LECTURE NOTES FOR ALGEBRAIC GEOMETRY

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Contents

1. Notation

Throughout the notes: $\mathbb{N} = \{0, 1, 2, \ldots\}$ (the set of natural numbers); $\mathbb{Z} = \mathbb{N} \cup -\mathbb{N}$ (the set of integers); $\mathbb{Q} = \{a/b : a, b \in \mathbb{Z}, b \neq 0\}$ (the set of rational numbers); $\mathbb{R}$ is the set of real numbers, and $\mathbb{C} := \mathbb{R} + i\mathbb{R}$ is the set of complex numbers; here $i^2 = -1$. All fields are commutative. For k a field, $k[x_1, \ldots, x_n]$ denotes the polynomial ring with coefficients in k.

2. What is algebraic geometry ?

The strict definition of the algebraic geometry is the study of solutions of polynomial equations. But very rarely equations are explicitly written in a problem one may solve. More often, we know that *some set of equations exist*, and based on this we obtain information about the given questions. Here are some examples:

- (Bèzout's theorem) Consider two plane curves, given by polynomials of degree a and b. How many intersection points can they have ? (Answer: One expects ab points, but there may be fewer.)
- (Enumerative Geometry) Consider a cubic surface in space (i.e., given by a polynomial of degree 3). Does it contain any lines ? If yes, how many ? (Answer: for a sufficiently general cubic polynomial, there are 27 lines.)
- (Rationality) When can we expect to have a *parametrization* ? (E.g., for curves this says that all points are of the form $x = x(t), y = y(t)$, where t is the parameter, and $x(t), y(t)$ are rational functions in t).

  As an example, $x(t) = t^2; y(t) = t^3$ is a parametrization of the plane curve given by equation $x^3 - y^2 = 0$. However, the curve $x^3 - y^2 = 1$ has no such parametrization.
- (Dimension count) What is the 'dimension' of the solution set given by d polynomials in the n-space ? One may expect that this is $n-d$ (each equation decreases the degree of freedom by 1). However, this may not be the case. For instance, consider the space curve given parametrically by

  $$C = \{(x,y,z) = (t^3, t^4, t^5) : t \in \mathbb{C}\} \subset \mathbb{C}^3.$$

  This may also be described by equations

  $$C = \begin{cases} x^3 = yz \\ y^2 = xz \\ z^2 = x^2 y \end{cases}.$$

  Note that there are three equations, so we would expect only *two* equations. However, if we leave out the last equation, the z-axis satisfies the first two; similarly we cannot remove any of the other two equations.

  Further, one may actually show that the common locus of

  $$\begin{cases} x^3 = yz \\ y^2 = xz \\ z^2 = x^2 y + \varepsilon \end{cases}$$

  for small $\varepsilon \in \mathbb{C}$ is actually 0 dimensional.
- (Index theorems) Cut an algebraic object by sufficiently many hyperplane sections, until we get to a curve. What can we say about this curve ? (Is it connected ? Can we write a parametrization ? What is its genus ?)
- (Changing the field of definition) What happens if we look for solutions of a system of equation over a specific field ? For instance, consider the plane curve given by equation:

  $$x^2 + y^2 = 1.$$

Can we find complex solutions ? Real solutions ? *Rational* solutions ? (i.e. solutions over $\mathbb{Q}$). What if we change the equation to $x^3 + y^3 = 1$? (This is essentially Fermat's Last Theorem.)

- (Borel-Weil-Bott theorem) Consider an (irreducible) representation V of the group $\mathrm{SL}_2(\mathbb{C})$ (2×2 matrices with complex coefficients and determinant 1). Is there a way to naturally obtain V as functions on a space where SL_2 acts ? (Yes, V is the global sections - a.k.a. generalized functions - of a certain bundle on the complex projective line $\mathbb{P}^1$.)

3. Affine algebraic varieties

In this section we define affine varieties and the Zariski topology. Then we use the Hilbert's Nullstellensatz to give a dictionary between affine varieties and ideals in polynomial rings.

3.1. **From algebra to geometry.** Let k be a field. The (n-dimensional) *affine space* is defined by

$$\mathbb{A}^n := \mathbb{A}^n_k = \{(a_1, \dots, a_n) : a_i \in k\}.$$

Point-wise it is the same as the k-vector space k^n, but unlike in the vector space situation, 0 will not play a special role. The most basic object in algebraic geometry is the *affine (algebraic) variety*, defined next.

Definition 3.1. *Let S be a nonempty subset of $k[x_1, \dots, x_n]$. The* **affine (algebraic) variety** *$V(S)$ is defined by*

$$V(S) = \{a = (a_1, \dots, a_n) \in \mathbb{A}^n : P(a) = 0 \quad \forall P \in S\}.$$

If $S = \{P_1, \dots, P_k\}$ then we denote $V(P_1, \dots, P_k) := V(\{P_1, \dots, P_k\})$.

One may think of $V(S)$ as the common solutions of the system of polynomial equations with elements from S. Some examples of affine algebraic varieties:

- $\emptyset = V(1)$ and $\mathbb{A}^n = V(0)$;
- The solutions of a system of linear system is an algebraic variety;
- If $n = 1$, then the only possibilities for affine algebraic varieties are: $\emptyset, \mathbb{A}^1$, and a finite collection of points (cf. Lemma 3.1).
- If $n = 2$ then in addition we may also have *curves*, of the form $V(P)$, $P \in k[x, y]$. For instance:
  - A parabola in the plane: $V(y - x^2)$;
  - A cusp $V(x^2 - y^3)$;
- For $n = 3$ we may have surfaces such as $V(x^2 + y^2 + z^2 - 1)$ (a sphere).

Lemma 3.1. *Let $X \subset k$ be an algebraic variety. Then either $X = k$ or X consists of finitely many points.*

Proof. This follows because any nonzero univariate polynomial has finitely many roots. □

Proposition 3.2. *Let V and W be affine varieties. Then $V \cup W$ and $V \cap W$ are also affine varieties.*

Proof. Let $V = V(S_1)$ and $W = V(S_2)$ for some $S_1, S_2 \subset k[x_1, \dots, x_n]$ (not necessarily finite). We claim that

$$V \cap W = V(S_1 \cup S_2),$$
$$V \cup W = V(fg : f \in S_1, g \in S_2).$$

The first equality is immediate from definition. For the second, observe that the inclusion $\subset$ is clear. We need to prove the opposite inclusion. It is equivalent to proving that if

$a \notin V \cup W$, then $a \notin V(fg : f \in S_1, g \in S_2)$. If $a \notin V \cup W$ then there exist $f \in S_1$ and $g \in S_2$ such that $f(a) \neq 0$ and $g(a) \neq 0$. Then $f(a)g(a) = (fg)(a) \neq 0$, and we are done. □

Example 3.3. *(An example of a space which is* not *an algebraic variety.) Let*

$$C := \{(x, \sin(x)) : x \in \mathbb{R}\} \subset \mathbb{R}^2.$$

If this were an affine variety, then C is the common zero locus of some polynomials $P_i(x,y) \in k[x,y]$. But then $C \cap \{(x,0) : x \in \mathbb{R}\}$ is also an algebraic variety, with equations $P_i(x,y) = 0$ and $y = 0$ (by Proposition 3.2). Observe that this intersection consists of the infinite discrete set $\{(k\pi, 0); k \in \mathbb{Z}\}$. On the other side, the zero locus of the system $P_i(x,0) = 0$ is either $\emptyset$, the whole x-axis, or finitely many points on the x-axis. This is a contradiction.

(A shorter way to say this is that $C \cap \mathbb{R}$ is an affine variety in $\mathbb{R}$, identified to the x-axis. But we know the affine varieties in $\mathbb{R}$ from Lemma 3.1, and $C \cap \mathbb{R}$ is not of that form.)

Consider the function

$$V : \{ \text{ subsets of } k[x_1, \dots, x_n]\} \to \{ \text{ subsets of } \mathbb{A}^n\}; \quad S \mapsto V(S).$$

We study some properties of V.

Proposition 3.4. *(1) The function V is decreasing, i.e. for $S_1 \subset S_2$, $V(S_1) \supset V(S_2)$.*

(2) Let $\langle S \rangle$ be the ideal generated by S. Then $V(S) = V(\langle S \rangle)$.

(3) Let S be a nonempty subset in $k[x_1, \dots, x_n]$. There exist finitely many elements $P_1, \dots, P_s \in S$ such that $V(S) = V(P_1, \dots, P_s)$.

(4) For sets $S_i \in k[x_1, \dots, x_n]$ $(i \in I)$, $\bigcap V(S_i) = V(\bigcup S_i)$.

(5) For finitely many sets $S_1, \dots, S_p \subset k[x_1, \dots, x_n]$,

$$V(S_1) \cup V(S_2) \cup \ldots \cup V(S_p)$$

is an affine variety.

Proof. Part (1) follows by definition: if we impose more equations, the resulting variety will be smaller.

For part (2), $S \subset \langle S \rangle$, therefore $V(\langle S \rangle) \subset V(S)$. Conversely, take $x \in V(S)$ and $f \in \langle S \rangle$. Then f is a finite combination $f = \sum a_i P_i$ for some $P_i \in S$ (by definition), therefore $f(x) = 0$, proving the reverse inclusion.

Part (3) utilizes the Hilbert Basis Theorem Theorem 26.1: the ideal $\langle S \rangle = \langle P_1, \dots, P_s \rangle$ for some $P_i \in S$, therefore

$$V(S) = V(\langle S \rangle) = V(\langle P_1, \dots, P_s \rangle) = V(P_1, \dots, P_s).$$

For part (4), observe that since $V(S_i) \supset V(\bigcup S_i)$ (by part(1)), it follows that $\bigcap V(S_i) \supset V(\bigcup S_i)$. We prove the reverse inclusion. If $x \in V(\bigcup S_i)$, then for any $i \in I$, and any $P \in S_i$, $P(x) = 0$. Then $x \in V(S_i)$, and since i is arbitrary, it is also in the intersection $\bigcap V(S_i)$.

Part (5) follows by induction from Proposition 3.2. $\square$

Corollary 3.5. *Any finite set is an affine variety.*

Proof. One point is an algebraic variety. Then apply Proposition 3.4, part (5). $\square$

Definition 3.2. *Consider the collection of sets:*

$$\{\emptyset, \mathbb{A}^n\} \cup \{V(S) : S \subset k[x_1, \dots, x_n]\}$$

Proposition 3.4 implies that these sets satisfy the properties of closed sets of a topology on $\mathbb{A}^n$, called the **Zariski topology**.

If $X \subset \mathbb{A}^n$ is any subset, we will consider it as a topological space with induced topology.

Remark 3.1. *The Zariski topology is very different from the classical 'Euclidean' topology. For instance:*

- *The Zariski topology is not Hausdorff. In fact, if k is infinite, then* any *two open sets in $\mathbb{A}^n$ intersect. (This will come up later, in relation to irreducibility.)*
- *The Zariski topology does not behave well with respect to products. For instance $\mathbb{A}^2 = \mathbb{A}^1 \times \mathbb{A}^1$ but the Zariski topology on $\mathbb{A}^2$ is* not *the product topology. (Exercise.)*

A basis of open sets for the Zariski topology is given by **distinguished open sets**: for $f \in k[x_1, \dots, x_n]$, define

$$D_f := \mathbb{A}^n \setminus V(f) = \{a = (a_1, \dots, a_n) : f(a) \neq 0\}.$$

3.2. **From geometry to algebra.** We defined the map V sending

$$S \subset k[x_1, \dots, x_n] \mapsto V(S) \subset \mathbb{A}^n.$$

We now define a map in the opposite direction. For $\Sigma \subset \mathbb{A}^n$ (subset) define

$$I(\Sigma) = \{P \in k[x_1, \dots, x_n] : P(x) = 0 \ \forall x \in \Sigma\}.$$

Clearly $I(\Sigma)$ is an ideal in $k[x_1, \dots, x_n]$. Further, the ideals arising in this way must be *radical*: indeed if $f^n \in I(\Sigma)$ then $f(x)^n = 0$ for all $x \in \Sigma$, thus $f(x) \equiv 0$ on Σ.

There we defined a map

$$I : \{ \text{ subsets of } \mathbb{A}^n\} \to \{ \text{ radical ideals in } k[x_1, \dots, x_n]\}; \quad \Sigma \subset \mathbb{A}^n \mapsto I(\Sigma).$$

We would like to study I and the relation between I and V. This will lead to a 'dictionary' between geometry (affine varieties) and algebra (radical ideals in a polynomial ring), which sits at the foundation of Algebraic Geometry.

Example 3.6. *(a) $I(0)$ consists of all polynomials which vanish at $x = 0$. Any such polynomial must satisfy $P(0) = 0$, giving that $I(0) = \langle x_1, \dots, x_n\rangle$.*

(b) If k is infinite, then $I(\mathbb{A}^n) = (0)$. (Exercise.)

(c) Let $k = \mathbb{F}_p$, the field with p elements. Then for any $x \in \mathbb{A}^1_k$, $x^p - x = 0$, but obviously $x^p - x$ is not the zero polynomial.

The next proposition gives the first properties of the map I.

Proposition 3.7. *(a) I is decreasing, i.e. if $\Sigma_1 \subset \Sigma_2$, then $I(\Sigma_1) \supset I(\Sigma_2)$.*

(b) When restricted to affine varieties, $V \circ I = id$, i.e. for any affine variety $X \subset \mathbb{A}^n$, $V(I(X)) = X$.

Proof. Part (a) follows from definition. For (b), again from definition, $X \subset V(I(X))$. To prove the reverse inclusion, let $X = V(S)$, for some $S \subset k[x_1, \dots, x_n]$. Then $\langle S \rangle \subset I(X)$, therefore $V(I(X)) \subset V(S) = X$. □

Corollary 3.8. *When restricted to algebraic varieties, the function $\Sigma \mapsto I(\Sigma)$ is injective.*

Proof. This follows from Proposition 3.7. □

Remark 3.2. *In general. the map $I \mapsto V(I)$ is not injective. For instance, take $k = \mathbb{R}$. Then*

$$V(\langle x^2 + 1 \rangle) = V(1) = \emptyset.$$

The issue here is that $\mathbb{R}$ is not algebraically closed; we will see that I is bijective *for k algebraically closed.*

3.3. **Hilbert's Nullstellensatz: statement and consequences.** Recall that the radical of an ideal $I \subset k[x_1, \dots, x_n]$ is defined by $\sqrt{I} = \{f \in I : \exists n \in \mathbb{N}, f^n \in I\}$. An ideal is radical if $I = \sqrt{I}$.

The main result of this section is the following:

Theorem 3.1 (Hilbert's Nullstellensatz)**.** *Let k be algebraically closed. Then for any ideal $I \subset k[x_1, \dots, x_n]$,*

$$I(V(I)) = \sqrt{I}.$$

Before we prove this theorem, we list some consequences.

Corollary 3.9. *The functions*

$$V : \{ \text{ radical ideals in } k[x_1, \dots, x_n]\} \to \{ \text{ affine varieties in } \mathbb{A}^n\}; \quad J \mapsto V(J);$$

$$I : \{ \text{ affine varieties in } \mathbb{A}^n\} \to \{ \text{ radical ideals in } k[x_1, \dots, x_n]\}; \quad \Sigma \mapsto I(\Sigma).$$

are inverse to each other, i.e. for any radical ideal J, $I(V(J)) = J$ and for any affine variety Σ , $V(I(\Sigma)) = \Sigma$. This gives a one-to-one, inclusion reversing, correspondence, between radical ideals in $k[x_1, \dots, x_n]$ and affine algebraic varieties in $\mathbb{A}^n$.

Proof. We already proved in Proposition 3.7 that $V(I(\Sigma)) = \Sigma$. The fact that $I(V(J)) = J$ for radical ideals J is a consequence of the Nullstellensatz (Theorem 3.1). □

Remark 3.3. *One may wonder if there is a more general correspondence, where the ideals are not necessarily radical. That leads to theory of (Noetherian) affine schemes.*

Remark 3.4. *Once we know about morphisms of affine varieties, we will see that the equivalence above may be upgraded to an equivalence of contravariant functors between the category of*

$$\{ \textit{ finitely generated } k\textit{-algebras } \} = \{k[x_1, \ldots, x_n]/I\}$$

to

$$\{ \textit{ (reduced) affine varieties with morphisms}\} = \{\mathrm{Spec}(k[x_1, \ldots, x_n]/I)\}.$$

Corollary 3.10. *There is a one-to-one correspondence between maximal ideals in $k[x_1, \ldots, x_n]$ and points in $\mathbb{A}^n$. Any maximal ideal is of the form $\langle x_1 - a_1, \ldots, x_n - a_n\rangle$ for some $(a_1, \ldots, a_n) \in \mathbb{A}^n$, and the correspondence is given by*

$$\langle x_1 - a_1, \ldots, x_n - a_n\rangle \leftrightarrow (a_1, \ldots, a_n).$$

Proof. By Corollary 3.9, the maximal ideals corresponds to minimal, non-empty, affine varieties, i.e. single points. These ideals are precisely of the form stated. □

Example 3.11. *Let $n = 1$, and let $I \subset k[x]$ be an ideal. Since any ideal in $k[x]$ is principal, $I = \langle P(x)\rangle$. Because k is algebraically closed*

$$P(x) = (x - a_1)^{n_1} \cdot \ldots \cdot (x - a_s)^{n_s},$$

for some $n_i \in \mathbb{N}$. Then the Nullstellensatz says that

$$I(V(P)) = I(\{a_1, \ldots, a_s\}) = \langle (x - a_1) \cdot \ldots \cdot (x - a_s)\rangle = \sqrt{\langle P\rangle}.$$

Example 3.12. *Take $n = 2$. If $P(x, y) \in k[x, y]$ is a non-constant irreducible polynomial then $I(V(P)) = \langle P\rangle$ and $V(P)$ is a curve. (We will see this curve is irreducible.)*

3.4. **Proof of the Nullstellensatz.** We start by proving a weaker version of Nullstellensatz.

Theorem 3.2 (Weak Nullstellensatz)**.** *Let k be algebraically closed and let I be an ideal such that $I \neq k[x_1, \ldots, x_n]$. Then $V(I) \neq \emptyset$. Further, if I is maximal, then $V(I)$ is a single point in $\mathbb{A}^n$.*

The theorem has the following interpretation. Consider a system of polynomial equations

$$\begin{cases} P_1(x_1, \ldots, x_n) = 0 \\ P_2(x_1, \ldots, x_n) = 0 \\ \vdots \\ P_s(x_1, \ldots, x_n) = 0. \end{cases}$$

If there exist $a_1, \ldots, a_s \in k[x_1, \ldots, x_n]$ such that $\sum a_i P_i = 1$, then obviously the system has no solution. The theorem says that the converse is also true.

Proof of Theorem 3.2. Since $I \neq k[x_1, \ldots, x_n]$, we may find a maximal ideal J which contains I. Since $V(I) \supset V(J)$, we may assume I is maximal to start with. Define:

$$K := k[x_1, \ldots, x_n]/I.$$

This is a k-algebra and since I is maximal $K \supset k$ is also a field.

Lemma 3.13. $K = k$.

Proof. (In the case k is uncountable.) We will prove this in the case when k is uncountable. The main steps for the general proof are outlined in Remark 3.5 below.

