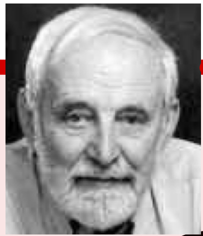
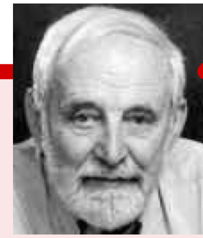
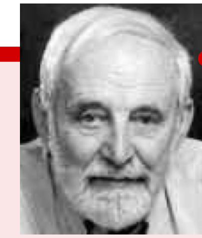
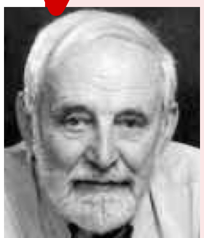
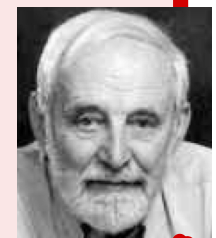
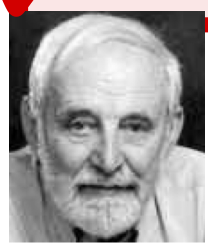
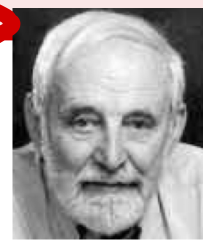
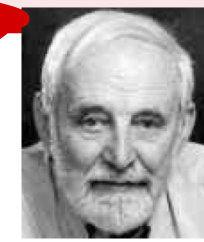




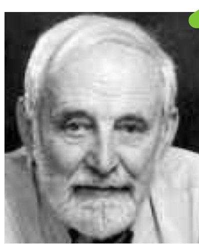


Bott Periodicity, algebro-geometrically

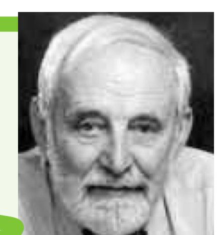


Monday September 27, 2021



Schubert Seminar



joint work with

Hannah Larson

and

Jim Bryan

at a crossroads of mathematics

What is Bott Periodicity?

- one of the fundamental results in topology and K-theory

- so fundamental that it looks like it comes from algebraic geometry

You may have heard of it in the following context:

Let $O(\infty)$ be the "infinite orthogonal group"

$$\pi_0(O(\infty)) = \mathbb{Z}/2$$

$$\pi_1(O(\infty)) = \mathbb{Z}/2$$

$O(100)$ "approximates" $O(\infty)$.

$$\pi_2(O(\infty)) = 0$$

$$\pi_3(O(\infty)) = \mathbb{Z}$$

$$\pi_4(O(\infty)) = 0$$

$$\pi_5(O(\infty)) = 0$$

$$\pi_6(O(\infty)) = 0$$

$$\pi_7(O(\infty)) = \mathbb{Z}$$ and then it repeats mod 8.

(and similarly for $Sp(\infty)$)

More fundamentally,

(don't worry, I'll explain soon)

$$\begin{tikzcd} BU \arrow[r, bend right, "\Omega'"'] & U \arrow[l, bend right, "\Omega'"'] \end{tikzcd}$$

"based loops", not differentials

$$\begin{tikzcd}
& O/U \arrow[dl, "\Omega'"'] & \\
U/Sp \arrow[d, "\Omega'"'] & & O \arrow[ul, "\Omega'"'] \\
BSp \arrow[d, "\Omega'"'] & & BO \arrow[u, "\Omega'"'] \\
Sp \arrow[dr, "\Omega'"'] & & U/O \arrow[u, "\Omega'"'] \\
& Sp/U \arrow[ur, "\Omega'"'] &
\end{tikzcd}$$

Plan for the hour:

first third:

"complex" Bott periodicity:

$$\Omega^2(BU) \xrightarrow{\sim} BU$$

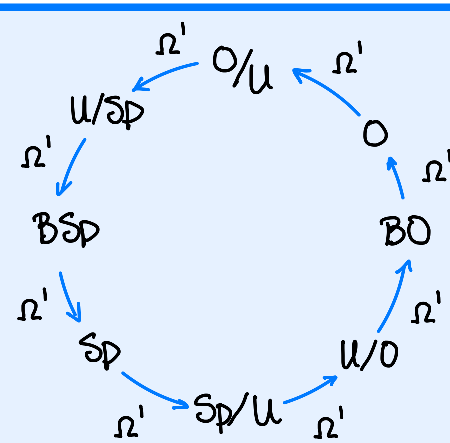
statement and proof

with Hannan Larson

second third: defining the words in the statement and proof

final third:

"real" Bott periodicity



with Jim Bryan

Our first goal:

Bott Periodicity, take one

$$\begin{tikzcd} BU \arrow[r, bend left, "\Omega^1"] & U \arrow[l, bend left, "\Omega^1"] \end{tikzcd}$$

Cheerful Fact

$GL_{\mathbb{C}}(n)$ deformation retracts onto $U(n)$.

