Combinatorics of the Cell Decomposition of Affine Springer Fibers

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Contents

Overview

Notation

- $K := \mathbb{C}((\epsilon))$ and $\mathcal{O} = \mathbb{C}[[\epsilon]]$
- Fix $n \geq 1$
- $V := K^n$ be a K-vector space
- $G := GL_n(K)$ (alternatively $SL_n(K)$)
- $P := GL_n(\mathcal{O})$ (alternatively $SL_n(\mathcal{O})$)
- $B := \{g \in P \mid g$ evalulated at $\epsilon = 0$ is uppertriangular$\}$ (Iwahori subgroup)

Interpretation

- Consider elements of V as infinite $\mathbb{C}$ tuples

$$\sum_{i \geq N_1} a_i \epsilon^i \qquad \sum_{i \geq N_2} b_i \epsilon^i \qquad \sum_{i \geq N_3} c_i \epsilon^i$$

$$(\ldots, a_{-1}, b_{-1}, c_{-1}, a_0, b_0, c_0, a_1, b_1, c_1, \ldots)$$

- ϵ acts by shifting n spots
- If $\{e_i\}_{i=1}^n$ is a $\mathbb{C}$ basis of $\mathbb{C}^n$, then

$$e_{i+kn} = \epsilon^k e_i$$

is a $\mathbb{C}$ basis of V

Affine Grassmannian

Definition ([1, 4])

The *affine Grassmannian* $\mathcal{G}_n$ for the group GL_n is the moduli space of $\mathcal{O}$-submodules M of V such that

(a) M is $\mathcal{O}$-invariant

(b) M has rank n as a $\mathcal{O}$-module

(c) There exists N such that $\epsilon^{-N}\mathcal{O}^n \supset M \supset \epsilon^N\mathcal{O}^n$

(d) $\dim_{\mathbb{C}} \epsilon^{-N}\mathcal{O}^n/M = \dim_{\mathbb{C}} M/\epsilon^N\mathcal{O}^n$

Note $\mathcal{G}_n \cong G/P$.

Affine Flag

Definition

The *affine flag ind-variety* $\mathcal{F}_n$ is the space of collections

$$M_0 \supset M_1 \supset \cdots \supset M_n = \epsilon M_0$$

such that

1. For all i, M_i satisfies $(a), (b)$, and (c) for $\mathcal{G}_n$
2. $M_0 \in \mathcal{G}_n$
3. $\dim_{\mathbb{C}} M_i / M_{i+1} = 1$

Note $\mathcal{F}_n \cong G/B$

Affine Springer Fiber

- Consider the single shift of the $\mathbb{C}$ basis of V

$$N = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \epsilon & 0 & 0 & \dots & 0 \end{bmatrix}$$

- Let $m > n$, $\gcd(m, n) = 1$ and $m = nk + b$
- $T = N^m$ and $I_n + T \in G$

Definition

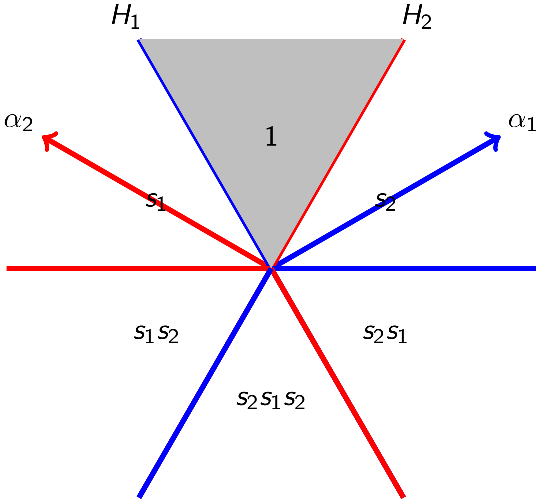
Let T be a semiregular nil-elliptic endomorphism. The *Affine Springer Fiber* is the set of fixed points of T over $\mathcal{F}_n$. Equivalently, if $u = I_n + T$, then the Affine Springer Fiber is $\mathcal{F}^u = \mathcal{F}_{m/n}$.