If the extension $K \supset k$ is algebraic, we are done, since k is algebraically closed. Therefore we assume that there exist $t \in K$ which is transcendent over k. Then we have field extensions

$$k \subsetneq k(t) = \{\frac{P(t)}{Q(t)} : P, Q \in k[x]\} \subset K.$$

We claim that the elements

$$S := \{\frac{1}{t-a} : a \in k\} \subset k(t)$$

are linearly independent over k. Indeed, assume that there exists a non-trivial finite combination

$$\sum \frac{c_i}{t-a_i} = 0; \quad c_i \in k.$$

We may assume that a_i's are distinct (otherwise collect the like terms). Consider a coefficient c_{i_0}, and multiply both sides by $\frac{(t-a_1)\cdot\ldots\cdot(t-a_s)}{t-a_{i_0}} \in k[t]$ to obtain

$$c_{i_0}(t-a_1)\cdot\ldots\cdot(t-a_{i_0-1})(t-a_{i_0+1})\cdot\ldots(t-a_s) = (t-a_{i_0})H(t), \tag{3.1}$$

where $H(t) \in k[t]$. Since t is transcendental, there is a well defined evaluation k-algebra homomorphism $\psi : k[t] \to k$ sending $t \mapsto c_{i_0}$. Applying ψ to both sides gives $c_{i_0} = 0$. This proves the linear independence claim.

On the other side, by the definition of K, it is a finitely generated k-algebra, therefore as a vector space it must have a countable basis. Therefore it cannot contain the linearly independent set S, which contains uncountably many elements. This implies that there are no elements $t \in K$ which are transcendental over k. This finishes the proof of the Lemma. $\square$

Since $k = k[x_1, \ldots, x_n]/I$, it follows that $x_i + I = a_i$ for $1 \leq i \leq n$, and $a_i \in k$. Equivalently, $x_i - a_i \in I$, implying that $\langle x_1 - a_1, \ldots, x_n - a_n\rangle \subset I$. Since both ideals are maximal, this forces equality, and finishes the proof of the theorem. $\square$

Remark 3.5. *(A high-brow proof of Lemma 3.13, for arbitrary $k = \overline{k}$.) By Noether normalization Lemma (see Theorem 26.2 below), if $trdeg_k K/k = N$ there exist algebraically independent elements $y_1, \ldots, y_N$ such that $k \subset k[y_1, \ldots, y_N] \subset K$ and $K/k \subset k[y_1, \ldots, y_N]$ is an integral extension.*

Now we use another result in commutative algebra, stating that for an integral extension of integral domains $A \subset B$, if B is a field, then A is also a field; see Lemma 26.1. We apply this to the integral extension $k[y_1, \ldots, y_N] \subset K$; then $k[y_1, \ldots, y_N]$ is a field. This forces $N = 0$, thus K/k is an integral extension, and since k is algebraically closed, $k = K$.

Proof of the strong Nullstellensatz Theorem 3.1. We need to show that $\sqrt{J} = I(V(J))$. The inclusions $\sqrt{J} \subset I(V(J))$ follows from definition.

To prove the converse, let $J = \langle f_1, \ldots, f_s \rangle$ for some $f_i \in k[x_1, \ldots, x_n]$. We need to prove the following. Let $g \in k[x_1, \ldots, x_n]$; if $g(a) = 0$ for any a such that $f_1(a) = \ldots = f_s(a)$, then $g^N \in J$ for some number N. Define the ideal:

$$B := \langle f_1, \ldots, f_s, 1 - g x_{n+1} \rangle \subset k[x_1, \ldots, x_n, x_{n+1}].$$

We first show that we must have that $B = k[x_1, \ldots, x_n, x_{n+1}]$. Indeed, if $B \neq \langle 1 \rangle$, then there exists a maximal ideal $\mathfrak{m} \supset B$. By the Weak Nullstellensatz Theorem 3.2, it follows that

$$\mathfrak{m} = \langle x_1 - a_1, \ldots, x_n - a_n, x_{n+1} - a_{n+1} \rangle.$$

Then

$$f_1(a_1, \ldots, a_n) = \ldots = f_s(a_1, \ldots, a_n) = 1 - a_{n+1} g(a_1, \ldots, a_n) = 0.$$

But the first s equalities imply that $g(a_1, \ldots, a_n) = 0$, which is impossible.

We now consider the situation when $B = k[x_1, \ldots, x_n, x_{n+1}] = \langle 1 \rangle$. Then there exist $h_i \in k[x_1, \ldots, x_{n+1}]$ such that

$$h_{s+1}(x, x_{n+1})(1 - x_{n+1} g(x)) + \sum h_i(x, x_{n+1}) f_i(x) = 1.$$

This is an identity in $k[x_1, \ldots, x_n, x_{n+1}]$. We specialize to $x_{n+1} = \frac{1}{g(x)}$ to obtain

$$\sum h_i(x, \frac{1}{g(x)}) f_i(x) = 1.$$

After clearing denominators (which are simply powers of g), we obtain an identity of the form

$$\sum h_i'(x) f_i(x) g(x)^{n_i} = g(x)^N.$$

This shows that $g^N \in \langle f_1, \ldots, f_s \rangle = J$, and we are done. □

3.5. **Irreducibility. Let k be an arbitrary field**. In this section we explore the first properties of affine varieties, in relation to the algebra/geometry dictionary. We start with the following definition.

Definition 3.3 (Coordinate ring)**.** *Let $X \subset \mathbb{A}^n$ be an affine variety and $I := I(X)$. The ring*

$$\Gamma(X) := k[x_1, \ldots, x_n]/I$$

is called the **coordinate ring** *of X.*

In a sense which will become precise later, $\Gamma(X)$ is the ring of (algebraic) regular functions on X. Some examples (to avoid arithmetic issues, here we assume k is of characteristic 0):

- $\Gamma(\mathbb{A}^n) = k[x_1, \ldots, x_n]$;
- If $C_1 = V(y - x^2) \subset \mathbb{A}^2$ is a parabola, $\Gamma(C_1) = k[x,y]/\langle y - x^2 \rangle$;
- If C_2 is the union of x and y axes in $\mathbb{A}^2$, then $\Gamma(C_2) = k[x,y]/\langle xy \rangle$.

The parabola and the union of lines are somehow different: the parabola is made of 'one part', the other is a union of two parts. We want to make this precise.

Definition 3.4. *A topological space X is called* **reducible** *if it can be written as $X = X_1 \cup X_2$ where $X_1, X_2 \subsetneq X$ are closed subsets. If X is not reducible, it is called* **irreducible**.

- The real affine line $\mathbb{A}^1 = \mathbb{R}$ with *Euclidean* topology is reducible: $\mathbb{R} = (-\infty, 0] \cup [0, \infty)$;
- The affine line $\mathbb{A}^1_k$ over any infinite field with *Zariski* topology is irreducible. (Because the non-trivial closed sets are just finite sets of points.)
- The union of lines C_2 from above is obviously reducible, but one can show that the parabola is irreducible.

Proposition 3.14. *Let X be an irreducible space. Then the following hold:*

(1) If $\emptyset \neq U \subset X$ is an open subset, then U is dense in X.
(2) Any two nonempty subsets in X intersect.
(3) Any non-empty subset $U \subset X$ (with induced topology) is irreducible.

Proof. Write $X = \overline{U} \cup (X \setminus U)$. Since X is irreducible one of the two closed subsets must be X, forcing $X = \overline{U}$.

We leave (2) and (3) as exercises. $\square$

Proposition 3.15. *Let $X \subset \mathbb{A}^n$ be an affine variety (with Zariski topology) and let $I(X)$. Then the following are equivalent:*

(1) *X is irreducible;*
(2) *The ideal $I(X)$ is a prime ideal;*
(3) *The coordinate ring of X, $\Gamma(X)$ is an integral domain.*

Proof. Obviously (2) $\Leftrightarrow$ (3), so we will prove (1) $\Leftrightarrow$ (2).

"$\Rightarrow$" Assume that X is irreducible, and take $f, g \in k[x_1, \ldots, x_n]$ such that $fg \in I(X)$. Then $\langle fg \rangle \subset I(X)$, therefore $X = V(I(X)) \subset V(fg) = V(f) \cup V(g)$. It follows that

$$X = (X \cap V(f)) \cup (X \cap V(g)).$$

By irreducibility it follows that either $X \subset V(f)$ or $X \subset V(g)$, i.e. $f \in I(X)$ or $g \in I(X)$.

"$\Leftarrow$" Assume that $I(X)$ is prime, and write $X = X_1 \cup X_2$ where X_1, X_2 are closed. First observe that

$$I(X_1 \cup X_2) = I(X_1) \cap I(X_2) \tag{3.2}$$

If $X \neq X_i$, by injectivity of the map I (Corollary 3.8), it follows that there exists $F_i \in I(X_i) \setminus I(X)$. Then

$$F_1 F_2 \in I(X_1) \cap I(X_2) = I(X_1 \cup X_2) = I(X),$$

and by hypothesis this implies that either F_1 or F_2 is in $I(X)$, a contradiction. $\square$

As a consequence of Proposition 3.15, the parabola $V(y - x^2) \subset \mathbb{A}^2$ is irreducible, since $\langle y - x^2 \rangle$ is a prime ideal.

Any affine variety is a *finite* union of irreducible components:

Theorem 3.3. *Let X be an affine variety. Then X can be written as*

$$X = X_1 \cup \ldots \cup X_s \tag{3.3}$$

where each $X_i \subset X$ is closed and irreducible affine variety. If we require that $X_i \nsubseteq X_j$ for any pair i, j then the decomposition (3.3) is unique up to permuting factors.

Proof. Exercise. $\square$

The (unique) varieties X_i from the decomposition (3.3) are called **irreducible components** and the decomposition itself is the decomposition into irreducible components. In commutative algebra, the analogue of this decomposition is called the **primary decomposition** of an ideal. For explicit ideals I, there are algorithms to obtain this, and there is software which will find the primary decomposition.

3.6. Morphisms (part I).

Definition 3.5. *Let $X \subset \mathbb{A}^p, Y \subset \mathbb{A}^s$ be two affine varieties and $F = (f_1, \ldots, f_s) : X \to Y$ a map between them, where $f_i : X \to k$. We say that F is a is a* **morphism***, if $f_i \in \Gamma(X)$. We say that F is an* **isomorphism** *if there exists a morphism $G : Y \to X$ such that $F \circ G = id_Y$ and $G \circ F = id_X$.*

In other words, morphisms $F : X \to Y$ are restrictions of morphisms $F : \mathbb{A}^p \to \mathbb{A}^s$ such that $F(X) \subset Y$. This may sound rather restrictive; it is, in other categories, but it turns out to be fine in the algebraic category.

An equivalent way to state this definition is to require that f_i's are polynomials in $k[x_1, \ldots, x_s]$ such that for any $Q \in I(Y) \subset k[x_1, \ldots, x_s]$,

$$Q(f_1(x_1, \ldots, x_p), \ldots, f_s(x_1, \ldots, x_p)) = 0 \quad \forall (x_1, \ldots, x_p) \in X \Longleftrightarrow Q(f_1, \ldots, f_s) \in I(X). \tag{3.4}$$

This definition implies that F is continuous in Zariski topology.

For a morphism $F = (f_1, \ldots, f_s) : X \subset \mathbb{A}^p \to Y \subset \mathbb{A}^s$ we may define a k-algebra homomorphism $F^* : \Gamma(Y) \to \Gamma(X)$ by

$$F^*(\overline{x_i}) = \overline{f_i}, \quad \forall 1 \leq i \leq s,$$

where the bars mean that we're taking images inside the corresponding quotient rings.

Observe that this morphism is well defined. For that, it suffices to show that $F^*(Q) \in I(X)$ for any $Q \in I(Y)$. Utilizing that F^* is an algebra homomorphism, and Equation (3.4), we obtain

$$F^*(Q) = Q(f_1(x), \ldots, f_s(x)) \in I(X),$$

as needed. We denote by $Hom(X, Y)$ the set of morphisms from X to Y.

Given a k-algebra homomorphisms $\varphi : \Gamma(Y) \to \Gamma(X)$ we may define a morphism $F_\varphi : X \to Y$ by

$$F_\varphi(x) = (f_1(x), \ldots, f_s(x))$$

where $f_i(x)$ are any lifts to $k[x_1, \ldots, x_p]$ of $\varphi(x_i)$. Clearly, $F_\varphi^* = \varphi$.

Lemma 3.16. *The assignments $F \mapsto F^*$ and $\varphi \to F_\varphi$ are compatible with composition, i.e.*

$$(F \circ G)^* = G^* \circ F^*; \quad F_{\varphi \circ \psi} = F_\psi \circ F_\varphi.$$

Proposition 3.17. *Let $X \subset \mathbb{A}^p, Y \subset \mathbb{A}^s$ be two affine varieties. There is a one-to-one correspondence between morphisms $F = (f_1, \ldots, f_s) \in Hom(X, Y)$ and k-algebra homomorphisms $Hom_{k-alg}(\Gamma(Y), \Gamma(X))$ given by $F \mapsto F^*$.*

Proof. One needs to check that for any $F \in Hom(X, Y)$, $F_{F^*} = F$, and that for any $\varphi \in Hom_{k-alg}(\Gamma(Y), \Gamma(X)$, $F_\varphi^* = \varphi$. We already checked the second equality. For the first, take $x \in X$, and assume that $F = (f_1, \ldots, f_s)$. Then $F^*(\overline{x_i}) = \overline{f_i}$ and

$$F_{F^*}(x) = (F^*(x_1), \ldots, F^*(x_s)) = (f_1(x), \ldots, f_s(x)) = F(x).$$

□

Remark 3.6. *For k algebraically closed, Proposition 3.17 says that the contravariant functor Γ from the category of* **affine varieties and morphisms** *to* **finitely generated, reduced, k-algebras and k-algebra homomorphisms**, *sending $X \mapsto \Gamma(X)$ is an equivalence of categories.*

We now record some algebraic properties of morphisms which correspond to geometric information. To start, Proposition 3.17 implies that:

Corollary 3.18. *A morphism $F : X \to Y$ is an isomorphism if and only if $F^* : \Gamma(Y) \to \Gamma(X)$ is a k-algebra isomorphism.*

In examples below we take char $k = 0$.

Example 3.19 (Add picture)**.** *Consider the parabola $C := V(y - x^2) \subset \mathbb{A}^2$. There is a morphism $F : \mathbb{A}^1 \to C$ defined by $F(t) = (t, t^2)$. This is an isomorphism, with inverse $G(x, y) = x$. In fact, F induces the isomorphism $F^* : k[x, y]/\langle y - x^2 \rangle \to k[t]$, $F(x) = t, F(y) = t^2$.*

Example 3.20 (Add picture)**.** *Consider the cusp $C = V(y^2 - x^3)$ and $F : \mathbb{A}^1 \to C$ defined by $F(t) = (t^2, t^3)$. This is a bijective morphism which is not an isomorphism.*

The bijectivity statement is clear. To prove it is not an isomorphism, observe that $F^* : \Gamma(C) \to k[t]$ *is given by* $F(x) = t^2, F(y) = t^3$*. Therefore the image of* F^* *is the subring* $k[t^2, t^3] \subsetneq k[t]$*, showing that* F^* *is not surjective.*

In fact, one may show that the rings $\Gamma(C)$ *and* $k[t]$ *are not isomorphic. Indeed,* $k[t]$ *is integrally closed, as it is a PID, but* $\Gamma(C)$ *is not. To see this, first observe that* F^* *above induces a ring isomorphism* $\Gamma(C) \simeq k[t^2, t^3]$*. The fraction field of* $\Gamma(C)$ *is* $k(t)$[1]*. Now* $t \in k(t)$ *is a solution of the monic polynomial* $X - t^2 \in k[t^2, t^3][X]$*, showing that* t *is integral over* $k[t^2, t^3]$*; but* t *is not in* $k[t^2, t^3]$*.*

Geometrically, the cusp has a singularity at $(0,0)$*, while* $\mathbb{A}^1$ *is non-singular. So they cannot be isomorphic in any category where a reasonable notion of smoothness is defined.*

A morphism $F : X \to Y$ is called **dominant** if $F(X)$ is dense in Y, i.e. $\overline{F(X)} = Y$.

Proposition 3.21. *Let* X, Y *be two affine varieties, and* $F : X \to Y$*.*
(a) F *is dominant if and only if* $F^* : \Gamma(Y) \to \Gamma(X)$ *is injective.*
(b) If X *is irreducible and* F *is dominant then* Y *is irreducible.*

Proof. (a) Assume that $F = (f_1, \ldots, f_s)$. For any $\overline{P(x)} \in \Gamma(Y)$, $F^*(\overline{P(x)}) = \overline{P(f_1, \ldots, f_s)}$. Then $\overline{P(x)} \in \ker(F^*)$ if and only if $P(f_1, \ldots, f_s) \in I(X)$, i.e. for all $x \in X$, $F(x) \in V(P) \cap Y$.

If F^* is not injective then P may be chosen outside $I(Y)$, therefore the open set $D_P = Y \setminus V(P)$ is non-empty, and disjoint from the image. Conversely, if the image is not dense then we can find an open set $U \subset Y$ such that $\mathrm{Im}(F) \subset Y \setminus U$. The latter is a nonempty closed subset not equal to Y, therefore there exists $P \in I(Y \setminus U) \setminus I(Y)$. Then $\overline{0} \neq \overline{P} \in \ker(F^*)$.

To prove (b), observe that by (a) $\Gamma(Y)$ is a subalgebra of $\Gamma(X)$. Since X is irreducible, $\Gamma(X)$ is a domain, therefore so must be $\Gamma(Y)$. $\square$

3.7. **Exercises.** 1. Prove that the topology on $\mathbb{A}^2 = \mathbb{A}^1 \times \mathbb{A}^1$ is not the product topology.

2. Prove that the distinguished open sets D_f form a basis of open sets for the Zariski topology.

3. (a) Prove that if k is infinite, then $I(\mathbb{A}^n) = (0)$. Find a counterexample for this in a finite field different from $k = \mathbb{F}_p$ from Example 3.6 above.
(b) Fix $a \in \mathbb{A}^n$. Find $I(a)$.
(c) Let $k = \mathbb{F}_p$ (the field with p-elements. Find $I(\mathbb{A}^1_k)$. Is it prime ?

4. A topological space X is called *Noetherian* if every descending chain of closed subsets $X \supsetneq X_1 \supsetneq X_2 \supsetneq \ldots$ is finite. Prove that every affine algebraic variety is Noetherian.

[1]This means that C and $\mathbb{A}^1$ are *birational*.

5. Prove Theorem 3.3. (One proof relies on the fact that affine varieties are Noetherian.)

4. Projective algebraic varieties

In the development of algebraic geometry, it quickly became apparent that *projective varieties* are much better behaved than the affine varieties. This is (partially) due to the fact that compact spaces behave better than non-compact ones. We start with the most basic projective variety.

