Math 2534 Solutions Homework 8 Spring 2018 (due March 21)

Show all work. Use complete sentences. Staple multiple sheets.

Problem 1:

Theorem: for all natural numbers. *If* $f(x) = \ln x,$ then the n$^{\text{th}}$ derivative $f^{(n)}(x) = \dfrac{(-1)^{n-1}(n-1)!}{x^n}$

(Remember that 0! = 1)

Proof: I will verify that the hypothesis is true for at least one value of n $\in$ N.

Consider n = 1 to get that $f^{/} = \dfrac{1}{x}$, and $f^{/} = \dfrac{(-1)^{n-1}(n-1)!}{x^n}$

Consider n = 2 to get that $f^{//} = \dfrac{-1}{x^2}$, and $f^{//} = \dfrac{(-1)^{2-1}(2-1)!}{x^2}$

Now assume the hypothesis is true from n = 2 up to k so that $f^{(\text{k})}(x) = \dfrac{(-1)^{k-1}(\text{k}-1)!}{x^k}$ and prove true for k + 1 by showing that $f^{(\text{k}+1)}(x) = \dfrac{(-1)^k \, \text{k}!}{x^{k+1}}$.

Consider the k + 1 term

$$f^{(\text{k}+1)}(x) = \left(f^{(k)}\right)^{/} = \left(\frac{(-1)^{k-1}(\text{k}-1)!}{x^k}\right)^{/} = (-1)^{k-1}(\text{k}-1)!(-\text{k}\,x^{-k-1}) = (-1)^{k-1}(-1)(\text{k}-1)!(\text{k})x^{-(k+1)}$$

$$= \frac{(-1)^k k!}{x^{k+1}}$$

Problem 2: Given the recursive sequence $a_1 = a_2 = 1$ with $a_n = (a_{n-1})^2 + a_{n-2}$ for n $\geq$??

a) Find the least value of n where n is an natural number. $n \geq 3$

b) Find the next 4 terms in this sequence

$a_3 = (a_2)^2 + a_1 = 2$

$a_4 = (a_3)^2 + a_2 = 5$

$a_5 = (a_4)^2 + a_3 = 27$

$a_6 = (a_5)^2 + a_4 = 734$

Problem 3: Given the sequence 1, 7, 49, 343, …..,

a) Find the function sequence representation f(n)

   $f(n) = \quad f(n) = 7^{n-1}$

b) Find the recursive representation a_n

   $a_1 = 1,\ a_n = 7a_{n-1},$ n > 1

Also consider the following theorem:

Theorem: If the recursive sequence is given to be $a_1 = 1,\ a_n = 7a_{n-1},$ n > 1 and the function $f(n) = 7^{n-1}$, then $a_n = f(n)\ \forall n \in N$.

Proof by PMI: The hypothesis is true for at least one value of n. Consider n = 1 so that $a_1 = 1$ and $f(1) = 7^0 = 1$. We will also consider n = 2 where

$a_2 = 7a_1 = 7(1) = 7$ and $f(2) = 7^{2-1} = 7$.

Now assume the hypothesis is true from n = 2 up to some arbitrary value k so that $a_k = f(k)$ and prove true for k + 1 by showing that $a_{k+1} = f(k+1)$ where $f(k+1) = 7^k$.

For the body of the proof consider the k + 1 term.

$a_{k+1} = 7a_k = 7f(k)$ by the inductive assumption, so $a_{k+1} = 7f(k) = 7(7^{k-1}) = 7^k = f(k+1)$

Since we assumed true up to k and proved true for k + 1, the hypothesis is true for all natural numbers.

Problem 4: Given the recursive sequence: $a_1 = 1, a_2 = 1$ and $a_n = 2a_{n-1} + 3a_{n-2}, n \geq 3$,

Show that $a_n < 2(3^{n-2})$ for all $n \in N$, n > 2.

Proof: The hypothesis is true for at least one value of n. consider n = 3 where
$a_3 < 2(3^{3-2})$, since $a_3 = 2a_2 + 3a_1 = 5$ we have that 5 < 2(3) = 6 .
Also consider n = 4 to get
$a_4 < 2(3^{4-2})$, since $a_4 = 2a_3 + 3a_2 = 2(5) + 3(1) = 13$ we have that 13 < $2(3^2)$ = 18
Assume the hypothesis is true from n = 3 up to some arbitrary value k so that $a_k < 2(3^{k-2})$ and prove true for k + 1 by showing that $a_{k+1} < 2(3^{k-1})$.

