Factorial Schur functions represent equivariant quantum Schubert classes

Leonardo C. Mihalcea

April 17, 2005

Slides available at:
`www.math.lsa.umich.edu/~lmihalce`

Goals

1. Give a presentation by generators and relations of the equivariant quantum cohomology of the Grassmannian.

2. In the given presentation, find polynomial representatives for the EQ Schubert classes.

3. Explain the main result behind the proof: an equivariant quantum Pieri-Chevalley rule.

Classical cohomology ring of the Grassmannian

Let $X = Gr(p, m)$ be the Grassmannian of subspaces of dimension p in $\mathbb{C}^m$.

D denotes the $p \times (m-p)$ rectangle. A partition $\lambda = (\lambda_1, ..., \lambda_p) \subset D$ is given by a sequence $m - p \geqslant \lambda_1 \geqslant \lambda_2 \geqslant ... \geqslant \lambda_p \geqslant 0$ of integers.

$H^*(X)$ is a graded $\mathbb{Z}-$algebra with a $\mathbb{Z}$-basis consisting of Schubert classes $\{\sigma_\lambda\}_{\lambda \subset D}$.

The complex degree of σ_λ is $|\lambda| = \lambda_1 + ... + \lambda_p$.

Multiplication:

$$\sigma_\lambda \cdot \sigma_\mu = \sum c^{\nu}_{\lambda,\mu} \sigma_\nu$$

where $c^{\nu}_{\lambda,\mu}$ is the Littlewood-Richardson coefficient.

Quantum cohomology

$QH^*(X)$ is a graded $\mathbb{Z}[q]-$algebra, where q is an indeterminate of degree m.

$QH^*(X)$ has a $\mathbb{Z}[q]-$basis $\{\sigma_\lambda\}_{\lambda\subset D}$.

Multiplication:

$$\sigma_\lambda \star \sigma_\mu = \sum_{d\geqslant 0} \sum_{\nu} q^d c_{\lambda,\mu}^{\nu,d} \sigma_\nu,$$

where $c_{\lambda,\mu}^{\nu,d}$ is the (3-point,genus 0) **Gromov-Witten invariant** (GW). It counts the number of rational curves of degree d passing through general translates of Schubert varieties Ω_λ, Ω_μ and $\Omega_{\nu^\vee}$, where $\nu^\vee$ is the partition complementary to ν in D.

Equivariant cohomology

$T \simeq (\mathbb{C}^*)^m$ acts on X by the action induced by the $Gl(m)-$action.

The eq. coh. of a point is $\Lambda := \mathbb{Z}[T_1, ..., T_m]$.

$H_T^*(X)$ is a graded $\Lambda-$algebra, with a $\Lambda-$basis $\{\sigma_\lambda^T\}_{\lambda \subset D}$.

Multiplication:

$$\sigma_\lambda^T \cdot \sigma_\mu^T = \sum_\nu c_{\lambda,\mu}^\nu(t) \sigma_\nu^T$$

where the $c_{\lambda,\mu}^\nu(t)$ are homogeneous polynomials in Λ of degree $|\lambda| + |\mu| - |\nu|$.

If $|\lambda| + |\mu| - |\nu| = 0$ then $c_{\lambda,\mu}^\nu(t) = c_{\lambda,\mu}^\nu$.

Equivariant quantum cohomology

$QH^*_T(X)$ is a graded $\Lambda[q]-$algebra, where q is an indeterminate of degree m.

$QH^*_T(X)$ has a $\Lambda[q]-$basis $\{\sigma_\lambda\}_{\lambda\subset D}$.

Multiplication:

$$\sigma_\lambda \circ \sigma_\mu = \sum_{d\geqslant 0}\sum_\nu q^d c_{\lambda,\mu}^{\nu,d}(t)\sigma_\nu$$

where $c_{\lambda,\mu}^{\nu,d}(t)$ is the (3-point, genus 0) *equivariant GW-invariant* (A. Givental-B. Kim '95).