Overview

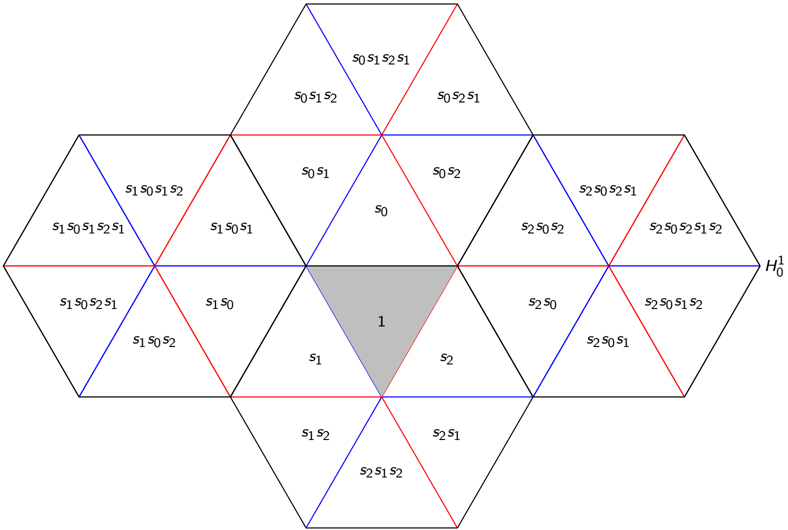
Affine Symmetric Group

- Let $E \subset \mathbb{R}^n$ be perpendicular to $(1, 1, \ldots, 1)$
- Let α_i be the ith simple coroot $(0, 0, \ldots, 1, -1 \ldots, 0)$ with 1 in the ith position, $1 \leq i < n$
- Let $H_i \subset E$ be the hyperplane perpendicular to α_i
- $\{H_i\}$ acts on E by reflection
  - Define s_i to be the reflection across H_i
- The complement of $\{H_i\}$, closed under reflections, form Weyl chambers
- The fundamental chamber is on the positive side of all H_i
  - Assigning $1 \in S_n$ to the fundamental chamber creates an isomorphism to S_n

Example $n = 3$



Affine Symmetric Group ($n = 3$)



Affine Symmetric Group

Definition (Affine Symmetric Group)

The affine symmetric group $\widetilde{S}_n$ is the group of words in $\{s_0, \ldots, s_{n-1}\}$ with the following relations:

- $s_i^2 = 1$
- $s_i s_j s_i = s_j s_i s_j$ if $i - j \not\equiv \pm 1 \pmod{n}$
- $s_i s_j = s_j s_i$ if $i - j \equiv \pm 1 \pmod{n}$ and $n > 2$

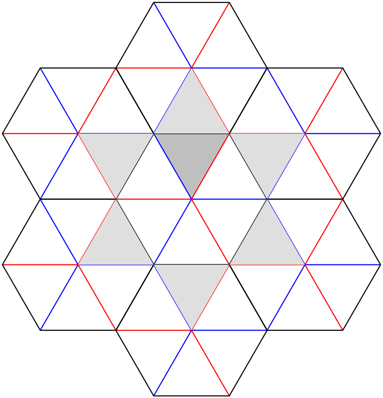
Affine Grassmannians

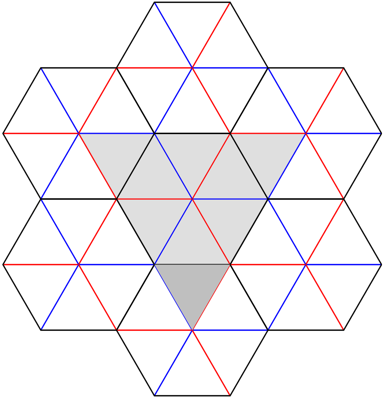
- Consider the cosets of $\widetilde{S}_n/S_n$

Definition

$\omega \in \widetilde{S}_n$ is *affine grassmannian* if it is the minimum length representative of ωS_n in $\widetilde{S}_n/S_n$.