4.1. **Projective spaces and subspaces.** Let V be a vector space over the field k. $n \in \mathbb{Z}_{>0}$. The **projective space** $\mathbb{P}(V)$ is defined by

$$\mathbb{P}(V) = \{\ell \subset V : \ell \subset V \text{ vector subspace }, \dim \ell = 1\}.$$

If we identify $V \simeq k^{n+1}$ (i.e. we choose a basis for V), then $\mathbb{P}(V)$ is denoted by $\mathbb{P}^n$. In this case $\mathbb{P}^n$ may also be regarded as the set of equivalence classes $[x_0 : \ldots : x_n]$ where $(x_0, \ldots, x_n) \in k^{n+1} \setminus 0$ and $(x_0, \ldots, x_n) \simeq (\lambda x_0, \ldots, \lambda x_n)$ for $\lambda \in k^*$. The coordinates $x_0, \ldots, x_n$ are called **projective coordinates**. If $\dim V = n+1$, then we define the **dimension** of $\mathbb{P}(V)$ to be $\dim \mathbb{P}(V) = n$.

A **linear projective (sub)space** of $\mathbb{P}(V)$ is a subset of the form $\mathbb{P}(W)$ where $W \subset V$ is a vector subspace. If $\dim W = 1$ then $\mathbb{P}(W)$ is a point; if $\dim W = 2$ then $\mathbb{P}(W)$ is called a projective line; if $\dim W = 3$ then $\mathbb{P}(W)$ is a projective plane, and so on.

Lemma 4.1. *Let $\mathbb{P}(V_1), \mathbb{P}(V_2) \subset \mathbb{P}(V)$ be projective subspaces. Then $\mathbb{P}(V_1) \cap \mathbb{P}(V_2) = \mathbb{P}(V_1 \cap V_2)$. In paricular, if $\dim \mathbb{P}(V_1) + \dim \mathbb{P}(V_2) \geq \dim \mathbb{P}(V)$, then $\mathbb{P}(V_1) \cap \mathbb{P}(V_2)$ is nonempty.*

Proof. By definition, $\mathbb{P}(V_1) \cap \mathbb{P}(V_2)$ is the set of lines $\ell \subset V_1 \cap V_2$, i.e. $\mathbb{P}(V_1 \cap V_2)$. If the dimension condition is satisfied then $\dim V_1 + \dim V_2 \geq \dim V + 1$, giving $\dim V_1 \cap V_2 \geq 1$. □

A **projective change of coordinates** is any transformation $u : \mathbb{P}(V) \to \mathbb{P}(V)$ given by a linear transformation $u \in \mathrm{GL}(V)$. For such transformations it is clear that $u(\mathbb{P}(W)) = \mathbb{P}(u(W))$, thus it sends a projective subspace to one of the same dimension.

By definition, the projective space is the set of lines in a vector space V. The set $\mathbb{P}^*(V)$ of *hyperplanes* $H \subset V$ is also a projective space, called the **dual projective space**. Indeed, the inclusion $H \subset V$ determines a line in V^* as the kernel of the projection $\ker(V^* \to H^*)$. This shows that the dual projective space is just the usual projective space $\mathbb{P}(V^*)$.

4.2. **Covering by affine spaces and Zariski topology.** Take $V = k^{n+1}$. For $0 \leq i \leq n$, define

$$U_i := \{[x_0 : \ldots : x_i : \ldots x_n] : x_i \neq 0\}.$$

Lemma 4.2. *(a) The sets U_i, for $0 \leq i \leq n$, cover $\mathbb{P}^n$, i.e. $\mathbb{P}^n = \bigcup_{i=0}^n U_i$.*

(b) There are bijections $\varphi_i : U_i \to \mathbb{A}^n$ defined by

$$[x_0 : \ldots : x_i : \ldots x_n] \mapsto (\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i-1}}{x_i}, \ldots, \frac{x_n}{x_i}).$$

The inverse is given by

$$(x_1, \ldots, x_n) \mapsto [x_1 : \ldots : 1 : \ldots : x_n].$$

Proof. Immediate from definitions. □

We may define the Zariski topology on $\mathbb{P}^n$ by identifying $U_i \simeq \mathbb{A}^n$, then require that a set $U \subset \mathbb{P}^n$ is open iff $U \cap U_i$ is open for each i. In particular, each U_i will be open in $\mathbb{P}^n$.

4.3. **Projective algebraic varieties.** A non-zero polynomial $F \in k[x_0, \ldots, x_n]$ is **homogeneous** if

$$F(\lambda x_0, \ldots, \lambda x_n) = \lambda^d F(x_0, \ldots, x_n), \forall \lambda \in k^*.$$

If F is homogeneous, the integer d is the degree of F. Any non-zero polynomial F may be written (uniquely) as $F = F_0 + \ldots + F_d$ where F_i's are homogeneous. We start with the following simple lemma.

Lemma 4.3. *Let $F \in k[x_0, \ldots, x_n]$ be a polynomial with $F = F_0 + \ldots + F_d$ the decomposition into irreducible components. Let $\bar{x} = [x_0 : \ldots : x_n] \in \mathbb{P}^n$. Then $F(x) = 0$ for all representatives of $\bar{x}$ if and only if $F_i(x') = 0$ for all i, and x' is any representative of $\bar{x}$.*

Proof. The $\Longleftarrow$ direction is clear. For the other direction, observe that any $\lambda \in k^*$ is a root of the polynomial equation

$$0 = F(\lambda x') = F_0(x') + \lambda F_1(x') + \ldots + \lambda^d F_d(x').$$

Since k is infinite, this forces that $F_0(x') = \ldots = F_d(x') = 0$, as required. □

The Lemma suggests that if one wants to look at zeros of polynomials which are independent of choices of projective coordinates, then it suffices to restrict to homogeneous polynomials.

Definition 4.1. *Let $S \subset k[x_0, \ldots, x_n]$ be a subset consisting of homogeneous polynomials. Define*

$$V_p(S) := \{[x_0, \ldots, x_n] : F(x_0 : \ldots : x_n) = 0, \forall F \in S\}.$$

A **projective algebraic variety** *is any subset of $\mathbb{P}^n$ of the form $V_p(S)$.*

The function V_p sending $S \mapsto V_p(S)$ has properties very similar to those from the affine case:

- V_P is decreasing;
- $V_p(S) = V_p(\langle S \rangle)$; in particular we may assume that S is finite;
- $V_p(0) = \mathbb{P}^n$;
- intersections of projective varieties are projective; finite unions of projective varieties are projective

The last property allows us to (re)define Zariski topology on $\mathbb{P}^n$, by requiring that $V_p(S)$ are closed subsets.

One difference is the behavior with respect to (homogeneous) maximal ideals. Let $\mathfrak{m} = \langle x_0, \dots, x_n \rangle \subset k[x_0, \dots, x_n]$. Then

$$V_p(\mathfrak{m}) = \emptyset.$$

Because of this, we will call $\langle x_0, \dots, x_n \rangle$ the **irrelevant ideal**. In the affine case, the irrelevant ideal is maximal, and it corresponds to the point $(0, \dots, 0) \in \mathbb{A}^{n+1}$. One may ask what are the ideals corresponding to points. Let $[a_0, \dots, a_n] \in \mathbb{P}^n$, and pick $a_i \neq 0$; w.l.o.g. $a_0 \neq 0$. Then

$$[a_0, \dots, a_n] = [1 : \frac{a_1}{a_0} : \dots : \frac{a_n}{a_0}] = V_p(x_1 - \frac{a_1}{a_0} x_0, \dots, x_n - \frac{a_n}{a_0} x_0).$$

One may go from an affine variety to a projective variety, by the process called **homogenization**. If $f(x_1, \dots, x_n) \in k[x_1, \dots, x_n]$, then we may define

$$F(x_0, \dots, x_n) = x_0^{\deg f} f(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}).$$

Note that this is a homogeneous polynomial. Then one may homogenize each of the generators of the ideal of the affine variety.

To illustrate, consider the parabola $P = V(x_2^2 - x_1)$ and the hyperbola $C = V(x_1 x_2 - 1)$. Recall that P and C are *not* isomorphic when regarded as affine varieties (because their coordinate rings are not isomorphic - see homework). The homogenizations of the two equations are:

$$F_1 := x_2^2 - x_1 x_0; \quad F_2 = x_1 x_2 - x_0^2.$$

Since F_1, F_2 are the same up to rearranging labels, $V_p(F_1)$ will be isomorphic to $V_p(F_2)$. Therefore, the **projective completion** of the parabola is the same as the projectivization of the hyperbola. But let's see what are the points at infinity are added in each case. We have

$$V(F_1) \cap (x_0 = 0) = \{[0 : 1 : 0]\} \quad ; V(F_2) \cap (x_0 = 0) = \{[0 : 1 : 0], [0 : 0 : 1]\}.$$

4.4. **Ideals of projective varieties.** For any subset $V \subset \mathbb{P}^n$, the ideal of V is

$$I_p(V) = \{F \in k[x_0, \dots, x_n] : F(x) = 0 \quad \forall x \in V\}.$$

Clearly $I_p(V)$ is a homogeneous ideal, i.e. it is generated by homogeneous polynomials.

As in the affine case, it is easy to check that the function I_p sending $V \mapsto I_p(V)$ has the following properties:

- $I_p(V)$ is a radical ideal;
- I_p is decreasing;
- $V_p(I_p(V)) = V$; in particular I_p is injective;
- V is irreducible if and only if $I_p(V)$ is a prime ideal.

4.5. **Cones and projective Nullstellensatz.** For a projective variety $X \subset \mathbb{P}^n$ there is the associated **cone** $C(X)$. If $\pi : k^{n+1} \setminus 0 \to \mathbb{P}^n$ is the projection, then

$$C(X) := \pi^{-1}(X) \cup \{0\}.$$

More generally, a subset $X \subset \mathbb{A}^{n+1}$ is called a **cone** if

$$(x_0, \dots, x_n) \in X \Longrightarrow \lambda(x_0, \dots, x_n) \in X, \forall \lambda \in k^*.$$

Lemma 4.4. *(a) Let $I \subsetneq k[x_0, \dots, x_n]$ be a homogeneous ideal and $X = V_p(I)$. Then $C(X) = V(I)$ in $\mathbb{A}^{n+1}$.*

(b) Let $\emptyset \neq X = V_p(I_p(X)) \subset \mathbb{P}^n$ (i.e. the projective variety X is given by the ideal $I_p(X)$). Then $I(C(X)) = I_p(X)$, i.e. the ideals of X and the cone over X coincide.

Proof. Exercise. □

Lemma 4.4 gives a one-to-one correspondence between projective varieties and cones over projective varieties, in such a way that the ideals of the sets in question are preserved.

Theorem 4.1 (Projective Nullstellensatz)**.** *Let $I \subsetneq k[x_0, \dots, x_n]$ be a homogeneous ideal and k be algebraically closed.*

(a) If $V_p(I) \neq \emptyset$ then $I_p(V_p(I)) = \sqrt{I}$.

(b) If $V_p(I) = \emptyset$ then $\sqrt{I} \supset \langle x_0, \dots, x_n \rangle$. That is, either $I = (1)$ or $\sqrt{I} = \langle x_0, \dots, x_n \rangle$.

Proof. Let $X = V_p(I)$. If $X \neq \emptyset$, by Lemma 4.4,

$$I_p(X) = I(C(X)) = I(V(I)) = \sqrt{I},$$

where the last equality is from the affine Nullstellensatz (Theorem 3.1). This proves (a).

For (b), if $X = \emptyset$ then since I is homogeneous, the variety $V(I) \subset \mathbb{A}^{n+1}$ is either empty or the single point $(0, \dots, 0)$. If $V(I) = \emptyset$ then $I = (1)$ by the affine Nullstellensatz. If $V(I) = (0, \dots, 0)$ then again by the affine Nullstellensatz,

$$I(V(I)) = \sqrt{I} = \langle x_0, \dots, x_n \rangle.$$

□

4.6. **Projective coordinate ring.** If $X = V_p(I) \subset \mathbb{P}^n$ then in analogy to the affine case one may associate the **projective coordinate ring**

$$\Gamma_p(X) = k[x_0, \dots, x_n]/I.$$

Since I is a homogeneous ideal, it follows that $\Gamma_p(X)$ is a homogeneous ring. We may choose the grading

$$(\Gamma_p(X))_i = \rho((k[x_0, \dots, x_n])_i),$$

where $\rho : k[x_0, \dots, x_n] \to k[x_0, \dots, x_n]/I$ is the natural projection. (But there are many other gradings, e.g. obtained by shifting this one.)

Another significant difference from the affine case is that the projective coordinate ring *depends* on the ambient projective space $\mathbb{P}^n$.

Example 4.5. *Consider the map $\nu_2 : \mathbb{P}^1 \to \mathbb{P}^2$ defined by*

$$[x_0 : x_1] \mapsto [x_0^2 : x_0 x_1 : x_2^2].$$

(This is a particular case of the Veronese embedding.*) It is easy to see that ν_2 is injective, and we will see later that $\mathbb{P}^1 \simeq \nu_2(\mathbb{P}^1)$. Further,*

$$\nu_2(\mathbb{P}^1) = V_p(x_1^2 - x_0 x_2);$$

(a smooth conic in $\mathbb{P}^2$). But

$$\Gamma_p(\mathbb{P}^1) = k[u,v] \not\simeq k[x_0, x_1, x_2]/\langle x_1^2 - x_0 x_2 \rangle = \Gamma_p(\nu_2(C)).$$

Indeed, the left is a polynomial ring, thus a UFD, while the second ring is not (x_1^2 has two factorizations). Geometrically, the affine variety of the left hand side is $\mathbb{A}^2$, but the right hand side is a cone over a smooth conic, which is singular at the origin.

Definition 4.2. *Let $0 \neq F \in \Gamma_p(X)$ of degree > 0. The* **distinguished open set** *$D_F^+ \subset X$ is defined by*

$$D_F^+ := \{x \in X : F(x) \neq 0\}.$$

Clearly D_F^+ is a non-empty open set in X.

As an example, $D_{x_i}^+ = U_i \simeq \mathbb{A}^n$ from §4.2.

Lemma 4.6. *The distinguished open sets D_F^+ form a basis for the Zariski topology of $X \subset \mathbb{P}^n$. In fact, any open subset of X is a finite union of distinguished open sets D_F^+.*

Proof. Exercise. □

4.7. **A word about morphisms of projective varieties.** A natural way to define a morphism $\varphi : \mathbb{P}^n \to \mathbb{P}^m$ is to specify an $(m+1)$-tuple of homogeneous polynomials $F_0, \dots, F_m \in k[x_0, \dots, x_n]$ of the same degree. If in addition the only common zero of the F_i's is $(0, \dots, 0)$, then

$$\varphi : \mathbb{P}^n \to \mathbb{P}^m; \quad \varphi(x) = [F_0(x) : \ldots : F_m(x)] \tag{4.1}$$

is a well defined map, and it is a morphism. Here are same examples:

The **Veronese embedding** $\nu_d : \mathbb{P}^n \to \mathbb{P}^N$ is defined by

$$\nu_d[x_0 : \ldots : x_n] = [\ldots : x_0^{d_0} \cdot \ldots \cdot x_n^{d_n} : \ldots], \tag{4.2}$$

where $\sum d_i = d$.

The **Segre embedding** $\nu_{a,b} : \mathbb{P}^a \times \mathbb{P}^b \to \mathbb{P}^{(a+1)(b+1)-1}$ is defined by

$$\nu_{a,b}([x_0 : \ldots : x_a], [y_0 : \ldots : y_b]) = [\ldots : x_i y_j : \ldots]. \tag{4.3}$$

It is not hard to see that the images of both ν_d and $\nu_{a,b}$ are projective varieties. For example, the image of

$$\nu_d : \mathbb{P}^1 \to \mathbb{P}^n; \quad [x_0 : x_1] \mapsto [x_0^n : x_0^{n-1} x_1 : \ldots : x_1^n],$$

is given by

$$\nu_d(\mathbb{P}^1) = \{[Z_0 : \ldots : Z_n] : \operatorname{rank} \begin{pmatrix} Z_0 & Z_1 & \ldots Z_{n-1} \\ Z_1 & Z_2 & \ldots Z_n \end{pmatrix} \leq 1\}.$$

A similar determinantal description may be given for the image of the Segre embedding. TODO.

Further, both maps are injective, and in fact are isomorphisms onto their images (to be proved later). As a consequence, the product of projective spaces is a projective variety.

However, asking for *all* morphisms to be of this form is too strong. For example, consider a potential map

$$\varphi : \mathbb{P}^2 \to \mathbb{P}^1; \quad [x_0 : x_1 : x_2] \mapsto [F_0(x) : F_1(x)],$$

where $F_0, F_1 \in k[x_0, x_1, x_2]$ are homogeneous polynomials of the same degree. We will see later that codim $V_p(F_0) \cap V_p(F_1) \leq 2$; in other words the common zero locus of F_0, F_1 will be nonempty in $\mathbb{P}^2$. A similar argument implies that there are *no non-constant morphisms* $\mathbb{P}^2 \to \mathbb{P}^1$. In practice, we will write morphisms form any projective variety $\varphi : Z \to \mathbb{P}^m$ as in Equation (4.1) above; we will allow common zero loci, but then we need to analyze whether one can use other relations (e.g. the equations of Z) to resolve the indeterminacies.

In general, a morphism will be defined by *local data*, i.e. $\varphi : Z \to \mathbb{P}^m$ is a morphism iff for any affine covering V_i of $\mathbb{P}^n$ and any affine covering U_i of Z, the restrictions of φ to $\varphi^{-1}(V_i) \cap U_j \to U_i$ give a morphism of *affine* varieties. More on this is discussed in §8 below.

This leads us to the next section.

4.8. **Exercises.** 1. Prove Lemma 4.4.

5. Sheaves, ringed spaces, and affine algebraic varieties

In this section we introduce the main objects of the 'classical' algebraic geometry, in their natural context. A *variety* will be a pair $(X, \mathcal{O}_X)$ of a topological space X and a sheaf $\mathcal{O}_X$ of regular functions on X, which is locally isomorphic to an affine variety. The affine varieties and and projective varieties are examples of these, but so will be any open subset of these. This generality is required to define basic objects such as regular functions of projective varieties, and morphisms of these.

We start with a simple example, that of defining a global regular function on $\mathbb{P}^1$. Write $\mathbb{P}^1 = U_0 \cup U_1$, where $U_i = \{[x_0 : x_i] : x_i \neq 0\}$. A regular function on U_0 is a polynomial $P(t) = \sum a_i t^i$, where $t = \frac{x_1}{x_0}$. Similarly a regular function on U_1 is a polynomial $Q(1/t) = \sum b_j(1/t^j) \in k[1/t]$. To define a global regular function, the two functions must agree on the common domain $U_0 \cap U_1$, given by $t \neq 0$. The condition that $P(t) = Q(1/t)$ forces that $P(t) = Q(t) \equiv a_0$, i.e. both are a constant.

This shows that to define a meaningful notion of regular functions on $\mathbb{P}^1$ we need to take into consideration local data. This is achieved by the notion of a *sheaf*.

5.1. **Sheaves.**

Definition 5.1. *Let X be a topological space. A* **presheaf** *$\mathcal{F}$ on X is given by the following data:*

- *For each open set $U \subset X$, a set $\mathcal{F}(U)$;*
- *For every pair of open sets $V \subset U$ in X, a restriction map $r_{V,U} : \mathcal{F}(U) \to \mathcal{F}(V)$,*

such that these satisfy the following compatibility conditions:

(1) *If $W \subset V \subset U$ are open sets, then $r_{WU} = r_{WV} r_{VU}$;*
(2) *$r_{UU} = id_{\mathcal{F}(U)}$.*

Denote by $f_{|V} := r_{VU}(f)$. The presheaf $\mathcal{F}$ is a **sheaf** *if in addition it satisfies the following gluing property:*

- *(patching axiom) For any open set U such that $U = \bigcup U_i$ is covered by some open sets U_i and for any $f_i \in \mathcal{F}(U_i)$ $(i \in I)$ such that*
$$(f_i)_{|U_i \cap U_j} = (f_j)_{|U_i \cap U_j}/,$$
there exists a unique $f \in \mathcal{F}(U)$ such that $f_{|U_i} = f_i$.