$c_{\lambda,\mu}^{\nu,d}(t)$ is a homogeneous polynomial in Λ of degree $|\lambda|+|\mu|-|\nu|-md$.

Properties of the coefficients $c^{\nu,d}_{\lambda,\mu}(t)$

Proposition 1 (A. Givental - B. Kim '95)

1. *If* $d = 0$ *then*
$$c^{\nu,0}_{\lambda,\mu}(t) = c^{\nu}_{\lambda,\mu}(t).$$

2. *If* $|\lambda| + |\mu| - |\nu| - md = 0$ *then*
$$c^{\nu,d}_{\lambda,\mu}(t) = c^{\nu,d}_{\lambda,\mu}.$$

The coefficients $c^{\nu,d}_{\lambda,\mu}(t)$ for which both $d > 0$ and $|\lambda| + |\mu| - |\nu| - md > 0$ are called **mixed**.

Factorial Jacobi-Trudi determinants

Let $h_1, ..., h_{m-p}$ respectively $e_1, ..., e_p$ be two sets of indeterminates. Define $t = (t_i)_{i\in\mathbb{Z}}$ by:

$$t_i = \begin{cases} T_{m+1-i}, & \text{if } 1 \leqslant i \leqslant m; \\ 0, & \text{otherwise .} \end{cases}$$

Define shifted indeterminates:

$$\tau^{-1}h_i = h_i + (t_{i+p-1} - t_0)h_{i-1},$$

$$\tau^1 e_j = e_j + (t_1 - t_{p-j+2})e_{j-1}.$$

By iterating this process one can define $\tau^{-s}h_i \in \Lambda[h_1, ..., h_{m-p}]$, $\tau^s e_j \in \Lambda[e_1, ..., e_p]$ for $s \in \mathbb{Z}_{\geqslant 0}$.

For $\lambda \subset D$ define

$$s_\lambda(t) = \det(\tau^{1-j}h_{\lambda_i+j-i})_{1\leqslant i,j\leqslant p}$$

$$\widetilde{s}_\lambda(t) = \det(\tau^{j-1}e_{\lambda'_i+j-i})_{1\leqslant i,j\leqslant m-p}$$

where $\lambda' = (\lambda'_1, ..., \lambda'_{m-p})$ is the partition conjugate to λ.

A presentation and EQ Giambelli formula

Theorem 2 *(a) There is a canonical isomorphism of* $\Lambda[q]-$*algebras*

$$\Lambda[q][h_1,...,h_{m-p}]/\langle E_{p+1},...,E_{m-1},E_m+(-1)^{m-p}q\rangle \longrightarrow QH_T^*(X)$$

where

$$E_k=\det(\tau^{1-j}h_{1+j-i})_{1\leqslant i,j\leqslant k}.$$

This isomorphism sends $s_\lambda(t)$ *to the Schubert class* σ_λ.

(b)(Dual version) There is a canonical isomorphism of $\Lambda[q]-$*algebras*

$$\Lambda[q][e_1,...,e_p]/\langle H_{m-p+1},...,H_{m-1},H_m+(-1)^{p}q\rangle \longrightarrow QH_T^*(X)$$

where

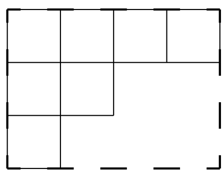
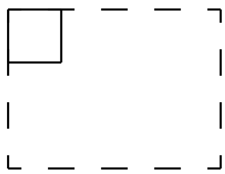
$$H_k=\det(\tau^{j-1}e_{1+j-i})_{1\leqslant i,j\leqslant k}.$$

This isomorphism sends $\widetilde{s}_\lambda(t)$ *to the Schubert class* σ_λ.

An equivariant quantum Pieri-Chevalley formula

In what follows $\mu \to \lambda$ means that $\lambda \subset \mu$ and $|\mu| = |\lambda| + 1$ and λ^- denotes the partition obtained from λ by removing $m-1$ boxes from (resp. to) its border rim.

