Curves and positroids in the Grassmannian

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Gromov-Witten varieties

- $X = \mathrm{Gr}(p, m)$ - transitive action of $\mathrm{GL}_m(\mathbb{C})$ acts transitively on X. Take B^+, B^- the upper/lower triangular matrices in $\mathrm{GL}_m(\mathbb{C})$.
- Closures of B^+-orbits $\leftrightarrow$ Schubert varieties Ω_λ
- Closures of B^--orbits $\leftrightarrow$ opposite Schubert varieties Ω_μ^{opp}.
- Fix $d \geq 0$. $\overline{\mathcal{M}}_{0,3}(X, d)$ compactifies the space of maps $f : (\mathbb{P}^1, pt_1, pt_2, pt_3) \to X$ such that $f_*[\mathbb{P}^1] = d[line]$.
- evaluation maps: $\mathrm{ev}_i : \overline{\mathcal{M}}_{0,3}(X, d) \to X$ given by $\mathrm{ev}_i(f) = f(pt_i)$.

Gromov-Witten varieties

Definition: Gromov-Witten variety

$$GW_d(\lambda, \mu) = \mathrm{ev}_1^{-1}\,\Omega_\lambda \cap \mathrm{ev}_2^{-1}\,\Omega_\mu^{opp}$$

Theorem (BCMP)

$GW_d(\lambda, \mu)$ is either empty or it is irreducible and unirational, with rational singularities. (This holds for any G/P.)

Y is unirational: $\exists F : \mathbb{P}^N \dashrightarrow Y$ dominant.
Y has rational singularities if $\exists$ desingularization $F : Z \to Y$ so that $F_*\mathcal{O}_Z = \mathcal{O}_Y$ and $R^i F_*\mathcal{O}_Z = 0, i > 0$.
Definition.

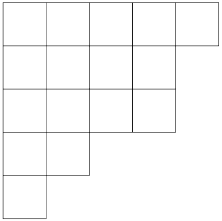
$$\Gamma_d(\lambda, \mu) = \mathrm{ev}_3(GW_d(\lambda, \mu))$$

- This is a subvariety of the Grassmannian;
- It is the union of all rational curves of degree d joining Ω_λ and Ω_μ^{opp}.

Example

$\Gamma_d(\lambda, \emptyset)$ is the union of all rational curves of degree d passing through Ω_λ.

Proposition.[Carrell-Peterson, Fulton-Woodward] $\Gamma_d(\lambda, \emptyset) = \Omega_{\lambda[-d]}$



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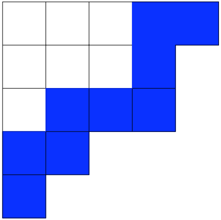

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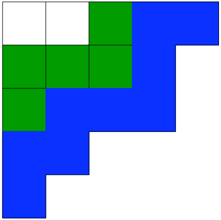

$\lambda[-2]$

Positroids

Definition.[Lusztig, Rietsch, Knutson-Lam-Speyer] Let R be a Richardson variety in the full flag manifold $Fl(n)$. A positroid is a projection $\pi(R)$, where $\pi : \mathrm{Fl}(m) \longrightarrow \mathrm{Gr}(p, m)$ is the projection.

Theorem.[Knutson-Lam-Speyer] Positroid varieties are normal and have rational singularities.

Gromov-Witten positroids

$$\begin{array}{ccc} \{K^{p-d} \subset V \subset S^{p+d}\} & \xrightarrow{\pi_1} & \mathrm{Gr}(p,m) = \{V\} \\ \downarrow \pi_2 & & \\ Y = \{K^{p-d} \subset S^{p+d} \subset \mathbb{C}^m\} & & \end{array}$$

$$\Omega_\lambda \subset \mathrm{Gr}(p,m) \text{ - Schubert variety}$$

$$Y_\lambda = \pi_2(\pi_1^{-1}\Omega_\lambda)$$

$$R_d(\lambda,\mu) = \pi_2^{-1}(Y_\lambda \cap Y_\mu^{opp}) \subset \mathrm{Fl}(p-d,p,p+d;m).$$

$R_d(\lambda,\mu)$ is a Richardson variety and $\pi_1(R_d(\lambda,\mu))$ is a GW positroid.

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Condition DIM. Say $\pi_1(R_d(\lambda,\mu))$ satisfies condition DIM if

$$\dim R_d(\lambda,\mu) = \dim \pi_1(R_d(\lambda,\mu)).$$

Definition of small quantum cohomology

$\mathrm{Gr}(p,m) = \{V \subset \mathbb{C}^m : \dim V = p\}$ - Grassmannian of p-planes in $\mathbb{C}^m$.

- $QH^*(\mathrm{Gr}(p,m))$ is a graded $\mathbb{Z}[q]-$algebra, where $\deg q = m$.
- $QH^*(X)$ has a $\mathbb{Z}[q]-$basis $\{[\Omega_\lambda]\}$ - the Schubert classes.