An element $f \in \mathcal{F}(U)$ is called a **section** over U.

Example 5.1. *If X is a topological space, then the assignment*
$$U \mapsto \{f : U \to K : f \text{ continuous function on } U\},$$
defines a presheaf of K-valued functions, denoted by $\mathcal{F}un(X; K)$.

More generally, if X, Y are topological spaces, then we may define a presheaf $\mathcal{F}un(X; Y)$ by
$$\mathcal{F}un(X, Y)(U) = \{f : U \to Y : f \text{ continuous function on } U\}.$$

The restriction map r_{VU} is given by the usual restriction of functions. Both are in fact sheaves, as a map $f : U = \bigcup U_i \to Y$ is continuous if and only if all $f_{|U_i}$ are continuous.

A **morphism** *of presheaves $\varphi : \mathcal{F} \to \mathcal{G}$ is a collections of maps $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ which commutes with restrictions:*

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi_U} & \mathcal{G}(U) \\ \big\downarrow{\scriptstyle r_{VU}} & & \big\downarrow{\scriptstyle r_{VU}} \\ \mathcal{F}(V) & \xrightarrow{\varphi_V} & \mathcal{G}(V) \end{array} \tag{5.1}$$

Example 5.2. *(A presheaf which is not a sheaf.) The sheaf of constant K-valued functions on a topological space X is a presheaf, but in general not a sheaf. Indeed, if X is disconnected, then constant sections over the components may not glue to a global section, unless all section values are the same.*

Definition 5.2. *(Stalks) Let $\mathcal{F}$ be a sheaf on X and $x \in X$. The collection $\mathcal{F}(U)$, with $x \in U$ forms an inverse system. Define*

$$\mathcal{F}_x := \varinjlim_{x \in U} \mathcal{F}(U).$$

This is called the **stalk** *of $\mathcal{F}$ at x.*

Example 5.3. *If $\mathcal{F}$ is the sheaf of continuous functions $f : U \to \mathbb{R}$, then the stalk at x consists of* **germs** *of continuous functions. Two functions f, g have the same germ at x if $f = g$ in a neighborhood of x.*

Definition 5.3. *(Sheafification) For simplicity, let's assume that $\mathcal{F}$ is a presheaf of K-valued functions on X. Then there exists a sheaf $\mathcal{F}^+ \supset \mathcal{F}$, called the* **sheafification** *of $\mathcal{F}$, defined by gluing local data, as follows:*

$$\mathcal{F}^+(U) = \{f : U \to K : \forall x \in U, \exists V \subset U \text{ open }, x \in V, \text{ and } g \in \mathcal{F}(V) \text{ s.t. } f_{|V} = g\}.$$

This will be a sheaf, by definition. One may actually show that $\mathcal{F}^+$ is the 'best possible sheaf obtained from $\mathcal{F}$', in the sense that if $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism to a sheaf $\mathcal{G}$, then it extends uniquely to a morphism $\varphi^+ : \mathcal{F}^+ \to \mathcal{G}$. (Exercise!)

5.2. **Ringed spaces.** Among the most important sheaves are the sheaves of regular functions. These are sheaves of rings, or, more precisely, sheaves of k-algebras. This means that the restriction maps r_{VU}, and the maps φ_U from diagram 5.1, are homomorphisms of k-algebras.

Definition 5.4. *A* **ringed space** *is a pair $(X, \mathcal{O}_X)$ where X is a topological space, and $\mathcal{O}_X$ is a sheaf of k-algebras. The sheaf $\mathcal{O}_X$ is called the* **structure sheaf** *of X.*

Assumption. From now on, we assume that k is algebraically closed, and that $\mathcal{O}_X$ is a sheaf of k-valued *functions* containing the constant functions.

Definition 5.5. *(Morphisms of ringed spaces.) Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two ringed spaces. A morphism of ringed spaces $\varphi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is given by a continuous map $\varphi : X \to Y$ such that*

$$\forall\ V \subset Y \text{ open, and } \forall g \in \mathcal{O}_Y(V), \quad g \circ \varphi \in \mathcal{O}_X(\varphi^{-1}(V)).$$

For $V \subset Y$ open, denote by

$$\varphi_V^* : \mathcal{O}_Y(V) \to \mathcal{O}_X(\varphi^{-1}(V)); \quad g \mapsto g \circ \varphi.$$

This is a homomorphisms of k-algebras.

Remark 5.1. *The assumption that $\mathcal{O}_X$ is a sheaf of functions means that the algebra homomorphisms φ_U commute with restriction homomorphisms, i.e. for any $V \subset U$, the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{O}_Y(U) \arrow[r, "\varphi_U^*"] \arrow[d, "r_{\varphi^{-1}(V)\varphi^{-1}(U)}"'] & \mathcal{O}_X(\varphi^{-1}(U)) \arrow[d, "r_{VU}"'] \\
\mathcal{O}_Y(V) \arrow[r, "\varphi_V^*"] & \mathcal{O}_X(\varphi^{-1}(V))
\end{tikzcd}$$

Without this assumption, the commutation of the diagram will be part of the data of what it means to be a morphism of ringed spaces.

Example 5.4. *If we consider $(X, \mathcal{O}_X)$ to be a differential manifold endowed with the sheaf of C^∞ functions on X, then the condition that $\varphi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism is equivalent to requiring that φ is a differentiable map.*

5.3. **Affine algebraic varieties as ringed spaces.** Let $X = V(I) \subset \mathbb{A}^n$ be an affine variety, where I is a radical ideal. We would like to define a sheaf of regular functions $\mathcal{O}_X$ so that $(X, \mathcal{O}_X)$ is a ringed space. We already know what the global regular functions should be:

$$\mathcal{O}_X(X) = \Gamma(X) = k[x_1, \ldots, x_n]/I.$$

In order to define the regular functions in $\mathcal{O}_X(U)$ for $U \subset X$ arbitrary open set, the strategy is to cover it by distinguished open sets,

$$U = D_{f_1} \cup \ldots \cup D_{f_p},$$

define regular functions $\mathcal{O}_X(D_{f_p})$, then 'glue' these functions to give regular functions on U. To make this precise we need a lemma.

Lemma 5.5. *Let X be a topological space, $\mathcal{U}$ a basis of open sets in X, and K a set. Assume we are given an assignment:*

$$U \mapsto \mathcal{F}(U) = \{f : U \to K : \text{ functions } \}, \quad \forall U \in \mathcal{U}$$

such that:

- *(Restriction) If $V \subset U$ are in $\mathcal{U}$, and $s \in \mathcal{F}(U)$, then $s_{|V} \in \mathcal{F}(V)$.*

- *(Gluing) If $U \in \mathcal{U}$ is covered by open sets $U_i \in \mathcal{U}$, and $s : U \to K$ is a function such that $s_{|U_i} \in \mathcal{F}(U_i)$, then $s \in \mathcal{F}(U)$.*

Then there exists a unique sheaf of K-valued functions $\overline{\mathcal{F}}$ such that

$$\overline{\mathcal{F}}(U) = \mathcal{F}(U), \quad \forall U \in \mathcal{U}.$$

Proof. Any open set $U \subset X$ is covered by open sets $U_i \in \mathcal{U}$. For $U = \bigcup_{i \in I} U_i$, define

$$\overline{\mathcal{F}}(U) = \{s : U \to K : \forall i, s_{|U_i} = s_i\}.$$

It is not hard to check that this assignment is independent of the way we write $U = \bigcup U_i$, and that this is a sheaf. □

Let $X = V(I)$ be an algebraic variety, considered as a topological space under Zariski topology. It has a basis of open sets

$$D_f := X \setminus V(f) = \{x \in X : f(x) \neq 0\}$$

for $f \in \Gamma(X)$. Lemma 5.5 allows us to define a sheaf by specifying its values on the sets D_f. Recall that $\mathcal{F}un(X; k)$ defined by

$$\mathcal{F}un(X; k)(U) = \{f : U \to k : \text{ continuous } \}$$

denotes the sheaf of continuous k-valued functions.

Theorem 5.1. *Let $X = V(I) \subset \mathbb{A}^n$ be an affine algebraic variety. Then there exists a unique sheaf of k-algebras on $\mathcal{O}_X$ on X, called the* **structure sheaf** *of X, such that for any $0 \neq f \in \Gamma(X)$,*

$$\mathcal{O}_X(D_f) := \Gamma(X)_f = \{\frac{P}{f^n} : P \in \Gamma(X), \quad n \geq 0\}.$$

Further, there is an injective map of sheaves of k-algebras $\mathcal{O}_X \to \mathcal{F}un(X; k)$, sending

$$\frac{P}{f^n} \mapsto \frac{P(x)}{f(x)^n}, \quad x \in D_f.$$

(For a brief review of definitions about localized rings, see §26.1. More details are given in [AM69, Ch. 3].)

Proof. We will apply Lemma 5.5. To start we will show that for $0 \neq f \in \Gamma(X)$ there is a well defined ring homomorphism

$$\mathcal{O}_X(D_f) \to \{f : D_f \to k : \text{ continuous } \}, \quad \frac{P}{f^n} \mapsto \frac{P(x)}{f(x)^n}.$$

To show well-defineness, we need to show that for each $0 \neq f \in \Gamma(X)$, the elements of $\Gamma(X)_f$ are functions on D_f. This follows because $\frac{P_1}{f^{n_1}} = \frac{P_2}{f^{n_2}}$ as elements in $\Gamma(X)_f$ if and only if

$$\exists p \in \mathbb{N} \quad f^p(P_1 f^{n_2} - P_2 f^{n_1}) = \overline{0} \in \Gamma(X).$$

Regarding $f = f(x)$ as a k-valued function on D_f, and since $f(x) \neq 0$ on D_f, this implies that

$$P_1(x)f(x)^{n_2} - P_2(x)f(x)^{n_1} = 0 \quad \forall x \in D_f, \tag{5.2}$$

therefore

$$\frac{P_1(x)}{f(x)^{n_1}} = \frac{P_2(x)}{f(x)^{n_2}} \in \mathcal{F}un(X;k)(D_f).$$

Next we show that this map is injective. Assume that

$$\frac{P_1(x)}{f(x)^{n_1}} = \frac{P_2(x)}{f(x)^{n_2}} \in \mathcal{F}un(X;k)(D_f).$$

Then Equation (5.2) holds for all $x \in D_f$, and regarding f, P_1, P_2 as polynomials in $k[x_1, \ldots, x_n]$, we obtain that

$$f(x)(P_1(x)f(x)^{n_2} - P_2(x)f(x)^{n_1}) = 0 \quad \forall x \in X.$$

Then the Nullstellensatz implies that

$$f(P_1 f^{n_2} - P_2 f^{n_1}) \in \sqrt{I} = I \iff f(P_1 f^{n_2} - P_2 f^{n_1}) = \overline{0} \in \Gamma(X).$$

The last condition means precisely that

$$\frac{P_1}{f^{n_1}} = \frac{P_2}{f^{n_2}} \in \Gamma(X)_f.$$

Next we show that the restriction and gluing conditions from Lemma 5.5 hold.

Restriction condition. Let $\overline{0} \neq f_1, f_2 \in \Gamma(X)$ such that $D_{f_1} \subset D_{f_2}$. This is equivalent to $V(f_2) \subset V(f_1)$, giving

$$f_1 \in I(V(f_1)) \subset I(V(f_2)) = \sqrt{\langle f_2 \rangle},$$

by Nullstellensatz. Then $f_1^p = g f_2$ in $\Gamma(X)$ for $g \in \Gamma(X)$ and some positive integer p. Clearly, $g \in D_{f_1}$. For any $P \in \Gamma(X)$, this gives the following identity in $\Gamma(X)$:

$$P f_1^p = P g f_2.$$

Therefore, as functions on D_{f_1},

$$\frac{P(x)}{f_2(x)} = \frac{P(x)g(x)}{g(x)f_2(x)} = \frac{P(x)g(x)}{f_1(x)^p},$$

showing that the restriction of $\frac{P}{f_2}$ to D_{f_1} is an element of $\Gamma(X)_{f_1}$.

Gluing condition. Let $U = D_f = \bigcup_{i=1}^p D_{f_i}$ be an open set, where $\overline{0} \neq f, f_i \in \Gamma(X)$. It follows that

$$V(f) = X \setminus D_f = V(f_1) \cap \ldots \cap V(f_p) = V(f_1, \ldots, f_p). \tag{5.3}$$

(See Proposition 3.4(iv).) Let $s : U \to k$ be a continuous function such that $s_i := s_{|D_{f_i}} \in \Gamma(X)_{f_i}$. We need to show that $s \in \Gamma(X)_f$. Write

$$s_i = \frac{P_i}{f_i^{n_i}} \in \Gamma(X)_{f_i}.$$

The intersection $D_{f_i} \cap D_{f_j} = D_{f_i f_j}$, and the equalities on the overlaps mean that

$$\frac{P_i(x)}{f_i(x)^{n_i}} = \frac{P_j(x)}{f_j(x)^{n_j}} \quad \forall x \in D_{f_i f_j}, \quad \forall i, j.$$

Possibly by multiplying the numerators by some powers of f_i's, we may assume that $n_i = n_j = N$ for all i, j. Then

$$P_i(x) f_j(x)^N - P_j(x) f_i(x)^N = 0 \quad \forall x \in D_{f_i f_j},$$

implying that

$$f_i(x) f_j(x)(P_i(x) f_j(x)^N - P_j(x) f_i(x)^N) = 0 \quad \forall x \in X.$$

As before, the Nullstellensatz implies that

$$(f_i f_j)(P_i f_j^N - P_j f_i^N) = \overline{0} \in \Gamma(X). \tag{5.4}$$

We are looking for $P \in \Gamma(X)$ and $m \in \mathbb{N}$ such that over D_{f_i},

$$s_i = \frac{P}{f^m} \Longleftrightarrow \frac{P_i}{f_i^N} = \frac{P}{f^m}.$$

By Nullstellensatz and Equation (5.3) applied to the ideal $V(f_1, \ldots, f_p) = V(f_1^{1+N}, \ldots, f_p^{1+N})$, there exists $m \in \mathbb{N}$ such that

$$f^m = \sum b_i f_i^{1+N}.$$

Take $P := \sum P_i b_i f_i$. Then for a fixed i_0 we have

$$P f_{i_0}^{1+N} = \sum b_i (f_i f_{i_0}) P_i f_{i_0}^N = \sum b_i (f_i f_{i_0}) P_{i_0} f_i^N = P_{i_0} f_{i_0} \sum b_i f_i^{1+N} = P_{i_0} f_{i_0} f^m$$

where the second equality follows from Equation (5.4). This implies that in $D_{f_{i_0}}$,

$$\frac{P(x)}{f(x)^m} = \frac{P_{i_0}(x)}{f_{i_0}(x)^N}$$

□

Remark 5.2. *It is possible that $D_{f_1} = D_{f_2}$ for different $f_1, f_2 \in \Gamma(X)$. Then*

$$\sqrt{\langle f_1 \rangle} = \sqrt{\langle f_2 \rangle}$$

and as in the beginning part of the proof above one checks that $\Gamma(X)_{f_1} = \Gamma(X)_{f_2}$.

Remark 5.3. *Finding sections of $\mathcal{O}_X(U)$ for arbitrary U is in general much more subtle. For instance, one can show that*

$$\mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2 \setminus 0) = \mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2).$$

This corresponds to the classical fact from complex analysis that a multivariable analytic function may be extended over sets of codimension ≥ 2.

5.4. **Affine algebraic varieties, revisited.** We are now in position to re-define, and slightly extend, the notion of affine algebraic variety from the beginning of the notes.

Definition 5.6. *An* **affine algebraic variety** *is a ringed space $(X, \mathcal{O}_X)$ isomorphic (as ringed spaces) with $(V, \mathcal{O}_V)$, where $V = V(I) \subset \mathbb{A}^n$ is an 'old' affine variety together with its structure sheaf.*

A morphism of affine algebraic variety is a morphism of ringed spaces.

One advantage to viewing affine algebraic varieties as ringed spaces is that this is an intrinsic notion, independent of coordinates. The next proposition shows that when viewed this way, the set of 'new' affine varieties is strictly larger than the 'old' set of affine varieties.

Proposition 5.6. *Let $V = V(I) \subset \mathbb{A}^n$ be an affine algebraic variety, and let $0 \neq f \in \Gamma(V)$. Then $(D_f, (\mathcal{O}_V)_{|D_f})$ is an affine algebraic variety.*

Proof. Consider a representative $\overline{f} \in k[x_1, \ldots, x_n]$ for the element $f \in \Gamma(X)$. Let $I = \langle f_1, \ldots, f_p \rangle$, and define an ideal $J \subset k[x_1, \ldots, x_n, x_{n+1}]$ by

$$J = \langle f_1, \ldots, f_p, 1 - x_{n+1}\overline{f} \rangle.$$

Define $\varphi : D_f \to V(J)$ by

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, \frac{1}{\overline{f}(x_1, \ldots, x_n)}).$$

Observe that φ is a continuous bijection. (One way to see continuity is to restrict to the basis of distinguished open sets.) The inverse is the projection $\pi : V(J) \to D_f$, $\pi(x_1, \ldots, x_n, x_{n+1}) = (x_1, \ldots, x_n)$, which is obviously continuous; thus φ is a homeomorphism. In particular, *as sets*, $\varphi(D_f) = V(J)$. We now need to show that J is a radical ideal. To do that we define a ring homomorphism

$$\varphi^* : k[x_1, \ldots, x_n, x_{n+1}]/J \to \Gamma(V)_f; \quad x_i \mapsto x_i (1 \leq i \leq n); \quad x_{n+1} \mapsto \frac{1}{f}.$$

This is induced from a similar ring homomorphism $k[x_1, \ldots, x_n, x_{n+1}]/J \to k[x_1, \ldots, x_n]_{\overline{f}}$, and it is easy to see it is well defined and surjective. We prove it is injective. Let $P(x_1, \ldots, x_n, x_{n+1}) \in k[x_1, \ldots, x_{n+1}]$ and write

$$P(x_1, \ldots, x_n, x_{n+1}) = \sum_j x_{n+1}^j P_j(x_1, \ldots, x_n).$$

Let $x = (x_1, \ldots, x_n)$ and let N be the largest power of the powers of x_{n+1}^j. Assume that $\varphi^*(P(x_1, \ldots, x_n, x_{n+1}) = 0$, i.e.

$$\sum_j \frac{f(x)^{N-j} P_j(x)}{\overline{f(x)}^N} = \overline{0} \in \Gamma(X)_f.$$

This means that

$$f(x)^s \sum_j f(x)^{N-j} P_j(x) = \sum a_i(x) f_i(x) \in I.$$

Regard this as an identity in $k[x_1, \ldots, x_{n+1}]$. After multiplying by x_{n+1}^{N+s}, and working modulo the ideal J (where $x_{n+1}^{N+s} f(x)^{N-j+s} = x_{n+1}^j$), this shows that the left hand side is in J, as claimed.

