Solutions to practice session 11/9/2018

1. Call an integer square-full if each of its prime factors occurs to a second power (at least). Prove that there are infinitely many pairs of consecutive square-fulls.

Hint: The numbers 8 and 9 form one such pair. Given a pair $(n, n+1)$ of consecutive square-fulls, find some way to build another pair of consecutive square-fulls.

Solution: If n and $n+1$ are square-full, so are $4n(n+1)$ and $4n(n+1)+1 = (2n+1)^2$ (the latter is a square, so its prime factors appear in even powers).

2. Prove that for any integer $n \geq 1$, $2^{2n} - 1$ is divisible by 3.

Solution: By induction. Base case $n = 1 : 2^{2\cdot 1} - 1 = 4 - 1 = 3$ is divisible by 3. Inductive step: Assume that $n \geq 1$ and $2^{2n} - 1$ is divisible by 3, so $2^{2n} - 1 = 3k$. Then

$$2^{2(n+1)} - 1 = 2^{2n+2} - 1 = 4 \cdot 2^{2n} - 1 = 3 \cdot 2^{2n} + (2^{2n} - 1) = 3 \cdot 2^{2n} + 3k,$$

and this is a multiple of 3.
Other solutions: Since the polynomial $x^{2n} - 1$ has root $x = -1$ (among others), we can factor $x^{2n} - 1 = (x+1)(x^{2n-1} - x^{2n-2} + \cdots - 1)$. Put $x = 2$. Some of you did it by factoring $2^{2n} - 1 = (2^n - 1)(2^n + 1)$ and considering this product $\mod 3$. One of the consecutive numbers $2^n - 1$, 2^n, and $2^n + 1$ must be divisible by 3. Certainly 2^n is not, so one of the other two factors must be divisible by 3.

3. (Putnam 2008, problem B1) What is the maximum number of rational points that can lie on a circle in $\mathbb{R}^2$ whose center is not a rational point? (A *rational point* is a point both of whose coordinates are rational numbers.) Answer: 2

Solution: There are at most two such points. For example, the points $(0, 0)$ and $(1, 0)$ lie on a circle with center $(1/2, x)$ for any real number x, not necessarily rational.

On the other hand, suppose $P = (a, b), Q = (c, d), R = (e, f)$ are three rational points that lie on a circle. The midpoint M of the side PQ is $((a + c)/2, (b + d)/2)$, which is again rational. Moreover, the slope of the line PQ is $(d - b)/(c - a)$, so the slope of the line through M perpendicular to PQ is $(a - c)/(b - d)$, which is rational or infinite.

Similarly, if N is the midpoint of QR, then N is a rational point and the line through N perpendicular to QR has rational slope. The center of the circle lies on both of these lines, so its coordinates (g, h) satisfy two linear equations with rational coefficients, say $Ag + Bh = C$ and $Dg + Eh = F$. Moreover, these equations have a unique solution. That solution must then be

$$g = (CE - BD)/(AE - BD)$$
$$h = (AF - BC)/(AE - BD)$$

(by elementary algebra, or Cramer's rule), so the center of the circle is rational.

For more details go to http://amc.maa.org/a-activities/a7-problems/putnamindex.shtml and download the solution to problem B1-2008.

4. Find the remainder when you divide $x^{81} + x^{49} + x^{25} + x^9 + x$ by $x^3 - x$.

Solution: Assume the quotient is $q(x)$ and the remainder is $r(x) = ax^2 + bx + c$. Plugging in the values $x = -1$, 0, 1 we get $r(-1) = -5$, $r(0) = 0$, $r(1) = 5$. Therefore $a = c = 0$, $b = 5$, and thus the remainder is $r(x) = 5x$.

5. Let $a_n = 10 + n^2$ for $n \geq 1$. For each n, let d_n denote the gcd of a_n and a_{n+1}. Find the maximum value of d_n as n ranges through the positive integers.

Solution: The answer is 41. Use

$$gcd(a_n, a_{n+1}) = gcd(a_n, a_{n+1} - a_n) = gcd(n^2 + 10, 2n + 1) = \cdots$$

(since $2n + 1$ is odd we can multiple the other argument by 4 without altering the gcd).

$$\cdots = gcd(4n^2 + 40, 2n + 1) = gcd((2n + 1)(2n - 1) + 41, 2n + 1) = gcd(41, 2n + 1) \leq 41.$$

The maximum is attained at $n = 20$.