- $Q^\vee := \{(x_1, \ldots, x_n) \in \mathbb{Z}^n \mid \sum_{i=1}^n x_i = 0\}$
- $\widetilde{S}_n \cong Q^\vee \rtimes S_n$
- If $\omega \in \widetilde{S}_n$ is affine grassmannian, then it is the identity or all its reduced words end with s_0





The Permutation

- Consider $\omega \in \widetilde{S}_n$ as a permutation $\mathbb{Z} \to \mathbb{Z}$
- $1 \mapsto id_{\mathbb{Z}}$
- We need only define the s_i action

$$(\omega s_i)(x) = \begin{cases} \omega(x+1) & \text{if } x \equiv i \pmod{n} \\ \omega(x-1) & \text{if } x \equiv i+1 \pmod{n} \\ \omega(x) & \text{otherwise} \end{cases}$$

- $\displaystyle\sum_{i=1}^{n} \omega(i) = \frac{n(n+1)}{2}$
- $\omega(x+n) = \omega(x) + n$
- Denoted by window notation $[\omega(1), \omega(2), \ldots, \omega(n)]$

Indexing Sets

- Recall $\mathcal{F}_n = G/B$ and $\mathcal{G}_n = G/P$

$$\mathcal{F}_n = \bigsqcup_{\omega \in \widetilde{S}_n} BwB/B$$

$$\mathcal{G}_n = \bigsqcup_{\omega \in \widetilde{S}_n / S_n} BwP/P$$

Overview

Notation

$$\begin{array}{ccccc}
\mathcal{F}_{m/n}^{\lambda} & \subseteq & \mathcal{F}_{m/n} & \subseteq & \mathcal{F}_n \\
\pi^{-1}\uparrow & & \downarrow\pi & & \downarrow\pi \\
C_\lambda & \subseteq & \mathcal{G}_{m/n} & \subseteq & \mathcal{G}_n
\end{array}$$

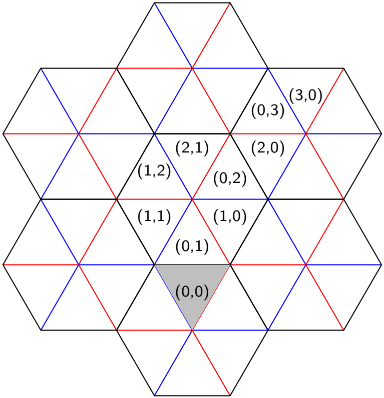
- $\pi : \mathcal{F}_n \to \mathcal{G}_n$ the natural projection ($\mathcal{G}_{m/n} := \pi(\mathcal{F}_{m/n})$)
- $\mathcal{G}_n$ has paving by Iwahori orbits $B\omega P/P$ where $\omega \in \widetilde{S}_n/S_n$
- $\mathcal{G}_{m/n}$ has a paving by $\mathcal{G}_{m/n} \cap (B\omega P/P)$
- The non-zero cells are indexed by partitions, called C_λ
- $\mathcal{F}_{m/n}^{\lambda} = \pi^{-1}(C_\lambda)$

Hikita Representation

- $P = \{(a_i) \mid a_i \in \mathbb{Z}^{\geq 0}, a_n = 0, a_1, \ldots, a_{n-1} \geq 0, a_{i+n} = a_i + 1\}$
- $\check{\lambda} : P \to Q^\vee$ is a bijection
- $a = \sum_{i=1}^{n-1} a_i$
- $s_{a+l} \cdot \check{\lambda}(a_1, \ldots, a_{n-1}) =$

$$\begin{cases} \check{\lambda}(a_1, \ldots, a_{l+1}, a_l, \ldots, a_{n-1}) & \text{if } l = 1, 2, \ldots, n-2, \\ \check{\lambda}(a_{n-1} - 1, a_1, \ldots, a_{n-2}) & \text{if } l = n-1 \text{ and } a_{n-1} \geq 1, \\ \check{\lambda}(a_1, \ldots, a_{n-1}) & \text{if } l = n-1 \text{ and } a_{n-1} = 0, \\ \check{\lambda}(a_2, \ldots, a_{n-1}, a_1 + 1) & \text{if } l = 0. \end{cases}$$