Now observe that I is radical if and only if $\Gamma(V)$ has no nilpotent elements. Then $\Gamma(V)_f$ has no nilpotent elements (cf. Lemma 26.4), showing that J is radical.

Finally, it remains to show that φ is an isomorphism of ringed spaces. This is left as an exercise. (Idea: reduce to distinguished open sets.) $\square$

Example 5.7. *Let $S := \{p_1, \ldots, p_k\} \subset \mathbb{A}^1$ be a finite set of points. Then $\mathbb{A}^1 \setminus S$ is affine. (Apply Proposition 5.6 to $f = \prod(x - p_i)$.)*

Example 5.8. *The group $\mathrm{GL}_n(\mathbb{C})$ is an affine algebraic variety, as it may defined as the distinguished open set $\mathrm{M}_n(\mathbb{C})_{\det}$.*

5.5. **The equivalence between morphisms and coordinate rings.** For an affine algebraic variety $(X, \mathcal{O}_X)$, denote by

$$\Gamma(X) := \mathcal{O}_X(X)$$

the **coordinate ring** of X.

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be affine algebraic varieties and $\varphi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ a morphism. By definition, $\varphi_Y^* : \Gamma(Y) \to \Gamma(X)$ is a k-algebra homomorphism, sending a regular element $g \in \Gamma(Y)$ to $\varphi_Y^*(g) = g \circ \varphi$.

Conversely, assume we are given a k-algebra homomorphism $F^* : \Gamma(Y) \to \Gamma(X)$. We show that this induces a morphism of algebraic varieties $F^{\#} : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$. First, as in Lemma 3.16 we obtain a continuous map $\varphi : X \to Y$ such that $\varphi^* = F^*$. Next, if $f \in \Gamma(Y)$ with $\overline{f} := F^*(f) \in \Gamma(X)$, then we have an induced homomorphism

$$\varphi^* : \Gamma(Y)_f \to \Gamma(X)_{\overline{f}}; \quad \frac{a}{f^n} \mapsto \frac{F^*(a)}{\overline{f}^n}.$$

This induces homomorphisms $\mathcal{O}_Y(D_f) \to \mathcal{O}_X(D_{\overline{f}})$. As in Lemma 5.5 this data extends uniquely to data giving a morphism of ringed spaces.

This argument proves:

Proposition 5.9. *Let X, Y be affine varieties. There is a one-to-one, direction reversing, correspondence*

$$Hom_k(X, Y) \longleftrightarrow Hom_k(\Gamma(Y), \Gamma(X)),$$

between morphisms of affine algebraic varieties and k-algebra homomorphisms of coordinate rings.

Example 5.10. *Consider $U = \mathbb{A}^2 \setminus 0$. One may show that*

$$\mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2 \setminus 0) = \mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2).$$

From this it follows that $\mathbb{A}^2 \setminus 0$ is not an affine variety. (If it were, the inclusion $\mathbb{A}^2 \setminus 0 \hookrightarrow \mathbb{A}^2$ would be an isomorphism, but it is not.)

6. Algebraic varieties

6.1. **Definition and basic properties.**

Definition 6.1. *A topological space X is* **quasi-compact** *if any open covering has a finite subcover.*

Note that we do not require that the space is Hausdorff: a quasi-compact space which is Hausdorff is called **compact**.

Lemma 6.1. *(a) Closed subsets of a quasi-compact space X are quasi-compact.*
(b) Finite unions of quasi-compact sets are quasi-compact.

Proof. Same as for the 'compact' situation, thus omitted. □

Example 6.2. *Affine algebraic varieties are quasi-compact but in general they are not compact.*

Lemma 6.3. *Any affine algebraic variety X is quasi-compact. (But in general it is not Hausdorff, thus not compact.)*

Proof. Exercise. □

Lemma 6.4. *Let X be an affine algebraic variety and $U \subset X$ open. Then U is quasi-compact.*

Proof. We may assume $X \subset \mathbb{A}^n$. Take $X \setminus U = V(f_1, \ldots, f_p)$ for $f_i \in \Gamma(X)$. Then $U = D_{f_1} \cup \ldots \cup D_{f_p}$. Each of D_{f_i} is affine by Proposition 5.6, thus quasi-compact. Then the claim follows from Lemma 6.1. □

Definition 6.2. *An* **algebraic variety** *is a quasi-compact ringed space $(X, \mathcal{O}_X)$ which is locally isomorphic to an affine variety, in the following sense:*
For any $x \in X$ there exists an open set U containing x such that $(U, (\mathcal{O}_X)_{|U})$ is an affine variety.
If $U \subset X$ is an open set which is isomorphic to an affine algebraic variety then U is called an **affine open set***.*

To simplify notation, when we speak about varieties, affine varieties, etc, instead of writing $(X, \mathcal{O}_X)$ we will simply write X, and we omit the associated sheaf of regular functions.

Lemma 6.5. *Let X be an algebraic variety. Then the affine open sets form a basis of open sets for the topology of X.*

Proof. Let $U \subset X$ be open. We need to show that U may be written as a union of affine open sets. By definition $X = \bigcup U_i$ where each U_i is affine, thus $U = \bigcup (U \cap U_i)$. It suffices to show that $U \cap U_i$ is a union of affine algebraic varieties. W.l.o.g. we may assume that $U_i \subset \mathbb{A}^n$ is a (closed) subset. Then $U_i \setminus (U_i \cap U)$ is again closed in $\mathbb{A}^n$, thus given by some equations $f_1, \ldots, f_j \in \Gamma(U_i)$. We deduce that

$$U \cap U_i = U_i \setminus (V(f_1) \cap \ldots \cap V(f_j)) = \bigcup_{i=1}^{j} (U_i \setminus V(f_j))$$

Each $U_i \setminus V(f_j)$ is a distinguished open set, thus affine by Proposition 5.6. ☐

Next two propositions show that open sets and closed sets of varieties are varieties.

Proposition 6.6. *Let X be an algebraic variety and $U \subset X$ is open. Then $(U, (\mathcal{O}_X)_{|U})$ is an algebraic variety.*

Proof. Quasi-compactness is an exercise. The restriction of a sheaf to an open set is always a sheaf, which finishes the proof. ☐

Example 6.7. *Proposition 6.6 and Example 5.10 shows that $\mathbb{A}^2 \setminus 0$ is a variety which is not an affine variety.*

Proposition 6.8. *Let X be an algebraic variety and $Z \subset X$ be a closed subset. For $V \subset Z$ open, define*

$$\mathcal{O}_Z(V)' := \{f : V \to k : \exists U \subset X \text{ open and } g : U \to k, \text{ s.t. } V = U \cap Z, g_{|V} = f\}.$$

Define $\mathcal{O}_Z(V) := \mathcal{O}_Z^{+}(V)'$ (the sheafification of $\mathcal{O}_Z(V)'$). Then $(Z, \mathcal{O}_Z)$ is an affine variety.

Remark 6.1. *In general $\mathcal{O}'_Z$ is only a presheaf: two restricted sections agreeing on an overlap $V_1 \cap V_2$ may not agree on the intersection $U_1 \cap U_2$ of the corresponding open sets $U_i \subset X$.*

Before starting the proof, we recall the definition of the sheafification in the case at hand. If $V \subset Z$ open,

$$\mathcal{O}_Z(V) := \{f : V \to k : \forall x \in V, \exists U \subset X \text{ open }, x \in U, \text{ and } g \in \mathcal{O}_X(U) \text{ s.t. } f_{|U \cap V} = g_{|U \cap V}\}.$$

Proof. Since Z is closed, it is quasi-compact (Lemma 6.1). Clearly $(Z, \mathcal{O}_Z)$ is a ringed space. It remains to show that if $U \subset X$ is open affine then $(U \cap Z, (\mathcal{O}_Z)_{|(U \cap Z)})$ is affine. For this, we may assume that $X \subset \mathbb{A}^n$ is affine and $Z \subset X$ is closed. We have two sheaves defined on Z: the sheaf $\mathcal{O}_Z$ above, and the usual sheaf of regular functions on Z (regarded as a closed subset of X), which we temporarily denote by $\mathcal{R}_Z$. We need to show that

$$\mathcal{O}_Z = \mathcal{R}_Z.$$

As in the proof of Lemma 6.5, one can show that the distinguished open set $D_f \cap Z$, where $f \in \Gamma(X)$, form a basis of open sets for the topology of Z. Then from Lemma 5.5 it suffices to show that the two sheaves agree on these distinguished open sets.

To start, take $f \in \Gamma(X)$ and $\overline{f}$ its image in $\Gamma(Z) = \mathcal{R}_Z(Z)$. If $f_{|Z} = 0$, there is nothing to do, as $D_{\overline{f}} = \emptyset$. Otherwise,

$$\mathcal{R}_Z(D_{\overline{f}}) = \{\frac{a}{\overline{f}^r} : a \in \Gamma(Z)\} \subset \mathcal{O}_Z(D_{\overline{f}}).$$

Conversely, take $V \subset Z$ open and $\overline{s} \in \mathcal{O}_Z(V)$. Since the distinguished open sets form a basis for the topology of Z, it follows that locally $\overline{s}$ is given by restrictions of sections

$$s_i \in \mathcal{O}_X(D_{g_i}) = \Gamma(X)_{g_i},$$

where $V = \cup(Z \cap D_{g_i})$ and $g_i \in \Gamma(X)$. The restrictions of s_i give sections $\overline{s}_i$ of $\mathcal{O}_Z$ over $D_{\overline{g}_i} = D_{g_i} \cap Z$. By definition, the restrictions $\overline{s}_i = (s_i)_{|D_{g_i} \cap Z}$ are in $\mathcal{R}_Z(D_{\overline{g}_i})$, and since both $\mathcal{O}_Z$ and $\mathcal{R}_Z$ are sheaves, it follows that

$$\mathcal{O}_Z(V) \subset \mathcal{R}_Z(V).$$

This finishes the proof. $\square$

Remark 6.2. *We shall see later that the proof above shows that there is a surjective homomorphism of sheaves $\mathcal{O}_X \to \mathcal{O}_Z$. The proof checks surjectivity on stalks of these sheaves.*

Example 6.9. *Let $Z \subset \mathbb{P}^2$ given by equations $x_0x_1 = 0, x_0(x_0 - x_2) = 0$. Then $Z = Z_1 \cup Z_2$ has two components, where $Z_1 \simeq \mathbb{P}^1$ is given by $x_0 = 0$ and $Z_2 = \{[1 : 0 : 1]\}$.*

Take any open set $U \subset \mathbb{P}^2$ such that $Z \subset U$. This is equivalent to requiring that $\mathbb{P}^2 \setminus U \subset \mathbb{P}^2 \setminus Z$, in particular, $(\mathbb{P}^2 \setminus U) \cap V(x_0) = \emptyset$. By Bézout this implies that $\mathbb{P}^2 \setminus U$ is a finite set, implying that

$$\mathcal{O}_{\mathbb{P}^2}(\mathbb{P}^2 \setminus U) = \mathcal{O}_{\mathbb{P}^2}(\mathbb{P}^2) = k,$$

i.e. constant sections. This implies that the restriction

$$\mathcal{O}_{\mathbb{P}^2}(\mathbb{P}^2 \setminus U) \to \mathcal{O}_Z(Z)$$

is **not** *surjective. (Indeed, $\mathcal{O}_Z$ contains more than constant sections, e.g. sections constant on each of the components.)*

Proposition 6.10 (Irreducible decomposition)**.** *Let X be an algebraic variety. Then $X = X_1 \cup \ldots \cup X_p$ where X_i is a closed irreducible algebraic variety and no X_i is included in another X_j.*

Proof. Since X is quasi-compact, it can be covered by finitely many affine varieties U_i. Each U_i has a decomposition by irreducible components (Theorem 3.3). After taking closures, and possibly removing redundant components one obtains the claimed decomposition. $\square$

Definition 6.3. *A closed subset of a variety with its induced sheaf of functions is called a* **subvariety***.*

Remark 6.3. *The notion of subvariety differs among authors. In English circles, a subvariety is usually closed and irreducible, while in textbooks such as that of Perrin, it is simply an intersection of an open and closed subset of a variety (with induced ringed space structure).*

6.2. **Local rings.** The sheaf of regular functions of an algebraic variety may be defined as an intrinsic object. The main tool to do this is the notion of **local ring**, which will turn to be closely related to the notion of stalks, and to rings of fractions.

Definition 6.4. *Let $(X, \mathcal{O}_X)$ be an algebraic variety, endowed with its sheaf of regular functions. Fix $x \in X$ and consider the set of pairs (U, f), where $x \in U \subset X$ is open and $f \in \mathcal{O}_X(U)$. Define the equivalence relation*

$$(U, f) \simeq (V, g) \Longleftrightarrow \exists x \in W \subset U \cap V \text{ open s.t.} f_{|W} = g_{|W}.$$

The equivalence class of (U, f) is called the **germ** *of f at x, denoted by f_x.*

The set of all germs f_x is denoted by $\mathcal{O}_{X,x}$, and it is called the **local ring** *of X at x.*

From a topological point of view, the set $\mathcal{O}_{X,x}$ is clearly 'local', as it depends only on the neighborhood of x. Algebraically, the 'local' terminology is justified by the following proposition.

Proposition 6.11. *The set $\mathcal{O}_{X,x}$ is a local k-algebra. More precisely, it is a k-algebra with the unique maximal ideal*

$$\mathfrak{m}_x := \{f_x \in \mathcal{O}_{X,x} : f(x) = 0\}.$$

Furthermore,

$$\mathcal{O}_{X,x}/\mathfrak{m}_x \simeq k.$$

Proof. Clearly $\mathcal{O}_{X,x}$ has a ring structure given by adding and multiplying germs:

$$(U, f) + (V, g) = (U \cap V, f + g); \quad (U, f) \cdot (V, g) = (U \cap V, f \cdot g).$$

(One should check these operations are well defined; this is left as an exercise.) Note that the 0 of this ring is represented by a germ of the function identically 0 on some neighborhood of x. There is a surjective ring homomorphism:

$$e_x : \mathcal{O}_{X,x} \to k; \quad (U, f) \mapsto f(x).$$

The kernel of e_x is $\ker(e_x) = \mathfrak{m}_x$, and by the first isomorphism theorem

$$\mathcal{O}_{X,x}/\ker(e_x) = \mathcal{O}_{X,x}/\mathfrak{m}_x \simeq k.$$

This implies that $\mathfrak{m}_x$ is a maximal ideal.

Furthermore, any element in $f_x \in \mathcal{O}_{X,x} \setminus \mathfrak{m}_x$ is invertible, with inverse $(1/f)_x$. Indeed, we may choose an open affine set $x \in U$ such that f is defined on U (cf. Lemma 6.5). The hypothesis that $f(x) \neq 0$ means that $x \in D_f$. Then the inverse $1/f \in \mathcal{O}_U(D_f)$ is a regular function on D_f, and it determines the germ $(1/f)_x$. This implies that $\mathfrak{m}_x$ is the unique maximal ideal, and finishes the proof. $\square$

Recall from Theorem 3.2 that if k is algebraically closed, there is a one-to-one correspondence between points $x \in \mathbb{A}^n$ and maximal ideals $\mathfrak{m}_x \subset k[X_1, \ldots, X_n]$ given by

$$x = (x_1, \ldots, x_n) \longleftrightarrow \langle X_1 - x_1, \ldots, X_n - x_n \rangle.$$

If J is any ideal in $k[X_1, \ldots, X_n]$, the ideals in the quotient ring $k[X_1, \ldots, X_n]/J$ are in one-to-one correspondence to the ideals in $k[X_1, \ldots, X_n]$ containing J. Combining the two gives a one-to-one correspondence between points $x \in V(J)$ and maximal ideals in $k[X_1, \ldots, X_n]/J$ given by

$$x \in V(J) \longleftrightarrow \mathfrak{m}_x/J \subset k[X_1, \ldots, X_n]/J.$$

Proposition 6.12. *Let X be an algebraic variety, $x \in X$, and let $x \in U \subset X$ be any affine open set, with coordinate ring $R := \Gamma(U)$. Let $\mathfrak{m}_x \subset R$ be the maximal ideal corresponding to x. Then the following hold:*

(a) There is an isomorphism $R_{\mathfrak{m}_x} \to \mathcal{O}_{X,x}$, which is local, i.e. it sends the (unique) maximal ideal in $R_{\mathfrak{m}_x}$ to $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$.

(b) There are one-to-one correspondences between the following three sets:

- *irreducible (closed) subvarieties X containing x;*
- *prime ideals of $R_{\mathfrak{m}_x}$;*
- *prime ideals $\mathfrak{p} \subset \mathfrak{m}_x \subset R$.*

Proof. W.l.o.g. we may assume that $U \subset \mathbb{A}^n$ and that $x = (x_1, \ldots, x_n)$. Elements of $R_{\mathfrak{m}_x}$ are (classes of) rational functions of the form $\frac{P}{Q}$ where $P, Q \in k[X_1, \ldots, X_n]$ and $Q \notin \mathfrak{m}_x$. This last condition means that $Q(x) \neq 0$; in particular, the germ Q_x is invertible in $\mathcal{O}_{X,x}$. Then we may define a ring homomorphism

$$\varphi_x : R_{\mathfrak{m}_x} \to \mathcal{O}_{X,x}; \quad \frac{P}{Q} \mapsto \frac{P_x}{Q_x}.$$

I leave it as an exercise to check that φ_x is an isomorphism. Furthermore,

$$\varphi_x^{-1}(\mathfrak{m}_x) = \{\frac{P}{Q} \in R_{\mathfrak{m}_x} : P(x) = 0\} \subset \mathfrak{m}_x R_{\mathfrak{m}_x}.$$

This shows that the isomorphism preserves the maximal ideals, therefore it is local.

To prove (b), note that the set of irreducible subvarieties of X containing x is in bijection to the set of irreducible subvarieties of U containing x. (This bijection is obtained by intersecting with U, and the inverse by taking closure in X.) By Nullstellensatz, the latter is in bijection to the set of prime ideals $\mathfrak{p}$ in $R = \Gamma(U)$ such that $\mathfrak{p} \subset \mathfrak{m}_x$. The last equivalence follows from the description of ideals in rings of fractions; see [AM69, Prop. 3.11]. □

Proposition 6.13. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of varieties. Let $x \in X$ and $y = f(x)$. Then f induces local homomorphism of local rings $f^\# : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$, i.e. a homomorphism such that $(f^\#)^{-1}(\mathfrak{m}_x) = \mathfrak{m}_y$.*

Proof. Exercise. (The definition of $f^{\#}$ follows from the diagram

$$\begin{tikzcd} f^{-1}(V) \arrow[r, "f"] \arrow[dr, dashed, "f^{\#}(\varphi)=\varphi\circ f"'] & V \arrow[d, "\varphi"] \\ & k \end{tikzcd}$$

where $y \in V \subset Y$ is open.) □

Remark 6.4. *Proposition 6.12 and Proposition 6.13 will be part of* definitions *of local rings of schemes, and morphisms of schemes.*

7. Projective algebraic varieties, revisited

The goal of this section is to put a variety structure on zero loci of homogeneous polynomials. The resulting structure will be called again a projective algebraic variety.

