Math 2534 Solution for Test 2A Spring 2011

Problem 1: Using PMI prove the following theorem. Justify each step. Indicate where you use the Inductive assumption.

Theorem: For all real numbers n > 3, $n+3<n!$

(20pts)

Proof: I will verify that the hypothesis is true for at least one value of n.

Consider n = 4, 7 < 4! Or 7 < 24

Now we will assume true for all natural numbers from n = 4 up to some arbitrary value k, ie: $k+3<k!$, and prove true for k+1. Ie: $(k+1)+3<(k+1)!$

Consider the k+1 term, (k+1)+3 = (k+3)+1 < k! +1 by the inductive assumption

This will give that (k+1)+3 = (k+3)+1 < k! +1 < k! + k < k! + k! = 2k! < (k+1)k! = (k+1)! Since k > 3.

Therefore I have assumed true for k and proved true for k + 1. Hence the hypothesis is true for all natural numbers greater than 3.

Problem 2: (16pts) Use **set Algebra** to prove the following. Justify each step.

Theorem: For any sets A and B,

$[(A\cup B)\cap A]^c - B^c = B - A$

Proof:

$[(A\cup B)\cap A]^c - B^c =$	given
$[(A\cup B)\cap A]^c \cap B^{cc} =$	equivalent form of difference
$[(A\cup B)\cap A]^c \cap B =$	double complement
$A^c \cap B =$	absorption
$B \cap A^c =$	commutativity
$B - A$	equivalent form of difference

$\therefore [(A\cup B)\cap A]^c - B^c = B - A$

Problem 3: Use **proof by elements** to prove the following. Justify each step

Theorem: For any sets A and B,

$P(A-B)\subseteq P(A)-P(B)$

(10pts)

As it turns out this is not a true statement: Consider the following counter example

$A=\{1,2\}$, B = {1,3} and $A-B=\{2\}$

$\text{P(A)}=\{\varnothing,\{1,2\},\{1\},\{2\}\}$, $\text{P(B)}=\{\varnothing,\{1,3\},\{1\},\{3\}\}$,

$P(A-B)=\{\varnothing,\{2\}\}$ and $P(A)-P(B)=\{\{1,2\},\{2\}\}$

If we correct the typo in the original statement we have the following:

$P(A-B) \subseteq [P(A)-P(B)] \cup \varnothing$

Proof:

$\forall x \in P(A-B) \to x \subseteq A-B$ by definition of Power sets

$\to x \subseteq A \wedge x \not\subset B$ by definition of difference

$\to x \in P(A) \wedge x \notin P(B)$

$\to x \in [P(A)-P(B)] \cup \varnothing$

$\therefore \forall x, P(A-B) \subseteq P(A)-P(B)$

Problem 4: Using PMI prove the following theorem. Justify each step. Indicate where you use the Inductive assumption.

Theorem: If $a_1 = a_2 = 1,\ a_n = 2a_{n-1} + a_{n-2}\ n > 2,$ then $a_n < 6a_{n-2},\ n \geq 5$ and $n \in N$

(20pts)

Proof: I will verify that the hypothesis is true for at least one value of n.

Consider n = 5, $a_5 < 6a_3$, so we have 17 < 6(3) =18

We also will need to verify for n = 6 which will give us 41 < 42 in order to have a valid truth set for the notation used in the body of the proof.

We will now assume true from n = 5 up to some arbitrary natural number k that $a_n < 6a_{n-2}$.

And prove true for k+1. We will show that $a_{k+1} < 6a_{n-1}$.

Consider the k+1 term

$a_{k+1} = 2a_k + a_{k-1} < 2(6a_{k-2}) + 6a_{k-3}$ by the inductive assumption

$= 6(2a_{k-2} + a_{k-3}) = 6a_{k-1}$

$\therefore a_{k+1} < 6a_{k-1}$

Therefore I have assumed true for k and proved true for k + 1. Hence the hypothesis is true for all natural numbers greater than 4.

Problem 5: Consider the set A of all divisors of 24, A = { 1, 2, 3, 4, 6, 8, 12, 24}, with the operations defined as follows: a♥b = LCM(a, b), a♦b = GCD(a,b) and the complement (or negation) is defined to be $a' = 24/a$. Show that the following properties of a Boolean Algebra are valid for arbitrary examples.

1) Is the following property true for 3♥(4♦8) = (3♥4)♦(3♥8) ?
2) Find the identities for each of the operations and justify your choice with an example.
3) Give an example that indicates that DeMorgan's will work.

4) Evaluate $6 \heartsuit (6' \diamondsuit 8) = ?$.

(14 pts)

1)

$$3 \heartsuit (4 \diamondsuit 8) \overset{?}{=} (3 \heartsuit 4) \diamondsuit (3 \heartsuit 8)$$

$$3 \heartsuit (4) = (12) \diamondsuit (24)$$

$$12 = 12$$

2) Identities $a \heartsuit 1 = a \quad \forall a \in A$

$a \diamondsuit 24 = a \quad \forall a \in A$

3) $(3 \heartsuit 4)' \overset{?}{=} 3' \diamondsuit 4'$

$$(12)' = 8 \diamondsuit 6$$

$$2 = 2$$

4) $6 \heartsuit (6' \diamondsuit 8) = 6 \heartsuit (4 \diamondsuit 8) = 6 \heartsuit (4) = 12$

Problem 6: Let the set A = {a, b, {b}, {a, b}}, set B = {b}, set C = {a, b}, D = {a, {a}}

Find the following: (20pts)

a) $A \cap D = \{a\}$

b) $A - C = \{\{b\},\{a,b\}\}$

c) $B \oplus D = \{a,b,\{a\}\}$

d) $B \times C = \{(b,a),(b,b)\}$

e) $P(D) = \{\emptyset, \{a,\{a\}\}, \{a\}, \{\{a\}\}\}$