Example



Relation to Partitions

- Recall $m = nk + b$
- $\delta = (k(n-1)+b-1, k(n-2)+b-1, \dots, k+b-1, b-1)$
- $\delta - \delta'$ is the partition below the diagonal $(0, n)$ to $(m, 0)$

Proposition ([2])

There is a bijection

$$\left\{(a_i) \in P \mid C_{\check{\lambda}(a_i)} \neq \emptyset\right\} \to \{\lambda \mid \lambda \textit{ is a partition}, \lambda \subseteq \delta - \delta'\}$$

$$(a_i) \mapsto \lambda(a_i) := \delta - (a_b, a_{2b}, \dots, a_{nb})$$

Overview

Parking Functions

- Denote $[n] = \{1, 2, \ldots, n\}$
- Let there be n parking spots on a one-way street
- $f : [n] \to [n]$ such that $f(i)$ is the ith car's parking preference
- A car will go to its preference, then take the next open spot
- f is a parking function if all can park without circling back
- Denote a parking function by $\llparenthesis f(1)\ f(2)\ f(3)\ \ldots\ f(n) \rrparenthesis$
- Consider $f(1) = 2, f(2) = 1, f(3) = 4, f(4) = 1 \longrightarrow \llparenthesis 2141 \rrparenthesis$

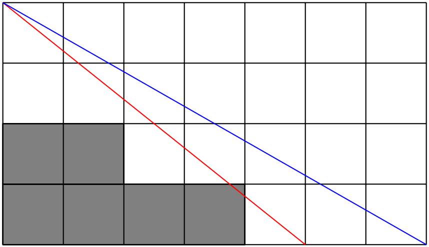
2	1	4	3
1	2	3	4

Parking Function Properties

- $f([n])$, when sorted as $a_1, a_2, \ldots, a_n$, obeys $a_i \leq i$
- Any permutation of a parking function is a parking function
- There are $(n+1)^{n-1}$ parking functions on domain $[n]$

$\mathcal{PF}_{m/n}$

- Let $n < m$ and $\gcd(m, n) = 1$
- $\mathcal{PF}_{m/n}$ - Parking functions whose Young diagram fits below the diagonal of an $n \times m$ box
  - $|\mathcal{PF}_{m/n}| = m^{n-1}$
- Consider the parking function $(\!|2040|\!)$



  - $(\!|2040|\!) \notin \mathcal{PF}_{5/4}$ but $(\!|2040|\!) \in \mathcal{PF}_{7/4}$

Mapping

- ${}^m\widetilde{S}_n \subset \widetilde{S}_n$ is the set of m-restricted permutations
- ${}^m\widetilde{S}_n = \{\omega \mid i < j \Rightarrow \omega(j) - \omega(i) \neq m\}$
- GMV created a map $\mathcal{SP} : {}^m\widetilde{S}_n \to \mathcal{PF}_{m/n}$

$$\omega \mapsto \mathcal{SP}_\omega$$

- Proven to be a bijection for $m = kn \pm 1$
- $\mathcal{SP}$ is conjectured to be a bijection for all m

$$\mathcal{SP}_\omega(i) = \#\{j > i \mid 0 < \omega(i) - \omega(j) < m\}$$

Example

$$\mathcal{SP}_\omega(i) = |\{j > i \mid \omega(i) - m < \omega(j) < \omega(i)\}|$$

- $n = 4,\ m = 7,\ \omega = [4, -2, 3, 5]$

⋯	-3	-2	-1	0	1	2	3	4	5	6	7	8	⋯
⋯	0	-6	-1	1	4	-2	3	5	8	2	7	9	⋯

- $\mathcal{SP}_\omega(1) = 3$
- $\mathcal{SP}_\omega(2) = 0$
- $\mathcal{SP}_\omega(3) = 1$
- $\mathcal{SP}_\omega(4) = 1$
- $\mathcal{SP}_\omega = (\!|3011|\!)$

Relation to the Affine Springer Fiber

Theorem ([3])

Consider the nil-elliptic operator T, where m is coprime to n. Then the corresponding affine Springer Fiber $\mathcal{F}_{m/n} \subset \mathcal{F}_n$ admits an affine paving by m^{n-1} affine cells.

