Practice problems 10/17/2018

1. (VT Regional Math Contest) A walker and a jogger travel along the same straight line in the same direction. The walker walks at one meter per second, while the jogger runs at two meters per second. The jogger starts one meter in front of the walker. A dog starts with the walker, and then runs back and forth between the walker and the jogger with constant speed of three meters per second. Let $f(n)$ meters denote the total distance travelled by the dog when it has returned to the walker for the n th time (so $f(0) = 0$). Find a formula for $f(n)$.

2. (Putnam 1958) If $a_0, a_1, \ldots, a_n$ are real numbers satisfying

$$\frac{a_0}{1} + \frac{a_1}{2} + \cdots + \frac{a_n}{n+1} = 0,$$

show that the equation $a_0 + a_1 x + \cdots a_n x^n = 0$ has at least one real root.

3. Prove that from ten distinct two-digit numbers, one can always choose two disjoint non-empty subsets, so that their elements have the same sum. Hint: pigeonhole principle.

4. Evaluate

$$I = \int_2^4 \frac{\sqrt{\ln(9 - x)}}{\sqrt{\ln(9 - x)} + \sqrt{\ln(3 + x)}} dx$$

5. Compute $\displaystyle \lim_{n \to \infty} \left\{ \frac{1}{n} + \frac{1}{n + 1} + \cdots + \frac{1}{2n - 1} \right\}$.