Practice session 11/9/2018

1. Call an integer square-full if each of its prime factors occurs to a second power (at least). Prove that there are infinitely many pairs of consecutive square-fulls.

Hint: The numbers 8 and 9 form one such pair. Given a pair $(n, n+1)$ of consecutive square-fulls, find some way to build another pair of consecutive square-fulls.

2. Prove that for any integer $n \geq 1$, $2^{2n} - 1$ is divisible by 3.

3. (Putnam 2008, problem B1) What is the maximum number of rational points that can lie on a circle in $\mathbb{R}^2$ whose center is not a rational point? (A *rational point* is a point both of whose coordinates are rational numbers.) Answer: 2

4. Find the remainder when you divide $x^{81} + x^{49} + x^{25} + x^9 + x$ by $x^3 - x$.

5. Let $a_n = 10 + n^2$ for $n \geq 1$. For each n, let d_n denote the gcd of a_n and a_{n+1}. Find the maximum value of d_n as n ranges through the positive integers.