Theorem (GMV [1])

There is a natural bijection between the affine cells in $\mathcal{F}_{m/n}$ and the affine permutations in ${}^m\widetilde{S}_n$. The dimension of the cell Σ_ω labeled by the affine permutation ω is equal to

$$\sum_{i=1}^{n} \mathcal{SP}_\omega(i).$$

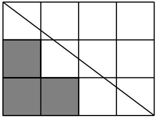
Overview

What do we know?

- $\mathcal{F}_{m/n}$ is paved by ${}^m\widetilde{S}_n$
- $\mathcal{G}_{m/n}$ is paved by a subset of $\widetilde{S}_n/S_n$
- ${}^m\widetilde{S}_n$ bijects with $\mathcal{PF}_{m/n}$
- Non-zero $\widetilde{S}_n/S_n$ bijects with P
- $\mathcal{F}_{m/n} \xrightarrow{\pi} \mathcal{G}_{m/n}$
- How to map from $\mathcal{PF}_{m/n} \to P$?

Extended Example

- $m = n + 1$ and $\lambda \subseteq \delta$

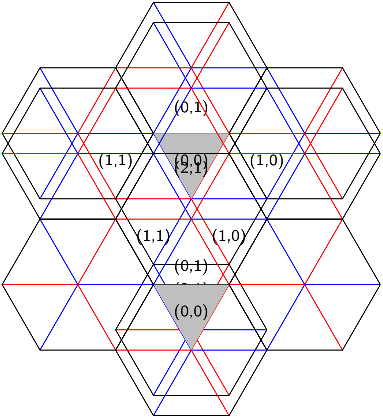


λ	 (2,1,0)	 (2,0,0)	 (1,1,0)	 (1,0,0)	$\emptyset$ (0,0,0)
$(a_i) = \delta - \lambda$	(0,0)	(0,1)	(1,0)	(1,1)	(2,1)
$PF_{(n+1)/n}$	(2,1,0) (2,0,1) (0,2,1) (1,2,0) (0,1,2) (1,0,2)	(2,0,0) (0,2,0) (0,0,2)	(1,1,0) (1,0,1) (0,1,1)	(1,0,0) (0,1,0) (0,0,1)	(0,0,0)

Extended Example

Convert (a_i) to $\widetilde{S}_n$

(0,0)	(0,1)	(1,0)	(1,1)	(2,1)
1	s_0	$s_2 s_0$	$s_1 s_0$	$s_2 s_1 s_2 s_0$



Useful Facts

- GMV extended the H_i^k notation

$$H_{i,j}^k = \{\bar{x} \in E \mid x_i - x_j = k\}$$

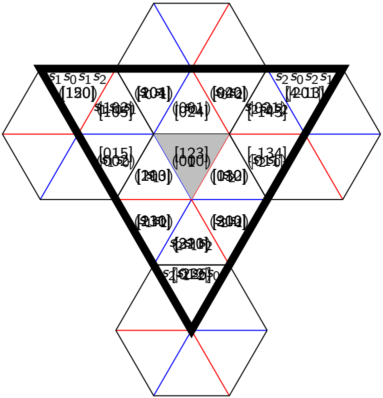
- $H_i^k \mapsto H_{i,i+1}^k$ for $1 \leq i < n$
- $H_0^k \mapsto H_{1,n}^k$
- $H_{i,j}^k = H_{j,i}^{-k} = H_{i+tn,j+tn}^k = H_{i,j-n}^{k-1}$
- D_n^m is the Sommers region bounded by $\{H_{i,i+m}^0 \mid 1 \leq i \leq n\}$

Lemma (GMV)

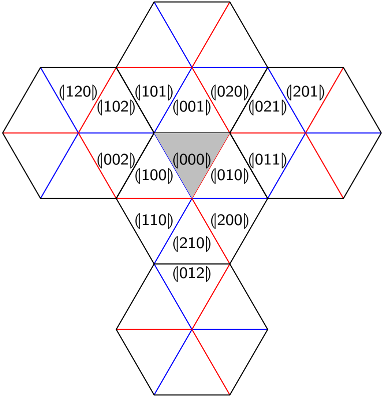
The set of alcoves $\{\omega(A_0) \mid \omega \in {}^m\widetilde{S}_n\}$ coincides with the set of alcoves that fit inside the region D_n^m.