Example: $p = 3, m = 7$.

λ  λ^- 

Theorem 3 *In $QH_T^*(X)$,*

$$\sigma_\lambda \circ \sigma_{(1)} = \sum_{\mu \to \lambda} \sigma_\mu + c^\lambda_{\lambda,(1)}(t)\sigma_\lambda + q\sigma_{\lambda^-}$$

where

$$c^\lambda_{\lambda,(1)}(t) = \sum_{i=1}^{p} T_{m-p+i-\lambda_i} - \sum_{j=m-p+1}^{m} T_j.$$

The last term is omitted if λ^- is not well-defined.

A characterization of $QH_T^*(X)$

Corollary 4 *Let A be a graded, commutative, associative $\Lambda[q]-$algebra with unit, where the degree of q is defined as usual. Assume that:*

1. A has an additive $\Lambda[q]-$basis $\{s_\lambda\}_{\lambda \in D}$ (graded as usual).

2. The equivariant quantum Pieri-Chevalley formula holds.

Then A is canonically isomorphic to $QH_T^*(X)$, as $\Lambda[q]-$algebras.

Remark: The divisor class $\sigma_{(1)}$ **does not** generate $QH_T^*(X)$.

Idea of the proof

We use the previous corollary. More precisely, we show that:

(a) The polynomials $s_\lambda(t)$ respectively $\widetilde{s}_\lambda(t)$ ($\lambda \subset D$) form a $\Lambda[q]-$basis for the corresponding presentations.

(b) The products $s_\lambda(t) \cdot s_{(1)}(t)$ respectively $\widetilde{s}_\lambda(t) \cdot \widetilde{s}_{(1)}(t)$ are given by the EQ Pieri-Chevalley formula.

Remarks:

1. In the equivariant setting, the factorial Schur functions can be recovered from a certain degeneracy locus formula.

2. The characterization of $QH^*_T(X)$ holds for any homogeneous space G/P. Does the result generalizes to this setting ?

A characterization of $c^{\nu,d}_{\lambda,\mu}(t)$

Theorem 5 *The coefficients $c^{\nu,d}_{\lambda,\mu}(t)$ are uniquely determined by:*

(a) *(homogeneity) $c^{\nu,d}_{\lambda,\mu}(t)$ is homogeneous* rational *function of degree $|\lambda| + |\mu| - |\nu| - md$.*

(b) *(multiplication by unit)*

$$c^{\lambda,d}_{\lambda,(0)}(t) = \begin{cases} 1 & \text{if } d = 0 \\ 0 & \text{otherwise} \end{cases}$$

(c) *(commutativity) $c^{\nu,d}_{\lambda,\mu}(t) = c^{\nu,d}_{\mu,\lambda}(t)$.*

(d) *(recurrence formula) For any $\lambda, \mu, \nu \subset D$ such that $\lambda \neq \nu$,*

$$c^{\nu,d}_{\lambda,\mu}(t) = \Big(\sum_{\delta\to\lambda} c^{\nu,d}_{\delta,\mu}(t) - \sum_{\nu\to\zeta} c^{\zeta,d}_{\lambda,\mu}(t)\Big)\Big/\sum_{i=1}^{p}(T_{m-p+i-\nu_i} - T_{m-p+i-\lambda_i}) + \Big(c^{\nu,d-1}_{\lambda^-,\mu}(t) - c^{\nu^+,d-1}_{\lambda,\mu}(t)\Big)\Big/\sum_{i=1}^{p}(T_{m-p+i-\nu_i} - T_{m-p+i-\lambda_i})$$

Comments about the Theorem

The proof is by an effective algorithm computing $c_{\lambda,\mu}^{\nu,d}(t)$.

The recurrence formula is obtained from the special associativity equation

$$\sigma_{(1)} \circ (\sigma_\lambda \circ \sigma_\mu) = (\sigma_{(1)} \circ \sigma_\lambda) \circ \sigma_\mu$$

using the equivariant quantum Pieri-Chevalley rule.