Multiplication:

$$[\Omega_\lambda] \star [\Omega_\mu] = \sum_{d \geqslant 0} \sum_{\nu} q^d \langle \Omega_\lambda, \Omega_\mu, \Omega_\nu^\vee \rangle_d [\Omega_\nu].$$

- $\langle \Omega_\lambda, \Omega_\mu, \Omega_\nu^\vee \rangle_d$ is the 3 point, genus 0 GW invariant.
- $\langle \Omega_\lambda, \Omega_\mu, \Omega_\nu^\vee \rangle_d$ equals the number of rational curves in X, passing through translates of Schubert varieties Ω_λ, Ω_μ and $\Omega_\nu^\vee$.

Cohomology class of some GW positroids

Theorem.[Buch-Kresch-Tamvakis, Postnikov, Knutson-Lam-Speyer] Assume condition DIM holds. Then:

1. $\Gamma_d(\lambda) = \pi_1(R_d(\lambda, \mu))$ so it is a positroid GW variety.
2. The class of $\Gamma_d(\lambda, \mu) \in H^*(\mathrm{Gr}(p, m))$ is

$$[\Gamma_d(\lambda, \mu)] = \sum \langle [\Omega_\lambda], [\Omega_\mu], [\Omega_\nu]^\vee \rangle_d [\Omega_\nu]$$

Moreover, condition DIM holds $\iff$ q^d appears in $[\Omega_\lambda] \star [\Omega_\mu] \iff \mu^\vee / d / \lambda$ is toric.

K-theory class of GW positroids

Theorem (B-C-M-P)

1. $\Gamma_d(\lambda,\mu) = \pi_1(R_d(\lambda,\mu))$ *is always a positroid GW variety.*
2. *The K-theory class of* $\Gamma_d(\lambda,\mu)$ *is given in terms of K-theoretic GW invariants:*

$$[\mathcal{O}_{\Gamma_d(\lambda,\mu)}] = \sum \langle [\mathcal{O}_{\Omega_\lambda}], [\mathcal{O}_{\Omega_\mu}], [\mathcal{O}_{\Omega_\nu}]^\vee \rangle_d [\mathcal{O}_{\Omega_\nu}]$$

Example. $X = \mathrm{Gr}(2,4), d = 1, \lambda = \mu = (2)$. it is known that $[\Omega_{(2)}] \star [\Omega_{(2)}] = [\mathcal{O}_{(2,2)}]$. No q^1 power, so DIM does not hold!

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$$\Gamma_1((2),(2)) = \Gamma_1(pt, \emptyset) = \Omega_{(1)}$$

This implies:

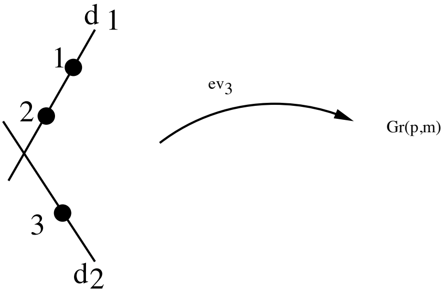
- $\langle [\mathcal{O}_{(2)}], [\mathcal{O}_{(2)}], [\mathcal{O}_\nu]^\vee \rangle_1 = 0$ if $\nu \neq (1)$;
- $\langle [\mathcal{O}_{(2)}], [\mathcal{O}_{(2)}], [\mathcal{O}_{(1)}]^\vee \rangle_1 = 1$.

Rational neighborhoods of GW positroids

In the study of QK(Gr(p, m)) the following variety arises naturally:

$$\Gamma_{d_1}(\lambda,\mu) \subset \Gamma_{d_1,d_2}(\lambda,\mu) \subset \mathrm{Gr}(p,m)$$

the union of all rational curves of degree d_2 passing through $\Gamma_{d_1}(\lambda,\mu)$.



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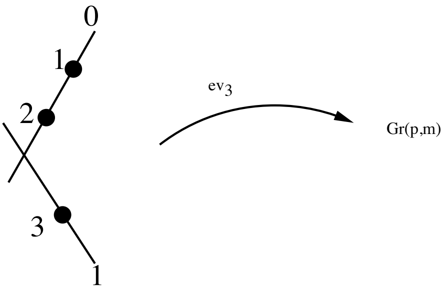

Example. $\Gamma_{0,1}(\lambda,\mu)$ = union of lines through $\Omega_\lambda \cap \Omega_\mu^{opp}$.

Open questions

1. Find geometric properties for $\Gamma_{d_1,d_2}(\lambda,\mu)$. Given a conjectural formula for K-class of $\Gamma_{d_1,d_2}(\lambda,\mu)$ we expect that this variety is normal and it has rational singularities.
2. Examples show:

$$\text{GW positroids} \subsetneq \{\text{ positroids }\}$$

$$\{\text{positroids }\} \text{ almost equal } \{\Gamma_{d_1,d_2}(\lambda,\mu)\}$$

   Is there a general statement ?
3. Other homogeneous spaces G/P ? Any connections to Lusztig stratification ?

Thank you!