7.1. **Graded rings and modules.**

A central notion in the study of projective varieties is that of *graded ring* and *graded module*. We briefly recall these, following [AM69, Ch. 10].

A **graded ring** R is a ring together with a family of subgroups $(R_n)_{n\geq 0}$ of the additive group R such that

- $R = \bigoplus R_n$;
- $R_i \cdot R_j \subset R_{i+j}$.

The hypotheses imply that R_0 is a subring and that each R_n is an R_0-module. If $R_+ := \bigoplus_{n>0} R_n$ then R_+ is an ideal of R.

Example 7.1. *(a) The polynomial ring $k[x_0, \dots, x_n]$ is a graded ring, with grading given by total degree.*

(b) For J a homogeneous ideal, the quotient $k[x_0, \dots, x-n]/J$ is a graded ring. If F is a homogeneous polynomial $F \in k[x_0, \dots, x_n]$, then the degree of the image of F in $k[x_0, \dots, x-n]/J$ is the minimal degree achieved by a representative of F.

A **graded module** M over a graded ring R is an M-module R together with a family of subgroups $(M_n)_{n\geq 0}$ such that

- $M = \bigoplus M_n$;
- $R_i \cdot M_j \subset M_{i+j}$.

Each M_n is an R_0-module. An element $x \in M$ is called **homogeneous** if $x \in M_n$ for some n; then n is the **degree** of x. Any $y \in M$ can be written uniquely as a finite sum $y = \sum_n y_n$; the y_n's are called the homogeneous components of y.

For M, N graded R-modules, a homomorphism of graded R-modules is an R-module homomorphism $f : M \to N$ such that $f(M_n) \subset N_n$ for all $n \geq 0$.

If R is a graded ring and $f \in R$ is a homogeneous element then we may define a grading on the localized ring R_f by setting

$$\deg \frac{r}{f^k} = \deg(r) - k \deg(f),$$

whenever r is a homogeneous element. Denote by $R_{(f)}$ the subring of elements in R_f of degree 0.

7.2. **The sheaf of regular functions on a projective variety.**

Let $X := V_p(J) \subset \mathbb{P}^n$ be the zero locus of a homogeneous (radical) ideal J. The coordinate ring $\Gamma_p(X) := k[x_0, \dots, x_n]/J$ is a graded ring. For a homogeneous polynomial $F \in k[x_0, \dots, x_n]$ define

$$D_F^+ := \{x \in X : F(x) \neq 0\}.$$

This is an open set in X, called (as before) a **distinguished open set**.

Definition 7.1. *Let $X := V_p(J) \subset \mathbb{P}^n$ and $F \in k[x_0, \dots, x_n]$ of degree > 0. There exists a unique sheaf of $k-$valued functions on X, denoted by $\mathcal{O}_X$, such that*

$$\mathcal{O}_X(D^+_F) := \Gamma_p(X)_{(F)} = \{\frac{\overline{P}}{\overline{F}^r} : \overline{P} \text{ homogeneous and } \deg \overline{P} = r \deg(\overline{F})\}.$$

Proof. The proof is based again on Lemma 5.5, and it follows the outline of Proposition 5.6. The only observation to make is that elements in $\Gamma_p(X)_{(F)}$ are indeed functions on X, independent of choices of representatives of points in $\mathbb{P}^n$. □

Remark 7.1. *The condition that $\deg(\overline{F}) > 0$ is needed. Indeed, if F is a constant, let's say $F = 1$, then $\Gamma(X)_{(F)}$ contains only constants, so $\mathcal{O}_X(D^+_F)$ consists of constant sections. But this does not lead to a sheaf (e.g. X may be disconnected and there are functions constant on each of the connected components).*

Proposition 7.2. *Let $X := V_p(J) \subset \mathbb{P}^n$. Then the ringed space $(X, \mathcal{O}_X)$ is an algebraic variety.*

Any ringed space which is isomorphic with $(V_p(J), \mathcal{O}_{V_p(J)})$ is called a **projective variety**.

Proof. We first reduce the statement to the situation when $X = \mathbb{P}^n$, therefore we assume for now that $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n})$ is an algebraic variety. Then any closed subset X of $\mathbb{P}^n$ is quasi-compact.

Next we need to check that X is covered by affine open sets. Clearly

$$\mathbb{P}^n = \bigcup_{i=0}^{n} D^+_{x_i}.$$

We will prove in Proposition 7.5 below that each $(D^+_{x_i}, (\mathcal{O}_{\mathbb{P}^n})_{|D^+_{x_i}})$ is isomorphic to the affine variety $(\mathbb{A}^n, \mathcal{O}_{\mathbb{A}^n})$. Then $X \cap D^+_{x_i}$ with restricted sheaf of functions is (isomorphic to) a closed variety of $\mathbb{A}^n$.

One subtlety is that one needs to check that the restriction of the sheaf $\mathcal{O}_X$ to $D^+_{x_i}$ coincides with the sheaf of regular functions on $X \cap D^+_{x_i}$ when regarded as a subvariety in $D^+_{x_i}$. This follows because the sheaf $\mathcal{O}_X$ is the sheafification of the restriction sheaf of regular functions $\mathcal{O}_{\mathbb{P}^n}$ (as in Proposition 6.8). □

7.3. **From affine to projective and back: homogenization and dehomogenization.** Consider the polynomial ring $k[x_0, \dots, x_n]$ and fix an indeterminate, x_0 for simplicity. We will define two operations (depending on x_0):

- (**dehomogenization**) For a homogeneous polynomial $F \in k[x_0, \dots, x_n]$ the dehomogenization is

$$F_\flat := F(1, x_1, \dots, x_n) \in k[x_1, \dots, x_n].$$

  This is a ring homomoprhism $k[x_0, \dots, x_n] \to k[x_1, \dots, x_n]$ (since the evaluation map is a ring homomorphism). One may also apply this to ideals: if $J = \langle P_i :$

$i \in I\rangle \subset k[x_0, \ldots, x_n]$ is a homogeneous ideal, its dehomogenization is

$$J_\flat := \langle (P_i)_\flat : i \in I\rangle \subset k[x_1, \ldots, x_n].$$

- (**homogenization**) Let $f \in k[x_1, \ldots, x_n]$ be a polynomial of total degree d, written as $f = f_d + f_{d-1} + \ldots + f_0$ and f_i is homogeneous of degree i. The homogenization of f is
$$f^\sharp := x_0^d f_0 + x_0^{d-1} f_1 + \ldots + f_d = x_0^d f(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}).$$
Observe that $(fg)^\sharp = f^\sharp g^\sharp$, but in general $(f+g)^\sharp \neq f^\sharp + g^\sharp$; in fact, the latter may not even be homogeneous. Thus homogenization is not a ring homomorphism. For an ideal $I \subset k[x_1, \ldots, x_n]$ the homogenization is defined by:
$$I^\sharp := \langle P^\sharp : P \in I\rangle.$$

Example 7.3. *Let $f = x_1^2 - x_2^2 - 1 \in k[x_1, x_2]$. Then $f^\sharp = x_1^2 - x_2^2 - x_0^2$ and $(f^\sharp)_\flat = f$.*

Example 7.4. *More generally, $(f^\sharp)_\flat = f$ and any homogeneous element in $F \in k[x_0, \ldots, x_n]$ may be written as $F = x_0^d f^\sharp$, for some $f \in k[x_1, \ldots, x_n]$, where x_0^d is the largest power of x_0 dividing F.*

Consider the set $U_0 = D^+_{x_0} \subset \mathbb{P}^n$ defined by $x_0 \neq 0$. There is a bijection

$$\varphi : \mathbb{A}^n \to U_0; \quad (x_1, \ldots, x_n) \mapsto [1 : x_1 : \ldots : x_n]$$

with inverse

$$\varphi^{-1} : U_0 \to \mathbb{A}^n; \quad [x_0 : \ldots : x_n] \mapsto \left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right).$$

The next proposition completes the proof of Proposition 7.2.

Proposition 7.5. *The map $\varphi : (\mathbb{A}^n, \mathcal{O}_{\mathbb{A}^n}) \to (U_0, (\mathcal{O}_{\mathbb{P}^n})_{|U_0})$ is an isomorphism of (affine) varieties.*

Proof. We need to show that φ is a homeomorphism (in Zariski topology), and that φ and φ^{-1} are morphisms of ringed spaces. Observe that as in the affine case, the distinguished open sets form a basis for the respective topologies. Therefore all the 'local' statements will be proved when restricted to these sets.

First observe that if $F \in k[x_0, \ldots, x_n]$ is homogeneous, then

$$\varphi^{-1}(D^+_F \cap U_0) = D_{F_\flat},$$

and for $f \in k[x_1, \ldots, x_n]$,

$$\varphi(D_f) = D^+_{f^\sharp} \cap U_0.$$

Since

$$(f^\sharp)_\flat = f \text{ and } (F_\flat)^\sharp = F/(x_0^d)$$

(for appropriate degree d), we deduce that φ is a homeomorphism. We now show that φ and φ^{-1} are morphisms of ringed spaces, i.e. they send regular sections to regular sections.

Using that $D_F^+ \cap D_{x_0}^+ = D_{x_0 F}^+$, one may check (exercise!) that:

$$\varphi^*\Big(\frac{P(x_0,\dots,x_n)}{(x_0F(x_0,\dots,x_n))^r}\Big) = \frac{P(1,x_1,\dots,x_n)}{F(1,x_1,\dots,x_n)^r}.$$

To define the map $(\varphi^{-1})^*$ observe that any fraction of the form $\frac{P(x_1,\dots,x_n)}{(f(x_1,\dots,x_n))^r}$ can be completed to a unique fraction of the form $\frac{\overline{P}(x_0,\dots,x_n)}{(x_0 f^\sharp)^{r'}}$ of degree 0, where $P^\sharp | \overline{P}$ and $r' \geq r$. (Multiply the denominator with appropriate power of $x_0 f^\sharp$, then the numerator with appropriate power of x_0. Then

$$(\varphi^{-1})^*\Big(\frac{P(x_1,\dots,x_n)}{(f(x_1,\dots,x_n))^r}\Big) = \frac{\overline{P}(x_0,\dots,x_n)}{(x_0 f(\frac{x_1}{x_0},\dots,\frac{x_n}{x_0}))^{r'}}.$$

and that

$$(\varphi^{-1})^*\Big(\frac{P(x_1,\dots,x_n)}{(f(x_1,\dots,x_n))^r}\Big) = x_0^{r\deg(f)-\deg(P)}\frac{x_0^{\deg P}P(\frac{x_1}{x_0},\dots,\frac{x_n}{x_0})}{(x_0^{\deg f}f(\frac{x_1}{x_0},\dots,\frac{x_n}{x_0}))^r}.$$

This finishes the proof. □

Corollary 7.6. *$\mathbb{P}^n$ is irreducible.*

Proof. This follows because $\bigcap_{i=0}^n D_{x_i}^+$ is open and dense in each $D_{x_i}^+ \simeq \mathbb{A}^n$, thus it must be irreducible, and its closure is the whole $\mathbb{P}^n$. □

Remark 7.2. *Using the Veronese embedding we will show later that if X is a projective variety and $D_F^+ \subsetneq X$ is a non-trivial distinguished open set, then D_F^+ is affine.*

Remark 7.3. *We will also show later that if X is an* connected *projective algebraic variety then $\mathcal{O}_X(X) = k$, i.e. the only global regular functions are the constants.*

We have seen that if $J \subset k[x_0,\dots,x_n]$ is a homogeneous ideal, then $V(J) \cap U_0 = V(J_\flat)$. The next result gives a geometric interpretation of the homogenization of an ideal.

Proposition 7.7. *(Homogenization yields projective closure.) Let $I \leq k[x_1,\dots,x_n]$, $X = V(I) \subset \mathbb{A}^n$ and $\overline{X} \subset \mathbb{P}^n$ its closure, where $\mathbb{A}^n$ is identified to the distinguished open set U_0. Then the following hold.*
(a) $\overline{X} = V_p(I^\sharp)$;
(b) If $I = \langle f \rangle$ is a nonzero principal ideal, then $\overline{X} = V_p(f^\sharp)$.

Proof. Obviously $V_p(I^\sharp)$ is closed and it contains X, thus $\overline{X} \subset V_p(I^\sharp)$. We now show that it is the smallest closed subset containing X. Let $Y \subset \mathbb{P}^n$ be any closed subset such that $Y \supset \overline{X}$. Write $Y = V_p(J)$ for some homogeneous ideal $J \subset k[x_0,\dots,x_n]$. We claim that $J \subset \sqrt{I^\sharp}$; this will imply that

$$V_p(I^\sharp) \subset V(J) = Y$$

proving that $V_p(I^\sharp) = \overline{X}$.

To prove the claim, take $F \in J$ and as in Example 7.4 write $F = x_0^d f^\sharp$ for some $f \in k[x_1, \dots, x_n]$. Since $x_0^d f^\sharp = 0$ on $\overline{X} \subset Y$ it follows that $f = 0$ on $X = \overline{X} \cap U_0$, i.e.

$$f \in I(X) = I(V(I)) = \sqrt{I}$$

by the Nullstellensatz. Then $f^p \in I$ for some $p \in \mathbb{Z}_{\geq 0}$ and

$$(f^p)^\sharp = (f^\sharp)^p \in I^\sharp$$

thus $f^\sharp \in \sqrt{I^\sharp}$. It follows that $F = x_0^d f^\sharp \in \sqrt{I^\sharp}$, giving that $J \subset \sqrt{I^\sharp}$, as claimed.

To prove (b), we observe first that

$$\langle f \rangle^\sharp = \langle \{(fg)^\sharp = f^\sharp g^\sharp : g \in k[x_1, \dots, x_n]\} \rangle = \langle f^\sharp \rangle.$$

Then by (a), $\overline{V(f)} = V(f^\sharp)$. $\square$

Example 7.8. *(Homogenization does not commute with generation.) Let $I = \langle x_1, x_2 - x_1^2 \rangle \subset k[x_1, x_2]$. The associated variety is $V(I) = \{(0,0)\}$ and by Proposition 7.7,*

$$V(I^\sharp) = \{[1:0:0]\} \subset \mathbb{P}^2.$$

However, the ideal generated by homogenizations $J = \langle x_1, x_2x_0 - x_1^2 \rangle$ has zero locus

$$V_p(J) = \{[1:0:0], [0:0:1]\} \neq V_p(I^\sharp).$$

Geometrically, $V(I)$ is the transversal intersection of a conic Q with an affine line L; this intersection contains one point, but $\overline{Q \cap L} \subsetneq \overline{Q} \cap \overline{L}$, since the latter contains a second point at infinity.

8. Morphisms (II)

The goal of this section is to give some examples of morphisms to affine varieties, and morphisms between projective varieties.

8.1. **Morphisms to affine varieties.**

Let $\varphi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a map.

Proposition 8.1. *The condition that φ is a morphism is local:*

- *φ is a morphism if and only if there exists an open cover $Y = \bigcup V_i$, and for each i, an open cover $\varphi^{-1}(V_i) = \bigcup U_{ij}$, such that the restriction $\varphi_i : U_{ij} \to V_i$ is a morphism.*

Proof. Immediate from definition, using that sections over U_{ij} glue to a section of $\mathcal{O}_X(\varphi^{-1}(V_i))$; cf. Definition 5.5. □

Informally, this says that if we have a *globally defined* map $\varphi : X \to Y$, the fact that it is a morphism may be checked in local charts. (This is similar to what happens in other situations, such as differentiable maps, or holomorphic functions.) One can simplify this statement in case Y is affine.

Proposition 8.2. *If $(Y, \mathcal{O}_Y)$ is an affine variety, then it suffices to check the morphism condition for global sections in $\Gamma(Y) = \mathcal{O}_Y(Y)$:*

- *φ is a morphism if and only if for all $f \in \Gamma(Y)$, $\varphi^*(f) = f \circ \varphi \in \mathcal{O}_X(X)$.*

Proof. If X is affine, this follows from Proposition 3.17. In general, cover X by affine varieties, and apply the previous proposition together with the affine situation. □

Example 8.3. *The projection $p : \mathbb{A}^{n+1} \setminus 0 \to \mathbb{P}^n$ is a morphism. (All components are global sections on $\mathbb{A}^{n+1} \setminus 0$, with no common zeros.) Note that we cannot extend this morphism over 0.*

In the simplest case, when $n = 1$, this map is sending a point $(x, y) \in \mathbb{A}^2$ to the line $[x : y] \in \mathbb{P}^1$. One may take a modification $\widetilde{\mathbb{A}^2} \to \mathbb{A}^2$ which over $(0, 0)$ has fibre all possible slopes of lines through $(0, 0)$; in other words, replace $(0, 0) \in \mathbb{A}^2$ by a $\mathbb{P}^1$. This will lead to the notion of the **blow-up** *of a point, and to a morphism $\widetilde{\mathbb{A}^2} \to \mathbb{P}^1$ extending $\mathbb{A}^2 \to \mathbb{P}^1$ over $(0, 0)$.*

Remark 8.1. *If $\varphi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism and $V \subset X$, $W \subset Y$ are subvarieties such that $\varphi(V) \subset W$, then the restriction $\varphi : V \to W$ is a morphism.*

8.2. **Morphisms to projective varieties.**

Many morphisms between projective varieties arise from the following construction.

Consider $b+1$ homogeneous polynomials *of the same degree* $F_0, \dots, F_b \in k[x_0, \dots, x_a]$. Then there is a well defined map:

$$\varphi : \mathbb{P}^a \setminus V_p(F_0, \dots, F_b) \to \mathbb{P}^b; \quad [x_0 : \dots : x_a] \mapsto [F_0(x) : \dots : F_b(x)].$$

Proposition 8.4. *The above map φ is a morphism.*

Proof. Cover the image $\mathbb{P}^b$ with the standard affine open sets: $\mathbb{P}^b = \bigcup D^+_{x_i}$. The preimage
$$\varphi^{-1}(D^+_{x_i}) = D^+_{F_i} \setminus V_p(F_0, \dots, F_b)$$
is an open set in the distinguished open set $D^+_{F_i}$. A global section on $D^+_{x_i}$ is of the form $\frac{G}{x_i^r}$ where $G \in k[x_0, \dots, x_b]$ is homogeneous of degree r. The pull-back via φ is
$$\varphi^*\Big(\frac{G}{x_i^r}\Big) = \frac{G(F_0, \dots, F_b)}{F_i^r},$$
which is clearly on $\mathcal{O}_{\mathbb{P}^a}(D^+_{F_i})$. □

Example 8.5 (Projections)**.** *Let $p \in \mathbb{P}^n$ be a point and let $H \subset \mathbb{P}^n$ be a hyperplane such that $p \notin H$. Then we may define the* **projection from** p **onto the hyperplane** H *to be the map*
$$\pi_p : \mathbb{P}^n \setminus \{p\} \to H; \quad q \mapsto \overline{pq} \cap H.$$
For example one may take H be given by equation $x_n = 0$, and $p = [0 : \dots : 0 : 1]$. Then the line containing p and a point $q = [x_0 : \dots : x_n]$ is of the form
$$\mathbb{P}(\langle (x_0, \dots, x_n), (0, \dots, 1)\rangle) = \{\langle ax_0, \dots, ax_{n-1}, ax_n + b\rangle : [a : b] \in \mathbb{P}^1\},$$
and the projection is defined by
$$\pi_p[x_0 : \dots : x_n] = [x_0 : \dots : x_{n-1}].$$
(Note that not all x_i's may be 0, as the points p, q are distinct.)