$\in GL(n)$

Reason: $M = QR$

upper triangular

unitary

(Use Gramm-Schmidt with the Hermitian inner product to turn the columns of M into a Hermitian basis.)

Homotope R to I.

What do I mean by GL?

$$GL(1) \hookrightarrow GL(2) \hookrightarrow GL(3) \hookrightarrow \cdots$$

$$\begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \qquad \begin{pmatrix} \boxed{\begin{matrix} \cdot & \cdot \\ \cdot & \cdot \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix}$$

GL: — colimit of GL(n)'s,

— think "infinite matrix, mostly identity"

$$\begin{pmatrix} 3 & 1 & 0 & 0 & \cdots \\ 2 & 0 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 1 & \\ & & \vdots & & \ddots \end{pmatrix}$$

— think "finite matrix, agnostic about the size"

$$\begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}$$

$\uparrow$

What is $BGL(n)$?

the moduli space of rank n vector bundles

Given X, the data of a rank n vector bundle on X is the data of a morphism $X \to BGL(n)$.

$$\begin{tikzcd}
pt \arrow[r] & BGL(n) \\
X \arrow[u] \arrow[ur] &
\end{tikzcd}$$

$pt \to BGL(n)$: a vector bundle on pt — a vector space U, say dim n.

X: a trivial vector bundle "$X \times U \to X$"

$BGL(n)$ is an Artin stack

Practice with the concept:

$BGL(n) \longrightarrow BGL(n+1)$ translates to:

take a rank n vector bundle and $\oplus$ a trivial bundle

$$\mathcal{E} \longmapsto \mathcal{E} \oplus \theta$$

Bott Periodicity:

$$\beta: \boxed{\Omega^2(BGL)} \xrightarrow{\sim} \boxed{BGL} \cancel{\times \mathbb{Z}}$$

What does Ω^2 mean?

"loops-2 with base point maps to ..."

"sphere with a marked point maps to ..."

"$\mathbb{CP}^1$ with a marked point p_∞ maps to ..."

"$\mathbb{P}^1$ with the marked point p_∞ maps to..."

(For experts: $\Omega^2 BGL$ "is" the affine Grassmannian.)

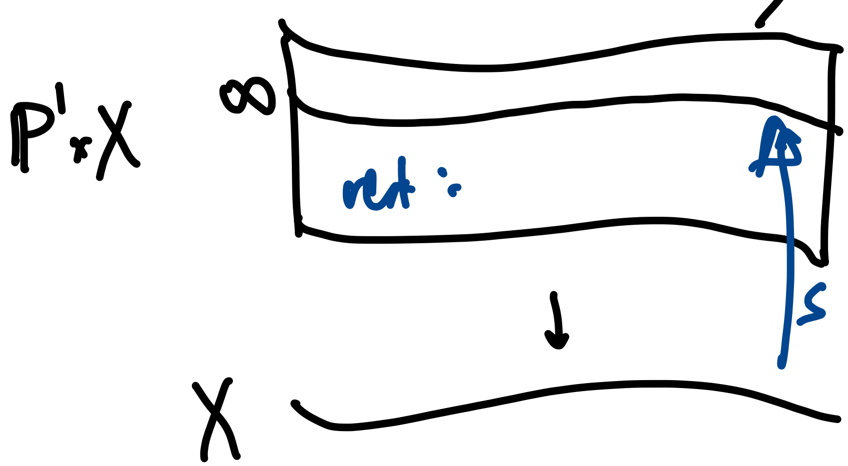
Goal: $\boxed{\text{define } \beta,}$ and show it is an isomorphism

Translation: I give you a map $X \to \Omega^2(BGL)$

$\downarrow \beta$

you tell me a map $X \to BGL$.

Translation: $\mathcal{E}$ vector bundle



tell me a bundle on X

$\boxed{s^*\mathcal{E} = \text{trivial bundle}}$

Answer if $\pi_* \mathcal{E}$ if $\mathcal{E}$ is "nonnegative"

$\equiv$ generated by global sections

$\equiv$ $\mathcal{E} = \underbrace{\mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_n)}$ $\quad a_i \geq 0$

But 1) we were agnostic about the rank of $\mathcal{E}$. What if we pushed forward $\mathcal{E} \oplus \mathcal{O}$ instead?

