Math 2534 Solution Homework 7 Spring 2018

Prove the following theorems using PMI. The write up needs to be complete by using sentences to explain and justify your presentation.

Theorem 1: $\forall n \in N,\ 1 + a + a^2 + a^3 + ... + a^{n-1} = \dfrac{a^n - 1}{a - 1}$ ($a \neq 0$ is some unknown real number.)

Proof: First we will verify that there is at least one value of n so that the hypothesis is true.

Consider n = 1 to get that $1 = \dfrac{a-1}{a-1}$ so we have that 1 = 1 and the hypothesis is valid for n = 1

Now consider n = 2 to get $1 + a = \dfrac{a^2 - 1}{a - 1} = \dfrac{(a-1)(a+1)}{a-1} = a + 1 = 1 + a$ and hypothesis is valid for n = 2.

Now we will assume that the hypothesis is true from n = 2 up to some arbitrary natural number k so that $1 + a + a^2 + a^3 + \ldots + a^{k-1} = (a^k - 1) / (a - 1)$.

Now we will prove that the hypothesis is true for k + 1. We will show that
$1 + a + a^2 + a^3 + \ldots + a^k = (a^{k+1} - 1) / (a - 1)$

Now consider the k + 1 term:
$1 + a + a^2 + a^3 + \ldots + a^{k-1} + a^k$ and by the inductive assumption we have that

$$[1 + a + a^2 + a^3 + \ldots + a^{k-1}] + a^k = \left[\frac{a^k - 1}{a - 1}\right] + a^k$$
$$= \frac{a^k - 1}{a - 1} + \frac{a^k(a-1)}{a-1}$$
$$= \frac{a^k - 1 + a^{k+1} - a^k}{a - 1}$$
$$= \frac{a^{k+1} - 1}{a - 1}$$

$$Therefore:\ 1 + a + a^2 + a^3 + \ldots + a^{k-1} + a^k = \frac{a^{k+1} - 1}{a - 1}$$

Since we assumed that the hypothesis is true up to k and we proved the hypothesis is true for k + 1, then the hypothesis is true for all natural numbers.

Theorem 2: $\forall n \in N,\ 3 \mid (4^n - 1)$

Proof: I will verify that the hypothesis is true for at least one value of n $\in$ N
Consider n = 1, We have that $4^1 - 1 = 3$ and there exist and integer q = 1 so that $3q = 4^1 - 1$ so by definition of divisible $3 \mid (4^1 - 1)$

Consider n = 2, We have that $4^2 - 1 = 15$ and there exist and integer q = 5 so that $3q = 4^2 - 1$ and by definition of divisible $3 \mid (4^2 - 1)$

Now assume that the hypothesis is true for each natural number between n = 2 up to some arbitrary natural number k. ie. $3 \mid (4^k - 1)$ and by definition of divisible there exist an integer f so that $3f = 4^k - 1$. In order to prove true for k + 1 there must exist an integer h so that $3h = 4^{k+1} - 1$ to satisfy the definition of divisible.

For the body of the proof we will consider the k+1 term,
$4^{k+1} - 1 = 4^k 4 - 1 = (4^k - 1) + (3)4^k = 3f + (3)(4^k)$ by the inductive assumption. Notice that 4^k is also an integer. Let $p = 4^k$ so that $4^{k+1} - 1 = 3f + (3)(4^k) = 3(f + p) = 3h$ where h = f + g is an integer. So by definition of divisible we have that $3 \mid (4^{k+1} - 1)$

Having assumed true up to k and proved true for k + 1, the hypothesis is true for all natural numbers.

Theorem 3: $\forall n \in N, n \geq 5,\ (n+1)! > 2^{n+3}$

Proof: I will verify that the hypothesis is true for at least one value of n $\in$ N
Consider n = 5 to get that $6! = 720$ and $2^8 = 256$ and clearly $6! > 2^8$

Now assume true from n = 5 up to k so that $(k+1)! > 2^{k+3}$ and prove true for k + 1 by showing that $(k+2)! > 2^{k+4}$.

Consider the k + 1 term $(k + 2)! = (k + 1)!(k + 2) > 2^{k+3}(k+2)$ by the inductive assumption.
So $(k + 2)! > 2^{k+3}(k+2) > 2^{k+3}(2) = 2^{k+4}$ since $k > 5 > 2$.
Having assumed true up to k and proved true for k + 1, the hypothesis is true for all natural numbers greater than 4.

Theorem 4: $\forall n \in N, n \geq 4,$ $2n + 3 \leq 2^n$

Proof: I will verify that the hypothesis is true for at least one value of n $\in$ N

Consider n = 4 to get that $2(4) + 3 = 11$ and $2^4 = 16$ and clearly $2(4) + 3 \leq 2^4$.

Now assume the hypothesis is true from n = 4 up to k so that $2k + 3 \leq 2^k$ and prove true for k + 1 by showing that $2(k+1) + 3 \leq 2^{k+1}$.

Consider the k + 1 term $2(k+1) + 3 = (2k + 3) + 2 \leq 2^k + 2 \leq 2^k + 2^k = 2(2^k) = 2^{k+1}$ by the inductive assumption and since $k > 3$ and $2 < 2^k$.

Having assumed true up to k and proved true for k + 1, the hypothesis is true for all natural numbers greater than 3.

Theorem 5: A jigsaw puzzle that has n pieces can be completed in using n – 1 fits.

Definition: A "fit" is defined to be one more puzzle piece added to already assembled puzzle pieces at a given point in time.

Proof: To verify that the hypothesis is true for at least one value of n, consider n = 1. Since there is just one piece there are no fits which gives us that when n = 1 pieces

then are n – 1 = 1-1 = 0 fits. If n = 2 pieces then 1 fit is required which means n – 1 = 2 – 1 = 1 fit is valid.

Now assume that for k pieces there are k – 1 fits and prove that for k + 1 pieces there will be

(k + 1) – 1 = k.

I we consider the k + 1 term, we have k + 1 pieces. This means that we have k pieces and one additional new piece. We have assumed that for the first k terms we will have k – 1 fits. When we add one more piece, we need 1 additional fit to give (k – 1) + 1 = k fits.

Since we assumed true up to k and proved true for k + 1, the hypothesis is true for all natural numbers.