Equivariant Quantum Schubert Calculus

Leonardo C. Mihalcea

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Classical cohomology

Notations:

1. $X = Gr(p, m)$, the Grassmannian of subspaces of dimension p in $\mathbb{C}^m$.

2. $D(p, m-p)$ the $p \times (m-p)$ rectangle.

- $H^*(X)$ - a graded $\mathbb{Z}-$algebra with a $\mathbb{Z}$-basis consisting of Schubert classes σ_λ.

  Here $\lambda = (\lambda_1, ..., \lambda_p)$ varies over the partitions included in $D(p, m-p)$ and the degree of σ_λ is $|\lambda| = \lambda_1 + ... + \lambda_p$.

- multiplication:

$$\sigma_\lambda \cdot \sigma_\mu = \sum c_{\lambda,\mu}^\nu \sigma_\nu$$

  where $c_{\lambda,\mu}^\nu$ is the Littlewood-Richardson coefficient (abbreviated LR).

Quantum cohomology

Notation: $QH^*(X)$.

- $QH^*(X)$ is a graded $\mathbb{Z}[q]-$algebra, where q is an indeterminate of degree m.

- it has a $\mathbb{Z}[q]-$basis $\{\sigma_\lambda\}$ where λ varies over the partitions included in $D(p, m-p)$.

- multiplication:

$$\sigma_\lambda \star \sigma_\mu = \sum_{d \geqslant 0} \sum_\nu q^d c^\nu_{\lambda,\mu}(d) \sigma_\nu$$

  where $c^\nu_{\lambda,\mu}(d)$ is the (3-point,genus 0) Gromov-Witten (GW) invariant, which counts the number of rational curves of degree d passing through general translates of Schubert varieties Ω_λ, Ω_μ and $\Omega_{\nu^\vee}$ (where $\nu^\vee$ is the partition dual to ν).

Equivariant cohomology

- $T \simeq (\mathbb{C}^*)^m$ acts on X by the action induced by the $Gl(m)-$action.

- $H_T^*(pt)$ (the $T-$equivariant cohomology of a point), is equal to the polynomial ring $\mathbb{Z}[t]$ where $t = (t_1, ..., t_m)$.

- $H_T^*(X)$ is a graded $\mathbb{Z}[t]-$algebra, with a $\mathbb{Z}[t]-$basis

$$\{\sigma_\lambda^T\}_{\lambda \subset D(p,m-p)}.$$

- multiplication:

$$\sigma_\lambda^T \cdot \sigma_\mu^T = \sum_\nu c_{\lambda,\mu}^\nu(t) \sigma_\nu^T$$

  where $c_{\lambda,\mu}^\nu(t)$ are homogeneous polynomials in $\mathbb{Z}[t]$ of degree $|\lambda| + |\mu| - |\nu|$.

- if $|\lambda| + |\mu| - |\nu| = 0$ one recovers the classical Littlewood-Richardson coefficients.

Equivariant quantum cohomology

Notation: $QH^*_T(X)$.

- $QH^*_T(X)$ is a graded $\mathbb{Z}[t][q]-$algebra, where q is an indeterminate of degree m.

- it has a $\mathbb{Z}[t][q]-$basis $\{\sigma_\lambda\}_{\lambda\subset D(p,m-p)}$.

- multiplication:

$$\sigma_\lambda \circ \sigma_\mu = \sum_{d\geqslant 0}\sum_{\nu} q^d c^{\nu}_{\lambda,\mu}(d;t)\sigma_\nu$$

  where $c^{\nu}_{\lambda,\mu}(d;t)$ is the (3-point, genus 0) *equivariant GW-invariant* (Givental-Kim).

- $c^{\nu}_{\lambda,\mu}(d;t)$ is a homogeneous polynomial in Λ of degree $|\lambda|+|\mu|-|\nu|-md$.

The coefficient $c^{\nu}_{\lambda,\mu}(d;t)$ when $d = 0$.

If $d = 0$ then

$$c^{\nu}_{\lambda,\mu}(0;t) = c^{\nu}_{\lambda,\mu}(t)$$

where $c^{\nu}_{\lambda,\mu}(t)$ is the equivariant coefficient.

Properties of the equivariant coefficients:

- $c^{\nu}_{\lambda,\mu}(t) \in \mathbb{Z}_{\geqslant 0}[t_1 - t_2, ..., t_{m-1} - t_m]$ (W. Graham [year], for any G/P).