Answer: $\boxed{\pi_*(\mathcal{E} \oplus \mathcal{O})} = \boxed{\pi_* \mathcal{E}} \oplus \boxed{\pi_* \mathcal{O}}$

$= \boxed{\pi_* \mathcal{E}} \oplus \boxed{\mathcal{O}}$

2) More seriously: What if $\mathcal{E}$ weren't positive?

Answer: twist it until it is!

Use $\pi_* \mathcal{E}(k)$ instead.

To make this work:

Proposition (false as stated):

If $\mathcal{E}(-1)$ is nonnegative then $\pi_* \mathcal{E}(-1) = \pi_* \mathcal{E}$.

Proof:

$$0 \to \mathcal{E}(-1) \to \mathcal{E} \to \mathcal{E}|_{P_\infty} \to 0.$$

$$0 \to \pi_* \mathcal{E}(-1) \to \pi_* \mathcal{E} \to \underbrace{V}_{\text{trivial bundle.}} \to 0$$

$\Omega^2 BGL \to BGL$

Magic:

This construction is basically reversible!

Why? How?

Leonardo Mihalcea to the rescue!

$\boxed{\text{Strømme}}$

Here come three pages explaining this.

(Stick with me, or at least come back after.)

Consider the vector bundles $\mathcal{E}$ on $\mathbb{P}^1$, satisfying:

- degree d, rank n $\qquad$ $\mathcal{E} = \bigoplus \mathcal{O}(+\ )$.
- "nonnegative": $h^1(\mathcal{E}(-1)) = 0 \Rightarrow$ $h^0(\mathcal{E}(-1)) = dn$ $\quad (A = H^0(\mathcal{E}(-1))$, $\quad h^0(\mathcal{E}) = (d+1)n$
- "trivialized @ ∞": fiber over ∞ is identified with U (of rank n)

So far: a dense open subset of $\Omega^2_d(BGL_n)$

Now add:

- choice of sections of $\mathcal{E}$ giving U:

$$H^0(\mathcal{E}) \twoheadrightarrow H^0(\mathcal{E}|_\infty) = U$$

(affine bundle)

Central theme of mathematics:

Turn complicated things into linear algebra

Strömme:

We can recover $\mathcal{E}$ from the following linear algebra data.

think: $H^0(\mathcal{E}(-1))$

think: A is a vector space of dimension dn

$\mathcal{E}|_{p_\infty}$ U is a fixed vector space of dimension n.

Choose two linear maps:

$$\begin{tikzcd} A \arrow[loop right, "\alpha"] \arrow[d, "j"] \\ U \end{tikzcd}$$

"quiver"

Let $\underline{A} = A \otimes \mathcal{O}_{\mathbb{P}^1}$, $\underline{U} = U \otimes \mathcal{O}_{\mathbb{P}^1}$ be trivial bundles on $\mathbb{P}^1$

$$\boxed{0 \to \underline{A}(-1) \xrightarrow{\sigma} \underline{A} \oplus \underline{U} \longrightarrow \mathcal{E} \longrightarrow 0}$$

$$\sigma = \begin{pmatrix} \mathrm{Id}_A x_0 - \alpha x_1 \\ j x_1 \end{pmatrix}$$

(open condition: coker σ should be a bundle)

Translation 1: The moduli space of such vector bundles is the space of such matrices (satisfying this explicit open condition) modulo the group $GL(A)$.

Translation 2:

$$\Omega^2_{\geq 0} BGL(n) = \text{open in affine bundle over } BGL(A)$$

Translation 3:

$$0 \to \pi^*(\pi_*\mathcal{E}(-1))(-1) \longrightarrow \pi^*\pi_*\mathcal{E} \longrightarrow \mathcal{E} \longrightarrow 0$$

Part II Making this make sense.

We want to work in a setting with some kind of algebro-geometric spaces

Properties:

- work over $\mathbb{Z}$, or $\mathbb{C}$, or k, or $\mathbb{R}$ (any reasonable base), and be able to change base
- if $/\mathbb{C}$, want to be able to take homotopy types (so as to generalize topological statements)
- also: Chow rings, Hodge theory, Grothendieck ring, étale homotopy, etc.
- we need Artin stacks, and colimits

In a nutshell:

- start with reasonable spaces (smooth irreducible nice Artin stacks, if you care)
- define when a map is "an isomorphism up to high codimension" / "highly connected": a class of morphisms "iso.codim$_k$".