One may use **elimination theory** *to show that if $X \subset \mathbb{P}^n$ is closed and $p \notin X$, then $\pi_p(X) \subset \mathbb{P}^{n-1}$ is closed.*

Example 8.6 (Conics)**.** *Let $\varphi : \mathbb{P}^1 \to \mathbb{P}^2$ be defined by*
$$\varphi[x : y] = [x^2 : xy : y^2]$$
By Proposition 8.4 this is a well defined morphism. Its image is the conic
$$C := \mathrm{Im}(\varphi) = V_p(Y^2 - XZ) \subset \mathbb{P}^2,$$
where X, Y, Z are the coordinates on $\mathbb{P}^2$. It is easy to check that φ is injective; in fact, φ is an isomorphism onto its image.

We construct the inverse using 'stereographic projection'. Let $U_0 = \{X \neq 0\}$ and $U_1 = \{Z \neq 0\}$, two distinguished open sets in $\mathbb{P}^2$. Observe that:
$$C = (C \cap U_0) \cup \{[0 : 0 : 1]\} = (C \cap U_1) \cup \{[1 : 0 : 0]\}.$$
Let $\mathbb{P}^1 = V_0 \cup V_1$, the usual decomposition into affine varieties. Then also observe that
$$\varphi^{-1}(C \cap U_0) = V_0; \quad \varphi^{-1}(C \cap U_1) = V_1.$$
Further,
$$\varphi[1 : s] = [1 : s : s^2]; \quad \varphi[s : 1] = [s^2 : s : 1].$$
Define $f_i : C \cap U_i \to V_i$ $(i = 0, 1)$ by
$$f_0[1 : s : s^2] = [1 : s]; \quad f_1[s^2 : s : 1] = [s : 1].$$

In the chart U_0, $s = Y/X$, therefore using the equations of C,

$$[1 : s : s^2] = [1 : Y/X : Y^2/X^2] = [1 : Y/X : XZ/X^2] = [X : Y : Z].$$

In other words, in the chart $C \cap U_0$, f_0 is defined by

$$f_0[X : Y : Z] = [X : Y].$$

A similar argument shows that in the chart $C \cap U_1$, $s = Y/Z$, and f_1 is given by

$$f_1[X : Y : Z] = [Y : Z].$$

Note that f_0 is not defined at $[0 : 0 : 1]$, while f_1 is not defined at $[1 : 0 : 0]$. We need to show that f_0 and f_1 glue, i.e. they agree on the overlap $C \cap U_0 \cap U_1 = C \cap \{XZ \neq 0\} = C \cap \{XYZ \neq 0\}$. To this end, observe that for $[X : Y : Z] \in C \cap \{XYZ \neq 0\}$,

$$[X : Y] = [XY : Y^2] = [XY : XZ] = [Y : Z].$$

This proves that φ^{-1} exists as a morphism, and it is given by gluing of f_0 and f_1.

Warning: *Even though $\varphi : \mathbb{P}^1 \to C$ is an isomorphism, it does not induce an isomorphism of projective coordinate rings. In fact, it induces the morphism:*

$$\Gamma_p(C) = k[X, Y, Z]/\langle Y^2 - XZ\rangle \to k[x, y] = \Gamma_p(\mathbb{P}^1); \quad X \mapsto x^2; \quad Y \mapsto xy; \quad Z \mapsto y^2.$$

Not only that this is not an isomorphism (it is not surjective), but the two rings are actually not abstractly isomorphic, e.g. because $k[X, Y, Z]/\langle Y^2 - XZ\rangle$ is not a UFD.

Example 8.7 (Veronese embedding)**.** *One may generalize the previous example as follows. Fix $d \in \mathbb{N}$ and define*

$$\nu_d : \mathbb{P}^n \to \mathbb{P}^{N-1}; \quad \nu_d[x_0 : \ldots : x_n] = [\ldots : x_0^{d_0} \cdot \ldots \cdot x_n^{d_n} : \ldots], \tag{8.1}$$

where $\sum d_i = d$. Here N equals the number of monomials of total degree d in $n + 1$ variables, i.e.

$$N = \binom{n+d}{n}$$

This is a morphism, and as in the previous example a local calculation shows it is actually an isomorphism onto its image. This image equals $V_p(J) \subset \mathbb{P}^{N-1}$, where

$$J = \ker(\nu_d^* = k[Z_0 : \ldots : Z_{N-1}] \to k[x_0 : \ldots : x_n]); \quad Z_i \mapsto x_0^{i_0} \cdot \ldots \cdot x_n^{i_n},$$

for $\sum i_j = d$. The variety $V_p(J) = \nu_d(\mathbb{P}^n)$ is called the **Veronese variety***.*

Example 8.8 (Veronese curves)**.** *If $n = 1$, $\nu_d : \mathbb{P}^1 \to \mathbb{P}^N$ is an example of a* **determinantal variety***:*

$$\nu_d(\mathbb{P}^1) = \{[Z_0 : \ldots : Z_n] : \operatorname{rank} \begin{pmatrix} Z_0 & Z_1 & \ldots & Z_{n-1} \\ Z_1 & Z_2 & \ldots & Z_n \end{pmatrix} \leq 1\}.$$

For example, for $d = 3$, one recovers the **twisted cubic** *$\nu_3(\mathbb{P}^1) \subset \mathbb{P}^3$:*

$$\{[U : V : Z : T] : \operatorname{rank} \begin{pmatrix} U & V & Z \\ V & Z & T \end{pmatrix} \leq 1\} = V_p(UZ - V^2, UT - VZ, VT - Z^2).$$

Example 8.9 (Veronese surfaces)**.** *Take $\nu_2 : \mathbb{P}^2 \to \mathbb{P}^5$. The image of ν_2 may be realized as a determinantal variety as follows. Consider symmetric matrices*

$$M = \begin{pmatrix} Z_0 & Z_3 & Z_4 \\ Z_3 & Z_1 & Z_5 \\ Z_4 & Z_5 & Z_2 \end{pmatrix}.$$

Then the Veronese surface is given by the condition that rank $M = 1$. *More generally, the image of the Veronese variety $\nu_2(\mathbb{P}^n) \subset \mathbb{P}^{\frac{(n+1)(n+2)}{2}-1}$ is given by the vanishing of the 2×2 minors of the $(n+1) \times (n+1)$ symmetric matrix with entry $Z_{i-1,j-1}$ in position (i,j). See [Harris].*

Example 8.10 (Coordinate free Veronese)**.** *There is also a coordinate free description of the Veronese embedding. If $\mathbb{P}^n = \mathbb{P}(E)$ for some vector space E, and for a fixed degree d,*

$$\nu_d : \mathbb{P}(E) \to \mathbb{P}(Sym^d E); \quad v \to v^d,$$

where $Sym^d(E)$ is the d-th symmetric power of E.

An important property of the Veronese embedding is that any hypersurface $V_p(F) \subset \mathbb{P}^n$ where F is homogeneous of degree d is the pull-back of a hyperplane via ν_d: if $F = \sum a_{i_0 \ldots i_n} x_0^{i_0} \cdot \ldots \cdot x_n^{i_n}$, then

$$V_p(F) = \nu_d^{-1}(H_F) \quad \text{where} \quad H_d = \sum a_{i_0 \ldots i_n} Z_{i_0 \ldots i_n}. \tag{8.2}$$

This observation is utilized to prove the following projective analogue of Proposition 5.6.

Proposition 8.11. *Let $F \in k[x_0, \ldots, x_n]$ be a homogeneous polynomial of degree d. Then $D_F^+ \subset \mathbb{P}^n$ is an affine variety.*

Furthermore, if $X \subset \mathbb{P}^n$ is a closed projective subvariety, then $X \cap D_F^+$ is affine.

Proof. From Equation (8.2) it follows that $D_F^+ = \nu_d^{-1}(D_{H_d}^+)$, where H_d is a linear form. Up to a projective change of coordinates, $V_p(H_d) \subset \mathbb{P}^{N-1}$ is a coordinate hyperplane, thus $D_{H_d}^+$ is affine. Since ν_d is an isomorphism over the image, it follows that D_F^+ is isomorphic to a closed subset of an affine variety, thus again affine.

Same argument also proves the statement involving the closed subvariety $X \subset \mathbb{P}^n$. □

8.3. **Images of morphisms.** Let $f : X \to Y$ be a morphism of varieties. Assume for simplicity that X, Y are irreducible. (Note that if X is irreducible, then so is the image $f(X)$.) A natural question is what type of object is the image of a morphism ? For example, is it always a subvariety of Y ? (Note: We will discuss preimages later on this class, in relation to systems of parameters, and expected dimension.)

The first is the following result, whose proof is deferred.

Proposition 8.12. *Let $f : X \to Y$ be a morphism of projective varieties. Then f is closed, i.e. for any $Z \subset X$ closed, $f(Z)$ is closed.*

There is a more general version of this, where f is only assumed to be **proper**; a morphism of projective varieties is proper.

However, consider the morphism $f : \mathbb{A}^2 \to \mathbb{A}^2$, $(x, y) \mapsto (x, xy)$. The horizontal line $y = y_0$ is sent to the line spanned by $(1, y_0)$, and the image of f is

$$f(\mathbb{A}^2) = (\mathbb{A}^2 \setminus \{y - \text{axis}\}) \cup \{(0, 0)\}.$$

This set is neither open, nor closed, around the origin. It is an example of a **constructible set**, defined next.

A **locally closed set** of a variety X is an intersection of an open set and a closed set in a variety X. A **constructible set** is a finite union of locally closed sets. One may check that equivalently, the class of constructible sets is the smallest class of objects containing open sets, closed sets, and finite intersections and complements of such.

Theorem 8.1 (Chevalley)**.** *Let $f : X \to Y$ be a morphism. Then $f(X)$ is constructible. Furthermore, f sends constructible subsets to constructible subsets.*

Proofs may be found in Mumford's 'Red Book', pag. 51, or in Harris' 'Algebraic Geometry', pag. 39, in the particular case when $Y = \mathbb{P}^n$.

9. Products

In this section we define the product of two varieties. We certainly want that $\mathbb{A}^r \times \mathbb{A}^s = \mathbb{A}^{r+s}$, but recall from a previous homework problem that the topology on $\mathbb{A}^{r+s}$ is not the product topology. It turns out that the correct way to define products is to use the categorical notion of product.

9.1. Categorical product.

Let $\mathcal{C}$ be a category; this consists of a class of **objects** $Ob(\mathcal{C})$, **morphisms** $F : X \to Y$ for $X, Y \in Ob(\mathcal{C})$ together with the abillity to compose morphisms so we recover the usual properties of composition (identities, associativity, etc). Let now $X, Y, Z \in Ob(\mathcal{C})$ be three objects. Given two morphisms $f : X \to Z$ and $g : Y \to Z$, a **product** $X \times_Z Y$ is any object in $\mathcal{C}$ equipped with two morphisms $\pi_X : X \times_Z Y \to X$, $\pi_Y : X \times_Z Y \to Z$, such that the following universal mapping property holds:

- For any object W and morphisms $p_X : W \to X$ and $p_Y : W \to Y$ so that $f \circ p_X = g \circ p_Y$ there exists a unique morphism $p : W \to X \times_Z Y$ such that $\pi_X \circ p = p_X$ and $\pi_Y \circ p = p_Y$.

$$\begin{tikzcd}
W \arrow[dr, dotted, "p"] \arrow[drr, bend left, "p_X"] \arrow[ddr, bend right, "p_Y"'] & & \\
& X \times_Z Y \arrow[r, "\pi_X"'] \arrow[d, "\pi_Y"] & X \arrow[d, "f"'] \\
& Y \arrow[r, "g"] & Z
\end{tikzcd}$$

The product $X \times_Z Y$, if it exists, is unique up to an isomorphism commuting with the projections.

Example 9.1. *Take $\mathcal{C} =$ `Sets` the category of sets and maps between sets. Take A, B, C sets and $f : A \to C, g : B \to C$. Then the product is*

$$A \times_C B = \{(a, b) : f(a) = g(b)\},$$

together with the projections to A and B respectively.

There is also the (dual) notion of categorical **coproduct**, which satisfies a similar universality property, except that *all arrows are reversed.*

Example 9.2. *Let $\mathcal{C} = k-$`Alg` be the category of k-algebras (k a field), and k-algebra homomorphisms. Let A, B, C, D be k-algebras and maps $A \to B, A \to C, B \to D$ and $C \to D$ such that the resulting diagram commutes. (In other words, B, C, D are A-algebras.) The morphisms $B \to D$ and $C \to D$ induce an A-bilinear k-algebra homomorphism $B \times C \to D$. The coproduct $B \times_A C$ is the algebra tensor product $B \otimes_A C$.*

(See e.g. [Lang, Ch. XVI, §6.)

$$\begin{tikzcd}
D & & \\
& B \otimes_A C \arrow[ul, dotted, "i"'] & C \arrow[l] \arrow[ull, bend right, "i_C"'] \\
& B \arrow[u] \arrow[uul, bend left, "i_B"] & A \arrow[l] \arrow[u]
\end{tikzcd}$$

9.2. **Products of varieties.** The main theorem of this section is the following:

Theorem 9.1. *Any two algebraic varieties X, Y have a product (as an algebraic variety).*

Note that there is no choice for what should be the underlying set of $X \times Y$: it's just the set theoretic product of X and Y. We need to realize this as a ringed space, locally isomorphic to an affine variety. To do this, the idea is to construct first products of affine varieties, then 'glue' these to get a variety structure.

Remark 9.1. *One may also ask where is Z from $X \times_Z Y$: in this case, Z is a point, regarded as an (affine) algebraic variety. Any variety maps to pt, and $X \times_{pt} Y$ is denoted simply by $X \times Y$.*

Proposition 9.3. *Let X, Y be affine algebraic varieties with coordinate rings $\Gamma(X)$ and $\Gamma(Y)$. Then the following hold:*

(1) *There is a product $X \times Y$ which is an affine algebraic variety with coordinate ring $\Gamma(X) \otimes_k \Gamma(Y)$;*
(2) *A basis for the topology of open sets is given by the distinguished open sets D_f, where*
$$f = \sum f_i(x) g_j(y); \quad f_i \in \Gamma(X), g_j \in \Gamma(Y).$$
(3) *The local ring $\mathcal{O}_{X \times Y,(x,y)}$ is the localization of $\Gamma(X) \otimes_k \Gamma(Y)$ at the maximal ideal $\mathfrak{m}_x \otimes \Gamma(Y) + \Gamma(X) \otimes \mathfrak{m}_y$.*

Proof. Let $X = V(f_1, \dots, f_n) \subset \mathbb{A}^r$ and $Y = V(g_1, \dots, g_m) \subset \mathbb{A}^s$. Then the set $X \times Y \subset \mathbb{A}^{r+s}$ is the zero locus of the polynomials f_i, g_j regarded in $k[x_1, \dots, x_r; y_1, \dots, y_s]$. Furthermore,

$$k[x_1, \dots, x_r; y_1, \dots, y_s]/\langle f_i; g_j \rangle \simeq k[x_1, \dots, x_r]/\langle f_i \rangle \otimes_k k[y_1, \dots, y_s]/\langle g_j \rangle = \Gamma(X) \otimes_k \Gamma(Y).$$

(There is a natural bilinear map

$$k[x_1, \dots, x_r]/\langle f_i \rangle \times k[y_1, \dots, y_s]/\langle g_j \rangle \to k[x_1, \dots, x_r; y_1, \dots, y_s]/\langle f_i; g_j \rangle,$$

which by the universal property induces a map from the tensor product. The latter is an isomorphism when regarded as k-vector spaces, so it must be an isomorphism in general.)

We now prove that $X \times Y$ is a categorical product. Obviously there are projections $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$, corresponding to maps $\Gamma(X) \to \Gamma(X \times Y)$,

$f(x) \mapsto f(x) \otimes 1$ and similarly for Y. Let now $p_X : Z \to X$ and $p_Y : Z \to Y$ be any morphism. Regarding $X \times Y$ as a product in `Sets`, we obtain a unique map $p : Z \to X \times Y$ which satisfies the required commutations. We need to show that p is a morphism. Since $X \times Y$ is affine, by Proposition 8.2, it suffices to show that $p^*(f(x) \otimes g(y)) \in \Gamma(Z)$. Since $\Gamma(X \times Y)$ is generated (as an algebra) by elements of the form $f(x) \otimes 1$ and $1 \otimes g(y)$, it suffices to consider these elements. But then

$$p^*(f(x) \otimes 1) = p^* \pi_X^*(f(x)) = p_X^*(f(x)) \in \Gamma(Z),$$

since $p_X : Z \to X$ is a morphism. Same argument applies to $1 \otimes g(y)$. This proves (1) and (2) at the same time.

We now turn to (3). By Proposition 6.12 we know that $\mathcal{O}_{X \times Y,(x_0,y_0)}$ is the localization of $\Gamma(X) \otimes \Gamma(Y)$ at the maximal ideal $\mathfrak{m}_{(x_0,y_0)}$ of functions vanishing at (x, y). Clearly $\mathfrak{m}_{x_0} \otimes \Gamma(Y) + \Gamma(X) \otimes \mathfrak{m}_{y_0} \subset \mathfrak{m}_{(x_0,y_0)}$. Conversely, let $h = \sum a_i(x) \otimes b_j(y) \in \mathfrak{m}_{(x_0,y_0)}$ such that $a_i(x_0) = \alpha_i$, $b_j(y_0) = \beta_j$. Then $\sum \alpha_i \otimes \beta_j = 0$, and

$$\begin{aligned} \sum a_i(x) \otimes b_j(y) &= \sum a_i(x) \otimes b_j(y) - \sum \alpha_i \otimes \beta_j \\ &= \sum (a_i(x) - \alpha_i) \otimes b_j(y) + \sum \alpha_i \otimes (g_j(y) - \beta_j); \end{aligned}$$

the last quantity is in $\mathfrak{m}_{x_0} \otimes \Gamma(Y) + \Gamma(X) \otimes \mathfrak{m}_{y_0}$. This finishes the proof. □

Corollary 9.4. $\mathbb{A}^r \times \mathbb{A}^s = \mathbb{A}^{r+s}$.

Proof. This repeats the first part of the proof: the projection morphisms from $\mathbb{A}^{r+s}$ to $\mathbb{A}^r$ and $\mathbb{A}^s$ induce a (unique) map $f : \mathbb{A}^{r+s} \to \mathbb{A}^r \times \mathbb{A}^s$. At the level of coordinate rings this gives an isomorphism $k[x_1, \dots, x_r] \otimes k[y_1, \dots, y_s] \simeq k[z_1, \dots, z_{r+s}]$, thus the desired isomorphism. □

Theorem 9.2. *Any two varieties $X \times Y$ have a product.*

Proof. The idea is to cover X and Y by affine varieties, then 'glue' the result, to form $X \times Y$. First of all $X \times Y$ exists *as a set*. We wish give it a variety structure.

