**Practice problems 10/17/2018**

1. (VT Regional Math Contest) A walker and a jogger travel along the same straight line in the same direction. The walker walks at one meter per second, while the jogger runs at two meters per second. The jogger starts one meter in front of the walker. A dog starts with the walker, and then runs back and forth between the walker and the jogger with constant speed of three meters per second. Let $f(n)$ meters denote the total distance travelled by the dog when it has returned to the walker for the $n$ th time (so $f(0) = 0$). Find a formula for $f(n)$.
The answer is $f(n) = 3 * \frac{3}{2}(1 + 5/2 + (5/2)^2 + \cdots (5/2)^{n-1}) = \frac{9((5/2)^n-1)}{2(5/2-1)} = 3((5/2)^n - 1)$.
One way to do see this is by a graphical construction (to be inserted later).

2. (Putnam 1958) If $a_0, a_1, \ldots, a_n$ are real numbers satisfying

$$\frac{a_0}{1} + \frac{a_1}{2} + \cdots + \frac{a_n}{n+1} = 0,$$

show that the equation $a_0 + a_1 x + \cdots a_n x^n = 0$ has at least one real root.

Hint: Consider an integral of $f(x) = a_0 + a_1 x + \cdots + a_n x^n$.
We have

$$\int_0^1 f(x)dx = \frac{a_0}{1} + \frac{a_1}{2} + \cdots + \frac{a_n}{n+1} = 0.$$

Now apply the Mean Value Theorem for integrals: $\exists\, \xi \in (0,1)$ such that $f(\xi) = \int_0^1 f(x)dx = 0$.
3. Prove that from ten distinct two-digit numbers, one can always choose two disjoint non-empty subsets, so that their elements have the same sum. Hint: pigeonhole principle.

(IMO 1972) A set of 10 elements has $2^{10} - 1 = 1023$ non-empty subsets. The possible sums of at most ten two-digit numbers cannot be larger than $10 \cdot 99 = 990$. There are more subsets than possible sums, so two different subsets $S_1$ and $S_2$ must have the same sum. If $S_1 \cap S_2 = \emptyset$, we are done. Otherwise remove the common elements and we get two disjoint subsets with the same sum.

4. Evaluate

$$I = \int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(3+x)}} dx$$

5. Compute $\lim_{n\to\infty}\left\{\frac{1}{n}+\frac{1}{n+1}+\cdots+\frac{1}{2n-1}\right\}$.
Hint: Use integrals to estimate the sum from above and below.

**Solution:** Let $s_n=\frac{1}{n}+\frac{1}{n+1}+\cdots+\frac{1}{2n-1}$. Then (if $n\geq 2$)

$$s_n\leq\int_{n-1}^{2n-1}\frac{dx}{x}=\ln x|_{n-1}^{2n-1}=\ln(2n-1)-\ln(n-1)=\ln\left(\frac{2n-1}{n-1}\right)$$

while, on the other hand,

$$s_n\geq\int_{n}^{2n}\frac{dx}{x}=\ln x|_{n}^{2n}=\ln(2n)-\ln(n)=\ln 2.$$

Hence $\ln 2\leq s_n\leq\ln[(2n-1)/(n-1)]$. Taking $n\to\infty$ gives $\ln 2$ for the right-hand side, hence the given limit is $\ln 2$.