# <u>Math 2534 Solution  Homework 12 Functions and Relations</u>

Show all work and staple multiple sheets.

**Problem 1:**

Given $A = \{a, b, c, d\}$, $F : X \to Y$ for X is the set $P(A)$] and Y =$\{0,1,2,3,4,5,6\}$,
Define F on all elements S in P(A) so that F(S) = n(S)  (ie. the number of elements in S)

a) Is F one to one?  Justify your conclusion
b)  Is F onto?   Justify your conclusion

**Solutions;**

$P(A) = \{\varnothing, \{a,b,c,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\},$
$\{c,d\}, \{a\}, \{b\}, \{c\}, \{d\}\}$

$F(\varnothing) = 0$

$F(\{a,b,c,d)\} = 4$

$F(\{a,b,c\}) = F(\{a,b,d\}) = F(\{a,c,d\}) = F(\{b,c,d\}) = 3$

$F(\{a,b\}) = F(\{a,c\}) = F(\{a,d\}) = F(\{b,c\}) = F(\{b,d\}) = F(\{c,d\}) = 2$

$F(\{a\}) = F(\{b\}) = F(\{c\}) = F(\{d\}) = 1$

The function is not onto since no domain value maps to 5 or 6.  F is not one to one since several domain values map to the same range value.

**Problem 2** : Explain the mistake in the following proof:

**Theorem:** If $f(x) = 5x^2 - 2$ for all integers, Then *f(x)* is one to one.

Proof:   Suppose any integer z  is given. Then by definition of a function, there is only one  possible value for *f(z)*, namely y = $f(z) = 5z^2 - 2$ . Hence *f (x)* is one to one.

**Solution:**

The author of this proof only verifies that  f(z) is a function. He does not address the definition of “one to one” at all.

**Problem 6:**

For each of the following relations defined on the set A = { 1,2,3,4,5}, determine if R is reflexive, symmetric and/or transitive.  Draw the directed graphs for each relation.  If R is an equivalence relation then represent A as a partition.

R1 = {(1,1), (2, 2), (2,3)(3, 2),(3,3), (3, 4),(4,3), (5,5), (4, 4)}  R, S,
Not Transitive since need (2, 4)

R2 ={(1,2),(1,4),(1,5)(2,4), (2,5), (3,4), (3,5), (4,5), (4,4)} T, Not R since no element relates to itself,  Not S since need (2,1)

R3 = {(1,3),(1,5),(2,4)(3,1), (3,5), (4,2), (5,1), (5,3)}  S, Not T since need (1,1), Not R since no element relates to itself.

R4 = {(1,1), (2, 2), (1,3), (1,5), (3,1), (3,3),(2, 4), (5,3), (3,5),(4, 2)(4, 4), (5,1)}
Not R and not Transitive since missing (5,5).   It is symmetric.

**Problem 4:** Prove or give a counter example;
**Theorem:**
If the relations S and R are each transitive then the union $S \cup R$ is also transitive.

Counter Example: Let S = { (1,2), (2,3),(1,3), (4, 5)} and R = { (1,6), (6,3),(1,3), (5, 7)}
Notice that S and R are each transitive but the union would not be transitive since it would not contain (4, 7)

**Problem 5:** Prove the following:
**Theorem:** A relation R is a symmetric relation on the set of all sets when for sets A,B, ARB if and only if there is a bijection from one set to other.

**Proof:** To verify that R is symmetric, it must be shown that for any sets A,B,
If ARB then BRA. Given that ARB, there exist a bijection *f* that maps A to B. We know that *f* is a bijection if and only if $f^{-1}$ exist (and is also a bijection). Therefore $f^{-1}$ maps B to A and BRA. Then R is symmetric.

**Problem 6: Modular Equivalences:** Let a, b be any integers and d is a positive integer where d > 1. The following statements are equivalent.

$a)$ $a \equiv b \bmod d$

$b)$ $a \bmod d = b \bmod d$

$c)$ $d \mid a - b$

$d)$ $b = dq + a[\bmod d]$, for $q \in Z$

**Part A: Prove the following:**
**Theorem:** If R is defined on the integers so that $a \equiv b \bmod 6$, then R is an equivalence relation on the integers when for integers a, b, $aRb$ iff $6 \mid a - b$.

**Proof:** In order to verify that R is an equivalence relation we must show that R is reflexive, symmetric and transitive.

To show that R is reflexive we must show that for all integers a, aRa and $6 \mid (a-a)$. In order for for this statement to be true there must exist an integer q so that 6q = a-a. This is true since q can be zero. Therefore by definition of divisible we have that aRa and R is reflexive.

To show that R is symmetric we must show that for all integers a, b, If aRb, then bRa. Since we know that aRb and $6 \mid (a-b)$. By definition of divisible we have that there exist an integer p so that 6p = a – b. If we multiply through with -1, we have that 6(-p) = b – a where –p is the required integer to satisfy the definition of divisible. Therefore bRa and R is symmetric.

To show that R is transitive we must show that for all integers a,b,c, If aRb and bRc, then aRc.

We are given that $6\left|(a-b)\right.$ and $6\left|(b-c)\right.$. By definition of divisible there exist integers m and n so that 6m = a – b and 6n = b – c, Now consider 6m + 6n = a – b + b – c = a – c. We now have that 6h = a – c where h = m + n is an integer an the definition of divisible is satisfied. Therefore aRc and R is transitive.
Since R has been shown to reflexive, symmetric and transitive, R is an equivalence relation.

**Part B:**
Zmod 6 partitions the integers as follows: $Z=[0]\cup[1]\cup[2]\cup[3]\cup[4]\cup[5]$
Label the following equivalence classes with the correct remainder representation.

$$[8],[-9],[24],[-22]$$

By definition of congruent relation we have that $a \equiv b \bmod 6$ which is equivalent $6\left|(a-b)\right.$ where $a$ takes on possible values 0 – 5 only.

$6\left|(a-8)\right.$ where $a$ = 2

$6\left|(a-(-9))\right.$ where $a$ = 3

$6\left|(a-24)\right.$ where $a$ = 0

$6\left|(a-(-22))\right.$ where $a$ = 2

$[8]=[2],[-9]=[3],[24]=[0],[-22]=[2]$