Solutions to practice session 11/30/2018

1. (Inspired by Putnam 1968, B6) Prove that a polynomial with only real roots and all coefficients equal to ± 1 has degree at most 3.

Solution: We may assume that the leading coefficient is $+1$. The sum of the squares of the roots of $x^n + a_1 x^{n-1} + \cdots + a_n$ is $a_1^2 - 2a_2$. The product of the squares of the roots is a_n^2. By the Arithmetic Mean-Geometric Mean inequality we have

$$\frac{a_1^2 - 2a_2}{n} \geq \sqrt[n]{a_n^2}$$

Since the coefficients are ± 1 that inequality is $(1 \pm 2)/n \geq 1$, hence $n \leq 3$. $n = 3$ is optimal; consider $x^3 - x^2 - x + 1$.

2. (Putnam 1974) Call a set of positive integers "conspirational" if no three of them are pairwise relatively prime. What is the largest number of elements in any conspirational subset of integers 1 through 16?

Solution: A conspirational subset of $S = \{1, 2, \ldots, 16\}$ has at most two elements from $T = \{1, 2, 3, 5, 7, 11, 13\}$, so it has at most $2 + 16 - 7 = 11$ numbers. On the other hand all elements of $S \setminus T = \{4, 6, 8, 9, 10, 12, 14, 15, 16\}$ are multiples of either 2 or 3, so adding 2 and 3 we obtain the following 11-element conspirational subset:

$$\{2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16\}.$$

Hence the answer is 11.

3. (From Putnam 1942, problem A-3) Is the following series convergent or divergent?

$$1 + \frac{1}{2} \cdot \frac{19}{7} + \frac{2!}{3^2}\left(\frac{19}{7}\right)^2 + \frac{3!}{4^3}\left(\frac{19}{7}\right)^3 + \frac{4!}{5^4}\left(\frac{19}{7}\right)^4 + \cdots$$

Solution: It is convergent. The ratio test gives

$$\frac{a_{n+1}}{a_n} = \frac{19}{7} \frac{1}{\left(1 + \frac{1}{n}\right)^n}$$

and $19/(7e) < 1$.

4. Prove that

$$|\sin(nx)| \leq n|\sin x|$$

for any real number x and positive integer n.

Solution: The case $n = 1$ is clear. Try $n = 2$. $|\sin(2x)| = 2|\sin x||\cos x| \leq 2|\sin x|$. This suggests to introduce cosines as factors in the proof of the inductive step.
Assume the inequality holds for $n = k$. Then

$$\begin{aligned} |\sin(k+1)x| &= |\sin(kx)\ \cos x + \sin x\ \cos(kx)| \leq |\sin(kx)||\cos x| + |\sin x||\cos(kx)| \\ &\leq |\sin(kx)| + |\sin x| \leq k|\sin x| + |\sin x| = (k+1)|\sin x|. \end{aligned}$$