Consider the k + 1 term.
$a_{k+1} = 2a_k + 3a_{k-1} < 2[2(3^{k-2})] + 3[2(3^{k-3})]$ by the inductive assumption.
so we have that $a_{k+1} < 2[2(3^{k-2}) + 3^{k-2}] = 2[3(3^{k-2})] = 2(3^{k-1})$.
We have assumed true up to k and proved true for k+1, the hypothesis is true for all natural numbers and n > 2.

Problem 5:

Theorem: If $a_1 = 1$ and $a_2 = 2$ and $a_n = a_{n-1} + 2a_{n-2}$ for all n ≥ 3 and $f(n) = 2^{n-1}$,

then $a_n = f(n)$ for all natural numbers..

Proof: To verify the Hypothesis is true for at least one value of n we will consider:
n = 1 to show that $a_1 = f(1)$ since it is given that $a_1 = 1$ and $f(1) = 2^{1-1} = 2^0 = 1$
n = 2 to show that $a_2 = f(2)$ since it is given that $a_2 = 2$ and $f(2) = 2^{2-1} = 2$.
n = 3 to show that $a_3 = f(3)$ since it is given that $a_3 = a_2 + 2a_1 = 2 + 2(1) = 4$
. and $f(3) = 2^{3-1} = 2^2 = 4$.
Now assume that the hypothesis is true from n = 3 up to some arbitrary natural number k so that $a_k = f(k)$ and prove true for k + 1 by showing for k + 1, $a_{k+1} = f(k+1)$ where $f(k+1) = 2^K$
For the body of the proof consider the k + 1 term:
$a_{k+1} = a_k + 2a_{k-1}$ by the definition of the a_n. By the inductive assumption we have that
$a_{k+1} = a_k + 2a_{k-1} = f(k) + 2f(k-1) = 2^{k-1} + 2(2^{k-2}) = 2^{k-1} + 2^{k-1} = 2(2^{k-1}) = 2^k$.
Since we have assumed true for k and proved true for k + 1, the hypothesis is true for all natural numbers.

Problem 6:

Theorem: Given the Fibonacci sequence f_n, $f_1 + f_3 + f_5 + ... + f_{2n-1} = f_{2n} \quad \forall n \in N$

Proof: The hypothesis is true for at least one value of n. consider n = 1 to get that $f_1 = f_{2(1)}$ which is 1=1. For n = 2, $f_1 + f_3 = f_{2(2)}$ which gives that 1 + 2 = 3.

Assume the hypothesis is true from n = 2 up to some arbitrary value k that

$f_1 + f_3 + f_5 + ... + f_{2k-1} = f_{2k}$ and then prove true for k + 1 by showing that

$f_1 + f_3 + f_5 + ... + f_{2k+1} = f_{2k+2}$

Now consider the k + 1 term:

$f_1 + f_3 + f_5 + ... + f_{2k+1} =$

$(f_1 + f_3 + f_5 + ... + f_{2k-1}) + f_{2k+1} =$

so by the inductive assumption we have that

$(f_{2k}) + f_{2k+1} =$

$f_{2k+1} + f_{2k} = f_{k+2}$ by the definition of Fibonacci Sequence.

We have assumed true up to k and proved true for k+1, the hypothesis is true for all natural numbers.

Problem 7: A group of people stand in line to purchase concert tickets. The first person in line is a women and the last person is a man. Use PMI to show that somewhere in the line a woman will always be directly in front a man.

Proof:

Using the Principle of Mathematical Induction for all natural numbers n > 1, verify that the hypothesis is true for at least one value of n. Let n = 2 where we have the first person is a women and the second person is a man by the sufficient condition. In this case the women is indeed directly in front of a women. Let us also consider n = 3. If the third person is a women and she inserts herself between the woman and man already there, she will be directly in front of the man. However if she places herself in front of the women there is still a women directly in front of the man where a women is head of the line and a man is at the end.

If the third person is a man and he inserts himself between the woman and man already there, He will be directly behind the women. However if he places himself behind the man, there is still a women directly in front of the man and the conditions are still met.

Now assume the hypothesis continues to be true up to some number k so that somewhere in line a woman is directly in front of a man. We will now prove true for k + 1 people in line. Now consider the k + 1 person to be added to the line. Suppose this person is a women. If she inserts herself in the spot in line where a women is directly in front of a man then she will be directly in front of this man. If she chooses another spot in line other than behind the last man in line, then there is still a women directly in front of a man and conditions are still met.

If the k + 1 person is a man and he inserts himself in the spot in line where a women is directly in front of a man, then he is directly behind this women in line. If he chose any other spot in line, other than in front of the first woman in line there is still a women directly in front of a man and conditions are still met.

Having assumed that the conjecture will hold up to k and demonstrated how it will continue to hold for k + 1, we have shown that this conjecture is true for all natural numbers n > 1.