Solutions to practice session 11/2/2018

1. (Leningrad Mathematical Olympiad, 1988) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, with $f(x) \cdot f(f(x)) = 1$ for all $x \in \mathbb{R}$. If $f(1000) = 999$, find $f(500)$.
Set $y = f(x)$. Then $f(y) = 1/y$ for all $y \in f(\mathbb{R})$. Hence $f(999) = 1/999$. Since f is continuous it takes all values between $1/999$ and 999. So $500 \in f(\mathbb{R})$. Hence $f(500) = 1/500$.

2. If $a, b, c \geq 0$, prove that $\sqrt{3(a+b+c)} \geq \sqrt{a} + \sqrt{b} + \sqrt{c}$.

Use the Schwarz inequality.

3. If p is an odd number not divisible by 3, then it is congruent to ± 1 modulo 6.
(a) Suppose that $p = 2k + 1 = 3s + 1$. Then $2k = 3s$, so s is even, $s = 2p$. Then $p = 6p + 1$ and so $p \equiv 1 \pmod 6$.
(b) Suppose that $p = 2k + 1 = 3s + 2$. Then $2k = 3s + 1$, which implies that $(k, s) = w(3, 2) + (-1, -1)$. So $k = 3w - 1$ and so $p = 6w - 1$. This is the case when $p \equiv -1 \pmod 6$.

4. Suppose $p > 3$ is prime. Show that $p^2 + 2$ cannot be prime. Hint: Use #3.

For $p > 3$ we have that p is odd and not divisible by 3. So it is congruent to ± 1 modulo 6. This implies that $p^2 + 2 \equiv 3 \bmod 6$. So $p^2 + 2$ is a multiple of 3 and cannot be prime.

5. Define a *domino* to be a 1×2 rectangle. In how many ways can an $n \times 2$ rectangle be tiled by dominoes?
Let x_n be the number of tilings of an $n \times 2$ rectangle. We have $x_1 = 1$, $x_2 = 2$. For $n \geq 3$ we can place the bottom domino horizontally and tile the rest of the rectangle in x_{n-1} ways, or we can place two vertical dominoes at the bottom and tile the rest in x_{n-2} ways. So $x_n = x_{n-1} + x_{n-2}$. The solution is the shifted Fibonacci sequence, $x_n = F_{n+1}$.