Math 2534 Solution Test 2 Solutions Spring 2018

Problem 1: : Use **PMI** to prove the following theorem and be clear where you use the inductive assumption. Justify all steps and tie your work together with complete sentences.
(18 pts)

Theorem: $\forall n \in N$ and $n \geq 3$, $(n+1)! > 1 + 2^n$

Proof:

The hypothesis is true for at least one value of n. Consider n = 3.
$(3 + 1)! = 4! = 24$ and $1 + 2^3 = 9$ so $24 > 9$

Now assume the hypothesis is true from n = 3 up to some arbitrary value k so that
$(k+1)! > 1 + 2^k$ and prove true for the k + 1 term by showing $(k+2)! > 1 + 2^{k+1}$

For the body of the proof, consider the k + 1 term which is (k + 2)!.

$(k + 2)! = (k + 2)(k + 1)! > (k+2)(1 + 2^k)$ by the inductive assumption.
Now we have that $(k + 2)! > (k+2)(1 + 2^k) > (2)(1 + 2^k)$ since $k +2 > k > 2$.
This gives the following results that $(k + 2)! > (2)(1 + 2^k) = 2 + 2^{k+1} > 1 + 2^{k+1}$ since $2 > 1$.

We have assumed the hypothesis is true up to k and proved true for k + 1. Therefore the hypothesis is true for all natural numbers greater than 2.

Problem 2: (14 pts) Prove the following using Set Algebra. Do not leave out steps in body of proof and state the properties used.

Theorem: For any sets A and B, $[(A^C \cap B^C)^C - A^C] \cup (A \cap \varnothing^C) = A$

Proof:

$[(A^C \cap B^C)^C - A^C] \cup (A \cap \varnothing^C) =$	given
$[(A^C \cap B^C)^C - A^C] \cup (A \cap U) =$	the Identity complement
$[(A^C \cap B^C)^C - A^C] \cup (A) =$	the Identity law
$[(A^{CC} \cup B^{CC}) - A^C] \cup (A) =$	DeMorgans law
$[(A \cup B) - A^C] \cup (A) =$	Double complement law
$[(A \cup B) \cap A^{CC}] \cup (A) =$	Difference law
$[(A \cup B) \cap A] \cup (A) =$	Double Complement law
$[A] \cup (A) =$	Idempotent law
A	Absorption Law

Therefore $[(A^C \cap B^C)^C - A^C] \cup (A \cap \varnothing^C) = A$

Problem 3: Prove the following using method of **Contradiction.**
(10pts)

Theorem: For all sets A, $A\times\varnothing=\varnothing$

Proof:

Assume that there exist at least one set A so that $A\times\varnothing\neq\varnothing$

By definition of Cartesian product,

for $(x,y)\in A\times\varnothing\rightarrow x\in A\wedge y\in\varnothing$ However this is a contradiction to the definition of $\varnothing$ which contain no elements. Therefore For all sets A, $A\times\varnothing=\varnothing$.

Problem 4: (12pts) Use proof by **elements** to verify that for all nonempty sets A, B, and D

Theorem: If $A\subseteq B,\ D^c\subseteq B^c$ Then $\mathrm{D}^c\subseteq A^c$. (**Justify each step of proof**)

Proof:

$\forall x\in D^C\rightarrow x\in B^C$ since $D^C\subseteq B^C$

$\rightarrow x\notin B$ by definition of complement

$\rightarrow x\notin A$ since $A\subseteq B$

$\rightarrow x\in A^C$ by definition of complement

therefore by definition of containment $D^C\subseteq A^C$

Problem 5: (18pts) Use PMI to prove the following theorem and be clear where you use the inductive assumption. Justify all steps and tie your work together with complete sentences.

Theorem: Given the Fibonacci Sequence $f_1 = 1, f_2 = 1, \ f_n = f_{n-1} + f_{n-2}$, n > 2, then

$$f_n = 3f_{n-3} + 2f_{n-4} \text{ , for all natural numbers } n > 4$$

Proof: To verify that hypothesis is true for at least one value of n,
consider n = 5 to see that $f_5 = f_4 + f_3 = 3 + 2 = 5$ and
$3f_{n-3} + 2f_{n-4} = 3f_2 + 2f_1 = 3(1) + 2(1) = 5$ so it is clear that $f_5 = 3f_2 + 2f_1$

also consider n = 6 so that $f_6 = f_5 + f_4 = 5 + 3 = 8$ and
$3f_{n-3} + 2f_{n-4} = 3f_3 + 2f_2 = 3(2) + 2(1) = 8$ so it is clear that $f_6 = 3f_3 + 2f_2$

Assume that the hypothesis continues to be valid from n = 6 up to some arbitrary value k so that $f_k = 3f_{k-3} + 2f_{k-4}$ and prove true for k + 1 by showing $f_{k+1} = 3f_{k-2} + 2f_{k-3}$.

In the body of the proof consider the k + 1 term:
$f_{k+1} = f_k + f_{k-1}$ by definition of Fibonacci sequence
So $f_{k+1} = f_k + f_{k-1} = (3f_{k-3} + 2f_{k-4}) + (3f_{k-4} + 2f_{k-5})$ by the inductive assumption.
This then gives us

$$f_{k+1} = (3f_{k-3} + 2f_{k-4}) + (3f_{k-4} + 2f_{k-5}) = 3(f_{k-3} + f_{k-4}) + 2(f_{k-4} + f_{k-5}) = 3f_{k-2} + 2f_{k-3}$$

By the Fibonacci sequence.

We have assumed the hypothesis is true up to k and proved true for k + 1. Therefore the hypothesis is true for all natural numbers n > 4.

Problem 6: (16pts) Let set A = {a, {b}, {a,b}}, B = {b}, C = {a, {b}, b} and D = {{a},a}

Part A) Find the following

$a)\ A \cap C = \{a, \{b\}\}$

$b)\ C - B = \{a, \{b\}\}$

$c)\ \mathrm{D} \cup C = \{a, \{a\}, \{b\}, b\}$

$d)\ B \times D = \{(b, \{a\}), (b, a)\}$

$e)\ A \oplus C = \{\{a, b\}, b\}$

$f)$ Power set $P(C - D) = P(\{\{b\}, b\}) = \{\varnothing, \{\{b\}, b\}, \{\{b\}\}, \{b\}\}$

Part B) (12pts) Indicate if the following is true or false and justify your response.

$a)$ $\{\{a\}\} \in$ P(D) True Since $\{a\} \in$ D

b) $\{b\} \in P(A)$ False Since b $\notin$ A

b) $\varnothing \subset$ P(B) True $\varnothing$ is a subset of every set

c) $\{\varnothing\} \in$ P(C) False Since $\varnothing \notin$ C

d) P(B) $\subset$ P(C) True since b $\in$ B and b $\in$ C and n(B) = 1 < n(C) = 3.

e) P(B) $\subset$ P(A) False A does not contain the element b and so P(A) will contain {b} which is in P(B)