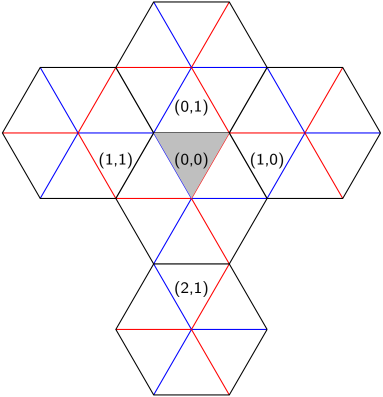
Extended Example

Label D_3^4 Convert to windows Apply $\mathcal{SP}$



The Projection





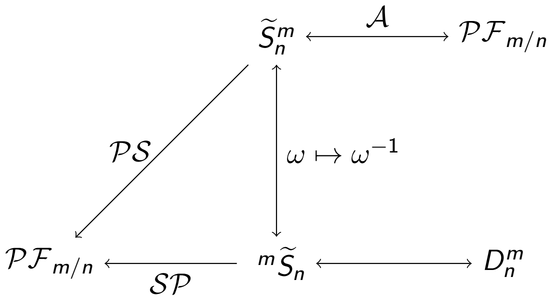
The Un-natural Mapping

- What is the combinatorial projection from $\mathcal{PF}_{4/3} \to P$?

P	(0,0)	(0,1)	(1,1)	(1,0)	(2,1)
$\mathcal{PF}_{m/n}$	$(\|000\|)$	$(\|001\|)$	$(\|002\|)$	$(\|011\|)$	$(\|012\|)$
	$(\|100\|)$	$(\|101\|)$	$(\|102\|)$	$(\|021\|)$	
	$(\|010\|)$	$(\|020\|)$	$(\|120\|)$	$(\|201\|)$	
	$(\|110\|)$				
	$(\|200\|)$				
	$(\|210\|)$				

The Conjectured Natural Mapping

- GMV also created $\mathcal{A} : \widetilde{S}_n^m \to \mathcal{PF}_{m/n}$ a bijection



The Conjectured Natural Mapping

m-Restricted	m-Stable	A map (PF)	Word of Restricted
[1, 2, 3]	[1, 2, 3]	[0, 1, 2]	[]
[2, 1, 3]	[2, 1, 3]	[1, 0, 2]	[1]
[1, 3, 2]	[1, 3, 2]	[0, 2, 1]	[2]
[2, 3, 1]	[3, 1, 2]	[1, 2, 0]	[1, 2]
[3, 1, 2]	[2, 3, 1]	[2, 0, 1]	[2, 1]
[3, 2, 1]	[3, 2, 1]	[2, 1, 0]	[1, 2, 1]
[0, 2, 4]	[0, 2, 4]	[0, 2, 0]	[0]
[2, 0, 4]	[0, 1, 5]	[2, 0, 0]	[0, 1]
[0, 4, 2]	[−1, 3, 4]	[0, 0, 2]	[0, 2]
[0, 1, 5]	[2, 0, 4]	[0, 1, 1]	[1, 0]
[1, 0, 5]	[1, 0, 5]	[1, 0, 1]	[1, 0, 1]
[1, 5, 0]	[1, −1, 6]	[1, 1, 0]	[1, 0, 1, 2]
[−1, 3, 4]	[0, 4, 2]	[0, 0, 1]	[2, 0]
[−1, 4, 3]	[−1, 4, 3]	[0, 1, 0]	[2, 0, 2]
[4, −1, 3]	[−2, 5, 3]	[1, 0, 0]	[2, 0, 2, 1]
[−2, 2, 6]	[4, 2, 0]	[0, 0, 0]	[2, 1, 2, 0]

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