- A closed, positive formula for $c^{\nu}_{\lambda,\mu}(t)$ (in the sense above) is known, in term of weighted *puzzles* (A. Knutson - T. Tao [year]).

The case $|\lambda| + |\mu| = |\nu| + md$

This is the case when the *polynomial degree* of $c^{\nu}_{\lambda,\mu}(d;t)$ is equal to zero. Then

$$c^{\nu}_{\lambda,\mu}(d;t) = c^{\nu}_{\lambda,\mu}(d)$$

where $c^{\nu}_{\lambda,\mu}(d)$ is the GW invariant. There are several algorithms to compute these invariants:

- the quantum Pieri and Giambelli formulae of A. Bertram [year].
- rim-hook algorithm of A. Bertram - W. Fulton - I. C.-Fontanine
- the reduction to two-step flag manifolds of A. Buch - A. Kresch - H. Tamvakis
- the toric tableau approach of A. Postnikov.
- using I. Coskun's degenerations.

Vanishing of certain coefficients $c^{\nu}_{\lambda,\mu}(d;t)$

The coefficients for which both $d > 0$ and $|\lambda| + |\mu| - |\nu| - md > 0$ are called **mixed**.

Lemma 1 *Let λ, μ, ν be three partitions included in $p \times (m-p)$ rectangle and let d be a positive integer. Suppose that $|\lambda| + d^2 > |\nu| + md$. Then $c^{\nu}_{\lambda,\mu}(d;t) = 0$.*

The lemma implies the vanishing of all mixed coefficients of the form $c^{\nu}_{\lambda,(1)}(d;t)$.

Proof: $d > 0$ and $c^{\nu}_{\lambda,(1)}(d;t)$ mixed implies that

$$|\lambda| + d^2 \geqslant |\lambda| + 1 > |\nu| + md.$$

This implies an equivariant quantum Pieri-Chevalley formula.

Equivariant quantum Pieri-Chevalley formula

Theorem 1 *The following formula holds in* $QH_T^*(X)$*:*

$$\sigma_\lambda \circ \sigma_{(1)} = \sum_{\mu \to \lambda} \sigma_\mu + c_{\lambda,(1)}^{\lambda}(t)\sigma_\lambda + q\sigma_{\lambda^-}$$

where

$$c_{\lambda,(1)}^{\lambda}(t) = \sum_{i=1}^{p} t_{m-p+i-\lambda_i} - \sum_{j=m-p+1}^{m} t_j.$$

- here $\mu \to \lambda$ means that $\lambda \subset \mu$ and $|\mu| = |\lambda| + 1$.

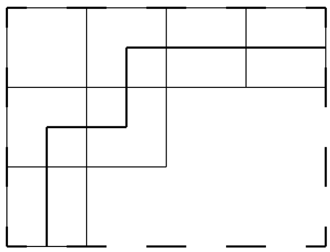
- λ^- is obtained from λ by removing $m-1$ boxes from its border rim. The last term is omitted if λ^- does not exist.

No mixed terms!

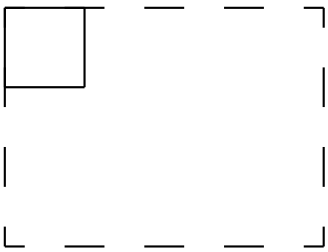
Examples of λ^-

$p = 3, m = 7$

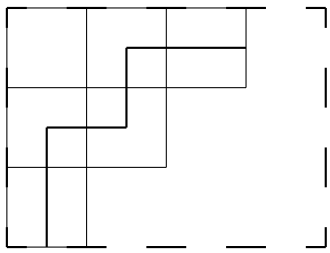
λ



λ^-



λ



λ^-does not exist.

An algorithm

In the next result $c^{\nu}_{\lambda,\mu}(d;t)$ are just homogeneous *rational* functions (not necessarily polynomials) of degree $|\lambda| + |\mu| - |\nu| - md$.

