Solutions to practice session 11/14/2018

1. Prove that $n! < \left(\dfrac{n+1}{2}\right)^n$, for $n = 2, 3\ldots.$ Hint: Use the Arithmetic Mean-Geometric Mean inequality.

Solution:

$$\sqrt[n]{a_1 \cdot a_2 \cdots a_n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n}, \qquad a_k \geq 0.$$

Solution: $n! \leq \left(\dfrac{1+2+\cdots+n}{n}\right)^n = \left(\dfrac{n(n+1)}{2n}\right)^n = \left(\dfrac{n+1}{2}\right)^n.$

The fact that $1 + 2 + \cdots + n = n(n+1)/2$ was used.

2. A spherical, 3−dimensional planet has center at $(0, 0)$ and radius 20. At any point of the surface of this planet, the temperature is $T(x, y, z) = (x + y)^2 + (y - z)^2$ degrees. What is the average temperature of the surface of this planet?

Solution: Consider the function

$$f(x, y, z) = T(x, y, z) + T(y, z, x) + T(z, x, y) = 4x^2 + 4y^2 + 4z^2$$

On the surface of the planet this function is constant and equal to $4 \cdot 20^2 = 1600$, and its average is 1600. Since the equation of a sphere with center at $(0, 0, 0)$ is invariant under rotations, the three terms on the left-side all have the same average value $\overline{T}$. Hence $1600 = 3\overline{T}$, so $\overline{T} = 1600/3$.

3. Show that the equation

$$n_1^4 + n_2^4 + \cdots + n_{14}^4 = 1599$$

has no solutions in nonnegative integers. Hint: Think mod 16.

Solution: Note that $n^4 \equiv 0$ or $1 \pmod{16}$ depending on whether n is even or odd. On the other hand $1599 \equiv 15 \pmod{16}$. So the equation can be satisfied only if the number of odd terms on the LHS is $15 \pmod{16}$. But that is impossible because there are only 14 terms on the LHS.