Definition: ① $V \to Y$ vector bundles, $\curvearrowleft$ sections $\Big\}$ $\in$ isocodim k

② $U \hookrightarrow Y$ open embeddings with $\operatorname{codim}(Y \setminus U) \geq k$ $\in$ isocodim k

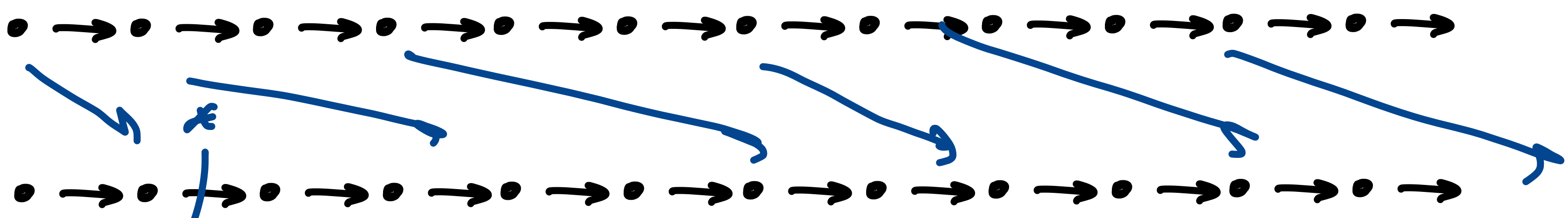
③ $\begin{array}{ccc} X & \longrightarrow & Y \\ & \searrow \; Z \; \swarrow & \end{array}$ 2 out of 3 in isocodim k $\Rightarrow$ 3rd is too.

② objects of our new category $\mathcal{G}_0$ are "Cauchy sequences" (convergent sequences):

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \longrightarrow \cdots$$

$f_n \in \text{isocodim}_{k(n)}$ with $k(n) \to \infty$

③ Morphisms in $\mathcal{G}_0$: maps of Cauchy sequences



commute up to $\mathbb{A}^1$

④ (Whitehead condition)

Maps of Cauchy sequences
that are in isocodim_k $\forall k$
are declared to be isomorphisms

This definition is contrived
to make the proof work without change.

(with Jim Bryan)

Bott Periodicity: the period-eight case.

$/\mathbb{C}$

$\phi : V \to V^\vee \qquad \phi^\vee = \pm \phi.$

$$\begin{tikzcd}
 & O/U \arrow[dl, "\Omega^2"'] & \\
BSp \arrow[dr, "\Omega^2"'] & & BO \arrow[ul, "\Omega^2"'] \\
 & Sp/U \arrow[ur, "\Omega^2"'] &
\end{tikzcd}$$

$\boxed{\Omega^2(Sp_{2n}/U_n)}$

Sp/U: $\quad Sp_{2n}/GL_n$ the isotropic Lagrangian Grassmannian

$\mathbb{P}^1 \to Sp_{2n}/GL_n$

$\mathbb{K}.$

$\mathbb{P}^1$

Maps $\mathbb{P}^1$ to Sp/GL:

$$\boxed{0 \to E^\vee \to \mathcal{U}\oplus\mathcal{U}^\vee \to E \to 0} \quad \text{self-dual}$$

$$\underbrace{\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}}_{\circlearrowleft \phi.}$$

bundles on $\mathbb{P}^1$

From this, I need to tell you a vector space with a <u>symmetric</u> pairing.

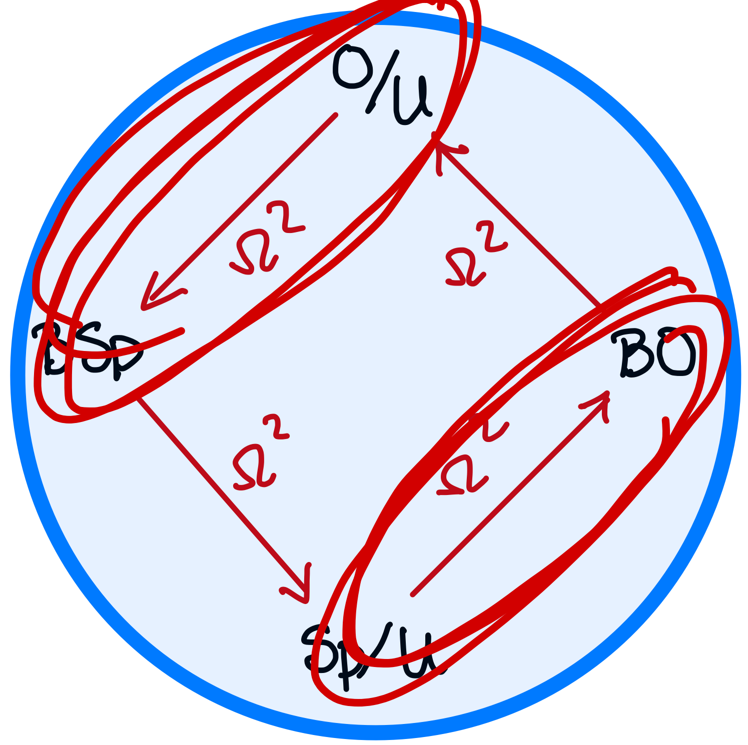
Maps $\mathbb{P}^1$ to Sp/GL:

$$0 \to \mathcal{E}^\vee \to \underline{U}\oplus\underline{U}^\vee \to \mathcal{E} \to 0 \quad \text{self-dual}$$

