applied inside the proof of our stationary phase theorem. In particular we use it to conclude that

$$|\det(DG(0))| = \Delta^{-1/2}$$

where $\Delta = 2^{-n}|\det(H_\phi(p))|$ and G is the diffeomorphism used in the proof coming from the Morse Lemma. Verify that this is true.