Theorem 2 *The coefficients $c^{\nu}_{\lambda,\mu}(d;t)$ are determined (algorithmically) by:*

(a) (multiplication by (0)*)*

$$c^{\lambda}_{\lambda,(0)}(d;t) = \begin{cases} 1 & \textit{if } d = 0 \\ 0 & \textit{otherwise} \end{cases}$$

(b) (commutativity) $c^{\nu}_{\lambda,\mu}(d;t) = c^{\nu}_{\mu,\lambda}(d;t)$

(c) (special associativity)

$$\sigma_{(1)} \circ (\sigma_\lambda \circ \sigma_\mu) = (\sigma_{(1)} \circ \sigma_\lambda) \circ \sigma_\mu \qquad (1)$$

for any $\lambda \neq \mu$.

A recurrence formula

Let

$$F_{\nu,\lambda}(t) \;=\; c^{\nu}_{(1),\nu}(t) - c^{\lambda}_{(1),\lambda}(t)$$

Given EQ Pieri-Chevalley the special associativity equation (1) is equivalent to:

$$F_{\nu,\lambda}(t)c^{\nu}_{\lambda,\mu}(d;t) = \Big(\sum_{\delta\to\lambda} c^{\nu}_{\delta,\mu}(d;t) - \sum_{\nu\to\zeta} c^{\zeta}_{\lambda,\mu}(d;t)\Big) + \Big(c^{\nu}_{\lambda^-,\mu}(d-1;t) - c^{\nu^+}_{\lambda,\mu}(d-1;t)\Big)$$

for any partitions λ, μ, ν and any nonnegative integer d.

- ν^+ is the partition obtained from ν by adding $m-1$ boxes to the rim-hook.

This formula, in the equivariant setting, was used by Knutson-Tao to derive their puzzle formula for $c^{\nu}_{\lambda,\mu}(t)$.

The induction

Recall the formula

$$F_{\nu,\lambda}(t)c^{\nu}_{\lambda,\mu}(d;t) = \Big(\sum_{\delta\to\lambda} c^{\nu}_{\delta,\mu}(d;t) - \sum_{\nu\to\zeta} c^{\zeta}_{\lambda,\mu}(d;t)\Big) + \Big(c^{\nu}_{\lambda^-,\mu}(d-1;t) - c^{\nu^+}_{\lambda,\mu}(d-1;t)\Big)$$

for any partitions λ, μ, ν such that $\lambda \neq \nu$. We use double induction:

- ascending on d.
- descending on the polynomial degree.

For a fixed d, it remains to investigate the cases when $\lambda = \mu = \nu$ and when $c^{\nu}_{\lambda,\mu}(d;t)$ has the maximum polynomial degree.

Case $\lambda \subsetneq \nu$

In the recurrence formula, the coefficient $c^{\nu}_{\lambda,\mu}(d;t)$ is determined by

- coefficients of smaller degree d.

- coefficients of the same degree d, but with larger λ.

- coefficients of the same degree d, but with smaller ν.

This implies that:

Lemma 2 *The coefficients $c^{\nu}_{\lambda,\mu}(d;t)$ such that either λ or μ is not included in ν are determined by those of degree $d-1$, or are equal to zero if $d=0$.*

An example

Take $X = Gr(2,4)$. Want to compute $c^{(2)}_{(2),(2)}(1;t)$. The recurrence formula yields

$$c^{(2)}_{(1),(2)}(1;t) = \frac{c^{(2)}_{(2),(2)}(1;t) + c^{(2)}_{(1,1),(2)}(1;t)}{T_1 - T_2} - \frac{c^{(1)}_{(1),(2)}(1;t)}{T_1 - T_2} + \text{ (deg } d = 0 \text{ terms)}$$

$$c^{(2)}_{(0),(2)}(1;t) = \frac{c^{(2)}_{(1),(2)}(1;t)}{T_1 - T_2} - \frac{c^{(1)}_{(0),(2)}(1;t)}{T_1 - T_2} + \text{ (deg 0)}$$

- $c^{(2)}_{(0),(2)}(1;t) = 0$ by hypothesis.

- $c^{(2)}_{(1,1),(2)}(1;t)$, $c^{(1)}_{(1),(2)}(1;t)$ and $c^{(1)}_{(0),(2)}(1;t)$ can be reduced to a combination of terms of degree $d = 0$.