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

σ

(bundles on $\mathbb{P}^1$)



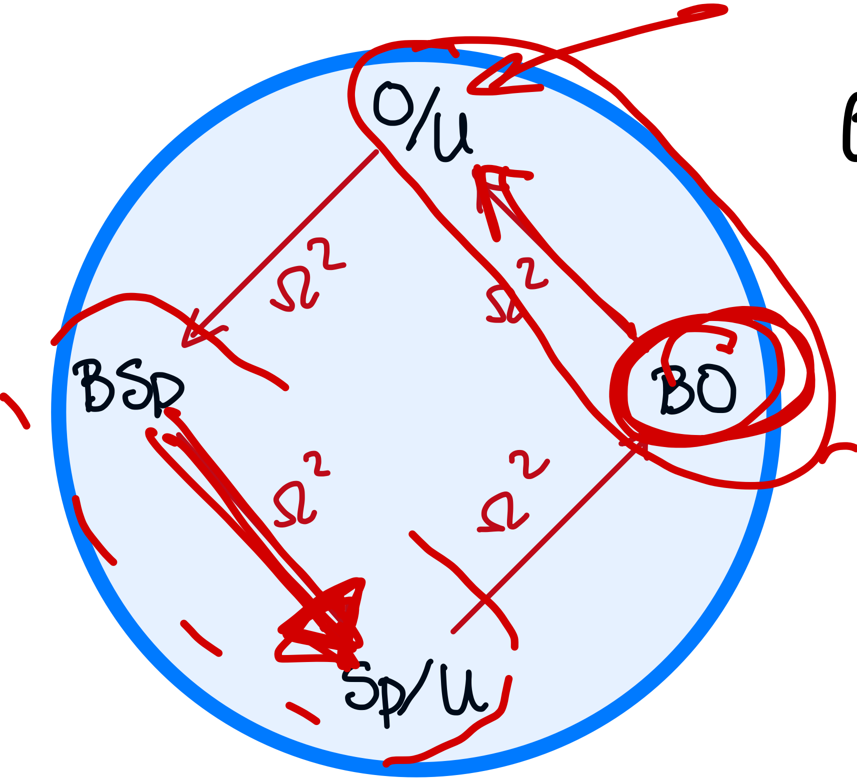
From this, I need to tell you a vector space with a symmetric pairing.

$$0 \to \mathcal{E}^\vee(-1) \to \underline{U}(-1)\oplus\underline{U}^\vee(-1) \to \mathcal{E}(-1) \to 0.$$

Take the long exact sequence in cohomology.

$$0 \to H^0(\mathcal{E}(-1)) \to \underbrace{H^1(\mathcal{E}^\vee(-1))}_{H^0(\mathcal{E}(-1))^\vee} \to \cdots)$$

$$A \to A^\vee$$



Given a map $\mathbb{P}^1 \to BO$ I need to tell you a map to the/an isotropic maximal Grassmannian

$$\begin{array}{c} \mathcal{E} \\ | \\ \mathbb{P}^1 \end{array} \quad \text{rank } n$$

$\mathcal{E} = \mathcal{O}(-3) \oplus \mathcal{O}(3)$

$\phi : \mathcal{E} \xrightarrow{\sim} \mathcal{E}^\vee \qquad \phi^\vee = \phi$

Restrict

On locus where $\mathcal{E}(k)$ is non-negative:

$$0 \to \mathcal{E}(-k-1) \xrightarrow{\cdot x_1^{2k}} \mathcal{E}(k-1) \to \mathcal{E}(k-1)/\mathcal{E}(-k-1) \to 0$$

$$0 \to H^0(\mathcal{E}(k-1)) \to H^0(\ \text{"}\ ) \to H^1(\mathcal{E}(-k-1)) \cong H^0(\mathcal{E}(k-1))^\vee$$

$H^0 = 0$

A

Conclusion:

The maps in Bott Periodicity, which often seem to be thought to need to be defined by analytic means, in fact are completely and naturally algebraic and elementary.