An algorithm for the coefficients $c^{\lambda}_{\lambda,\lambda}(d;t)$

Let $\alpha \subset \lambda$ (one should think at $\alpha = (0)$ or $\alpha = (1)$). Define a rational function in $\mathbb{Q}[t]$, denoted $R_{\lambda,\alpha}(t)$ as follows:

$$R_{\lambda,\alpha}(t) = \begin{cases} \sum \prod_{i=0}^{l-1} \frac{1}{F_{\lambda,\alpha^{(i)}}(t)} & \text{if } \lambda \neq \alpha \\ 1 & \text{if } \alpha = \lambda \end{cases}$$

In the case $\lambda \neq \alpha$, l denotes the nonnegative integer $|\lambda| - |\alpha|$, and the sum is over all chains of partitions

$$\lambda = \alpha^{(l)} \to \alpha^{(l-1)} \to ... \to \alpha^{(1)} \to \alpha^{(0)} = \alpha.$$

Lemma 3

$$c^{\lambda,d}_{\alpha,\lambda} = R_{\lambda,\alpha}(t) c^{\lambda,d}_{\lambda,\lambda} + \text{ (deg } d-1 \text{ terms)}$$

If $d = 0$ the degree $d-1$ part vanishes.

The case of maximal polynomial degree

Fix d and let $c^{\nu}_{\lambda,\mu}(d;t)$ of maximal polynomial degree. Then $\lambda = \mu = (m-p)^p$ (the full rectangle) and $\nu = (0)$. The recurrence relation yields

$$c^{(0)}_{(m-p)^p,(m-p)^p}(d;t) = \frac{c^{(0)}_{(m-p-1)^{p-1},\mu}(d-1;t)}{F_{(0),(m-p)^p}(t)} - \frac{c^{(m-p,1^{p-1})}_{(m-p)^p,\mu}(d-1;t)}{F_{(0),(m-p)^p}(t)}$$

which shows that the coefficient $c^{(0)}_{(m-p)^p,(m-p)^p}(d;t)$ is determined by coefficients of degree $d-1$, known by induction on d.

A consequence of the algorithm

Corollary 4 *Let $(A, \diamond)$ be a graded, commutative, associative $\mathbb{Z}[t][q]-$algebra with unit such that:*

1. A has an additive $\mathbb{Z}[t][q]-$basis $\{t_\lambda\}$ (graded as usual).

2. The equivariant quantum Pieri holds, i.e.

$$t_\lambda \diamond t_{(1)} = \sum_{\mu \to \lambda} t_\mu + c_{\lambda,(1)}^{\lambda}(t) t_\lambda + q t_{\lambda^-}$$

where the last term is omitted if λ^- does not exist.

Then A is canonically isomorphic to $QH_T^*(X)$, as $\mathbb{Z}[t][q]-$algebras.

Positivity

Theorem 3 *The coefficients* $c^{\nu}_{\lambda,\mu}(d;t)$ *are homogeneous polynomials in* $\mathbb{Z}[t]$ *such that*

$$c^{\nu}_{\lambda,\mu}(d;t) \in \mathbb{Z}_{\geqslant 0}[t_1 - t_2, ..., t_{m-1} - t_m].$$

- Case $d = 0$ (equivariant coefficients), conjectured by D. Peterson, proved by W. Graham.

- The proof of the positivity is purely geometrical. An interesting question is to derive positivity from the algorithm.

Further work

All the results generalize to the case G/P. We have obtained:

- EQ Chevalley - with no mixed terms.
- An algorithm to compute the structure coefficients - it is determined by the same equations (but now one has more divisor classes - one for each simple root not in the Weyl group of P).
- Positivity holds, and it is expressed in terms of negative simple roots.

A consequence in quantum cohomology of G/P

One gets an algorithm to compute the GW-invariants for G/P. So far, these coefficients have been computed using:

- Peterson comparison formula which writes a coefficient $c_{u,v}^{w}(d)$ (u, v, w minimal length representatives for W/W_P) on G/P as a coefficient, of possibly different degree, on G/B.

- polynomial representatives for quantum Schubert classes.

- Quantum Chevalley formula.

  (On G/B the (quantum) cohomology is generated by divisors).

Except for the EQ Chevalley, we don't use any of these.

Possible directions

- A. Knutson - T.Tao prove that their puzzles satisfy the equivariant restriction of the recurrence formula (X- Grassmannian). Are there EQ puzzles ?

- Find polynomial representatives for the EQ Schubert classes. In type A, A. Kirillov proposes some double Schubert polynomials, but one has to check if they satisfy the EQ Chevalley formula.