To start, we give $X \times Y$ a topology as follows: for any $U \subset X$ and $V \subset Y$ open affine, consider the distinguished open sets $D_{\sum f_i g_j} \subset U \times V$, where $f_i \in \Gamma(U), g_j \in \Gamma(V)$. We declare these to give a basis for the topology of $X \times Y$. (Observe that if $U' \subset X$ and $V' \subset Y$ are other open affines, and functions $\sum f_i' g_j'$, then we may find $U'' \subset U \cap U'$ and $V'' \subset V \cap V'$ such that $D_{(\sum f_i g_j) \cdot (\sum f_i' g_j')}$ is a distinguished set on $U'' \times V'' \subset (U \cap U') \times (V \times V')$.) Note that when restricted to products of affines $U \times V$, this recovers the topology of that product. Furthemore, we may also define a ringed space structure on $X \times Y$ by declaring that for $D_{\sum f_i g_j} \subset U \times V$,

$$\mathcal{O}_{X \times Y}(D_{\sum f_i g_j}) := (\Gamma(U) \otimes \Gamma(V))_{\sum f_j g_j}.$$

One may check that the conditions from Lemma 5.5 are satisfied, and this gives a ringed space structure to $X \times Y$. This space $X \times Y$ is quasi-compact, as $X \times Y$ may be written as a finite union of quasi-compact spaces $U_i \times V_j$, for U_i, V_j affine.

We now need to check that the resulting ringed space $X \times Y$ is the categorical product $X \times Y$. Take any morphisms $p_X : Z \to X$ and $p_Y : Z \to Y$ which satisfy the required commutativity requirements. We need to show that the induced set-theoretic map $p : Z \to X \times Y$ is a morphism. By Proposition 8.1, it suffices to check that for each $U \subset X$, $V \subset Y$ affine, the restriction $p : Z_{U,V} := p^{-1}(U \times V) \to U \times V$ is a morphism. Since $U \times V$ is affine, by Proposition 8.2 it suffices to check that for any $s \in \Gamma(U \times V)$, $p^*_{U,V}(s) \in \mathcal{O}_Z(Z_{U,V})$. This follows as in the proof of Proposition 9.3 because $\Gamma(U \times V)$ is generated by $f \otimes 1$ and $1 \otimes g$ for $f \in \Gamma(U)$, $g \in \Gamma(V)$, and since $p^*_{U,V}(f \times 1) = p^*_X(f)$, $p^*_{U,V}(1 \times g) = p^*_Y(g)$ are elements in $\mathcal{O}_Z(Z_{U,V})$. □

Remark 9.2. *In addition, one may show the following:*

(1) *If $U \times X$ is open, then $U \times Y$ will be an open subset of $X \times Y$.*
(2) *If $Z \subset X$ is a closed subvariety, then $Z \times Y$ will be a closed subvariety of $X \times Y$. (For this, it suffices to show that for any $U \subset X$ and $V \subset Y$ open affine, $(Z \cap U) \times Y$ is closed in $U \times V$.)*

For $a, b \in \mathbb{N}$ define $\nu_{a,b} : \mathbb{P}^a \times \mathbb{P}^b \to \mathbb{P}^{(a+1)(b+1)-1}$ by

$$\nu_{a,b}([x_0 : \ldots : x_a], [y_0 : \ldots : y_b]) = [\ldots : x_i y_j : \ldots]. \tag{9.1}$$

Note that this is a well defoned map, and an analogous proof as the one from Proposition 8.4 this is a morphism, called the **Segre embedding**. (Note that here $\mathbb{P}^a \times \mathbb{P}^b$ is already given a product variety structure.)

Lemma 9.5. *(a) The image of $\nu_{a,b}$ is given by the determinants of 2×2 minors in the matrix:*

$$\begin{pmatrix} Z_{0,0} & Z_{0,1} & \ldots & Z_{0,b} \\ Z_{1,0} & Z_{1,1} & \ldots & Z_{1,b} \\ \vdots & \vdots & \ldots & \vdots \\ Z_{a,0} & Z_{a,1} & \ldots & Z_{a,b} \end{pmatrix}$$

where $Z_{i,j}$ is the coordinate corresponding to $x_i y_j$. (This is another example of a determinantal variety.)

(b) Let $S_{a,b} := \nu_{a,b}(\mathbb{P}^a \times \mathbb{P}^b) \subset \mathbb{P}^{(a+1)(b+1)-1}$ be the image of the Segre embedding. Then $\nu_{a,b} : \mathbb{P}^a \times \mathbb{P}^b \to S_{a,b}$ is an isomorphism.

Proof. Exercise. □

Theorem 9.3. *The product of two projective varieties is projective.*

Proof. If $X \subset \mathbb{P}^a$ and $Y \subset \mathbb{P}^b$ are closed then so is $X \times Y \subset \mathbb{P}^a \times \mathbb{P}^b$ (cf. Remark 9.2). By Lemma 9.5, $\mathbb{P}^a \times \mathbb{P}^b$ is isomorphic to a projective variety, thus it is projective. □

9.3. **Separated varieties and the Haussdorff axiom.** The following is the analogue of the Haussdorff condition in algebraic geometry.

Definition 9.1. *An algebraic variety X is called* **separated** *if for all varieties Z and all morphisms $Z \underset{g}{\overset{f}{\rightrightarrows}} X$ the set $\{z \in Z : f(z) = g(z)\}$ is a closed subset of Z.*

Consider the diagonal map $\Delta : X \to X \times X$ given by $x \mapsto (x, x)$. This is the morphism induced by the universality property, where $p_1, p_2 : X \times X \to X$ are the projections:

$$
\begin{tikzcd}
X \arrow[dr, dotted, "\Delta"] \arrow[drr, bend left, "id"] \arrow[ddr, bend right, "id"'] & & \\
& X \times X \arrow[r, "p_2"'] \arrow[d, "p_1"] & X \arrow[d] \\
& X \arrow[r] & pt
\end{tikzcd}
$$

If X is separated, then by definition

$$\Delta(X) = \{(a, a) \in X \times X\} = \{(a, b) \in X \times X : p_1(a, b) = p_2(a, b)\}$$

is closed in $X \times X$. The converse is also true:

Proposition 9.6. *Let X be a variety. Then X is separated if and only if $\Delta(X) \subset X \times X$ is closed.*

Proof. We only need to show that if $\Delta(X)$ is closed, then X is separated. Let $f, g : Z \to X$ be two morphisms. Then there is an induced morphism $(f, g) : Z \to X \times X$, and

$$\{z \in Z : f(z) = g(z)\} = (f \times g)^{-1}(\Delta(X)).$$

The latter is a closed set by hypothesis. □

Example 9.7 (A non-separated variety: the line with 2 origins)**.** *The projective line $\mathbb{P}^1$ may be seen as the as gluing two copies $U_0, U_1 \simeq \mathbb{A}^1$ along $\mathbb{A}^1 \setminus \{0\}$ by the map $x \mapsto 1/x$. We have checked that $\mathbb{P}^1$ is an algebraic variety, and, for example, $\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1) = k$.*

We may also glue along $\mathbb{A}^1 \setminus \{0\}$ by $x \to x$; this gives a different variety structure. We call it X, and we may visualize it as 'the line with 2 origins'. For example, $\Gamma(X)$ will contain non-constant global sections.

$$
\begin{tikzcd}
& U_0 \arrow[dr, hook] & \\
\mathbb{A}^1 \arrow[ur, bend left, "\simeq"'] \arrow[dr, bend right, "\simeq"] & & X \\
& U_1 \arrow[ur, hook] &
\end{tikzcd}
$$

The variety X is not separated; indeed, take $i_1, i_2 : \mathbb{A}^1 \to X$ be the compositions. Then

$$\{x \in \mathbb{A}^1 : i_1(x) = i_2(x)\} = \mathbb{A}^1 \setminus \{0\},$$

which is not closed in $\mathbb{A}^1$.

Proposition 9.8. *The following hold:*

(1) *Closed subvarieties of separated varieties are separated.*
(2) *Products of separated varieties are separated.*
(3) *Let* $f : X \to Y$ *be a morphism with* Y *separated. Then the* **graph of** f
$$\Gamma_f = \{(x,y) : y = f(x)\} \subset X \times Y$$
is closed.
(4) *Affine varieties are separated.*
(5) *Projective varieties are separated.*

Proof. Part (1) follows from definitions, and part (2) from Remark 9.2.

For part (3), consider the morphism $(f, id) : X \times Y \to Y \times Y$. Since Y is separated, the diagonal $\Delta(Y)$ is closed (cf. by Proposition 9.6). Then $\Gamma_f = (f, id)^{-1}(\Delta(Y))$ is also closed. [2]

From the construction of products of affine varieties, it follows that the diagonal is closed, thus proving part (4).

For part (5), it suffices to assume that the projective variety X is a projective space, $X = \mathbb{P}^n$. Then for any $x, y \in \mathbb{P}^n$, there exists a hyperplane $H \subset \mathbb{P}^n$ such that $x, y \in \mathbb{P}^n \setminus H$. Since $\mathbb{P}^n \setminus H$ is affine, the conclusion that $\mathbb{P}^n$ is separated follows from the Lemma 9.9 below. □

Lemma 9.9. *Let* X *be an algebraic variety. Assume that for all* $x_1, x_2 \in X$ *there exists an open affine* $U \subset X$ *such that* $x_1, x_2 \in U$*. Then* X *is separated.*

Proof. Let $f, g : Y \to X$ be two morphisms, and let $Z := \{y \in Y : f(y) = g(y)\}$. Take $z \in \overline{Z}$, and $x_1 = f(z), x_2 = g(z)$. By assumption there exists an open affine set $V \subset X$ such that $x_1, x_2 \in V$. Let $U := f^{-1}(U) \cap g^{-1}(V) \subset Y$. Then $z \in U$, and U is open. The restrictions of $f, g : U \to V$ are again morphisms. Since V is affine, it is separated, thus $Z \cap U$ is closed in U, in particular, $\overline{Z} \cap U = Z \cap U$. Then $z \in Z \cap U$, thus $z \in Z$, proving that Z is closed. □

For further reading on separated varieties, close to approach from these notes, I recommend [Mum99, §6].

[2] As an aside, observe that $\Gamma_f = Image((id, f) : X \to X \times Y)$. Further, the restriction of the first projection from $X \times Y$ to X gives an isomorphism $\Gamma_f \to X$.

10. Dimension

The purpose of this section is to introduce the notion of dimension for algebraic varieties. We will then utilize it to study the expected dimensions of systems of polynomial equations, and dimension of generic fibers of morphisms.

10.1. **Topology.** We start with the following definition.

Definition 10.1. *Let X be a topological space. The* **dimension** *of X is the supremum n such that there exists a chain*

$$\emptyset \neq X_0 \subsetneq X_1 \subsetneq \ldots \subsetneq X_n \subseteq X$$

where each X_i is irreducible and closed in X.

If $Y \subset X$ is a subspace, the **codimension** $\mathrm{codim}_X Y := \dim X - \dim Y$.

We will denote by $\dim X$ the dimension of X. In many proof we will use the following properties of irreducible sets.

Lemma 10.1. *Let X be a topological space, $U \subset X$ an open, nonempty subset, and $Y \subset X$. Then the following hold:*

- *If X is irreducible then U is dense, and any two nonempty open sets intersect.*
- *If Y is irreducible, then $\overline{Y}$ is irreducible, and $Y \cap U$ is irreducible.*

Example 10.2. *One may show that for $\mathbb{A}^n$ with the Euclidean topology,* $\dim \mathbb{A}^n = 0$. *(Because the only irreducible closed sets are points.)*

Proposition 10.3. *Let $Y \subset X$. Then $\dim Y \leq \dim X$. Further, if Y is closed, irreducible and $Y \neq X$, then $\dim Y < \dim X$.*

Proof. Let $F_0 \subsetneq F_1 \subsetneq \ldots \subsetneq F_k = Y$ be a chain of closed irreducible subset of Y. Then each $\overline{F}_i$ is irreducible, and we have a chain of closed subsets $\overline{F}_0 \subset \overline{F}_1 \subset \ldots \subset \overline{F}_k = \overline{Y} \subset X$. Observe that $F_i = Y \cap \overline{F}_i$, thus the chain is strictly increasing, and it gives $k \leq \dim X$. If Y is closed an irreducible then to any chain as above we may add X, proving that $k < \dim X$. □

It is also easy to show that if $X = \bigcup X_i$ is a union of closed subsets X_i, then $\dim X = \sup \dim X_i$. One such union is obtained when X_i's are the irreducible components of X.

10.2. **Krull dimension and dimension of algebraic varieties.** We now turn to algebra and algebraic varieties. Let R be a ring. Recall that the **Krull dimension** of R is the maximum number n such that

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_n,$$

where each $\mathfrak{p}_i$ is a prime ideal in R. We will use the same notation $\dim R$ for the Krull dimension. A fundamental theorem in algebra calculates the dimension as a transcendence degree.

Theorem 10.1. *Let R be a finitely generated k-algebra which is an integral domain. Let K be the fraction field of R. Then the Krull dimension of R is the transcendence degree K/k.*

Proof. See ADDREF. $\square$

Example 10.4. *The Krull dimension of $k[x_1, \dots, x_n]$ equals to n. A chain of prime ideals is given by $0 \subsetneq \langle x_1 \rangle \subsetneq \langle x_1, x_2 \rangle \subsetneq \dots \subsetneq \langle x_1, \dots, x_n \rangle$.*

Definition 10.2. *Let X be an irreducible affine algebraic variety. Then the coordinate ring $\Gamma(X)$ is an integral domain, and the* **function field** *$K(X)$ is defined to the fraction field of $\Gamma(X)$.*

By Theorem 5.1 the function field of X coincides with the function field of any distinguished open set in X.

Proposition 10.5. *Let X be an irreducible algebraic variety, and $U \subset X$ a nonempty open set. Then the following hold:*

(1) *If X is affine then*
$$\dim X = \dim \Gamma(X) = \text{trdeg}_k K(X) < \infty.$$
(2) *For an arbitrary (irreducible) X, $\dim X = \dim U < \infty$.*

Proof. The equality $\dim X = \dim \Gamma(X)$ follows from the Hilbert's Nullstellensatz, which gives an order reversing correspondence between closed irreducible subvarieties of X and prime ideals in $\Gamma(X)$. The second equality, and the finiteness conclusion, follow from Theorem 10.1.

We prove the second part in several steps. First, assume that X is affine. Then we can pick a distinguished open set $D_f \subset U \subset X$. Since $K(D_f) = K(X)$, it follows from part (1) and Proposition 10.3 that
$$\dim X = \dim D_f \leq \dim U \leq \dim X.$$
For arbitrary X, this also proves that any two open affine open sets $U_1, U_2 \subset X$ must have the same dimension. Indeed, their intersection is non-empty, and one can find an open set $V \subset U_1 \cap U_2$; this satisfies:
$$\dim U_1 = \dim V = \dim U_2.$$
Let $r := \dim V$ for any $\emptyset \neq V \subset X$ open, affine. Find an open affine V such that $V \subset U$. Then
$$r = \dim V \leq \dim U \leq \dim X,$$
and it suffices to show that $\dim X = r$.

Take a chain of closed irreducible subsets in X: $F_0 \subset F_1 \subset \dots \subset F_n = X$, and take V open affine such that $V \cap F_0 \neq \emptyset$. Intersecting with V gives a chain $F_0 \cap V \subset \dots \subset F_n \cap V = V$. Each $F_i \cap V$ is closed and irreducible in V (since F_i are irreducible), and the sets $F_i \cap V$ are distinct and non-empty, because $F_i = \overline{F_i \cap V}$. This implies that $\dim X \leq \dim V$ and we are done. $\square$

Corollary 10.6. *(a)* $\dim \mathbb{A}^n = \dim \mathbb{P}^n = n$.
(b) Any variety of dimension 0 *is finite.*

Proof. The function field of $\mathbb{A}^n$ is $k(x_1, \ldots, x_n)$, which has transcendence degree n over k. To calculate the dimension of $\mathbb{P}^n$, we may take one of the distinguished open sets $U_i \simeq \mathbb{A}^n$, then apply Proposition 10.5.

For part (b), w.l.o.g. we may assume that X is irreducible, and we will show that X consists of a single point. Take a nonempty open affine set $U \subset X$; then U is also irreducible and $\dim U = 0$. By the Nullstellensatz it follows that U corresponds to a maximal ideal in a polynomial ring, thus it must be a point. Then $X = \overline{U}$ is also a point, and we are done. □

10.3. **Expected dimension of systems of equations.** The main result of this section is the following:

Theorem 10.2. *Let* X *be an irreducible variety,* $U \subset X$ *open,* $0 \neq g \in \mathcal{O}_X(U)$ *and* Z *an irreducible component of* $\{x \in U : g(x) = 0\}$*. Then* $\dim Z = \dim X - 1$*.*

Proof. Take $U_0 \subset U$ open affine such that $Z \cap U_0 \neq \emptyset$. Then $Z \cap U_0$ is closed and irreducible in U_0. By irreducibility and by the Nullstellensatz it corresponds to a prime ideal $\mathfrak{p}$ which contains the restriction $f := g|_{U_0} \in R := \Gamma(U_0)$, and it is *minimal* with this property. Since the $Z \cap U_0 \neq U_0$, $f \neq 0$. The result follows from Krull's Hauptidealsatz theorem, stated next. □

Theorem 10.3 (Krull's principal ideal theorem / Krull's Hauptidealsatz)**.** *Let* R *be a finitely generated algebra over* k *which is an integral domain. Let* $f \in R$ *and let* $\mathfrak{p}$ *containing* f *which is minimal among prime ideals containing* f *(this is called an* **isolated prime** *for* f*). If* $f \neq 0$ *then*

$$\dim R/\mathfrak{p} = \dim R - 1.$$

Proof. TODO: ADDREF. □

Definition 10.3. *Let* X *be an irreducible variety and let* $Z \subset X$ *be a closed subset. We say that* Z *has* **pure dimension** r *if all its irreducible components have dimension* r*. Similar for pure codimension* r*.*

With this terminology, Theorem 10.2 states that the zero locus of g has pure codimension 1.

Here's a converse to Theorem 10.2.

Proposition 10.7. *Let* X *be an irreducible variety and let* $Z \subset X$ *irreducible, closed, of codimension* 1*. Then for any open* $U \subset X$ *such that* $Z \cap U \neq \emptyset$*, and for any* $f \in \mathcal{O}_X(U)$ *such that* $f \equiv 0$ *on* Z*,* $Z \cap U$ *is an irreducible component of* $V(f)$*.*

Proof. Take W be an irreducible component of $V(f)$ such that $W \supset Z \cap U$. Then

$$\dim X > \dim W \geq \dim Z \cap U = \dim X - 1.$$

Since $Z \cap U$ is closed in W, it follows that $W = Z \cap U$. □

Example 10.8. *The hypothesis that X is irreducible is essential. For instance, take $X = V(xy) \subset \mathbb{A}^2$; this is the union of the coordinate axes. Now take*

$$f = x(x+y+1) \in \Gamma(X) = k[x,y]/\langle xy\rangle.$$

Then $V(f) = V(x) \cup \{(-1,0)\}$; this is not pure-dimensional.

Corollary 10.9. *Let X be an irreducible variety and let Z be an irreducible component of $V(f_1, \dots, f_r)$, with each $f_i \in \mathcal{O}_X(X)$. Then* codim $Z \le r$.

Proof. Induction on r. The case $r = 1$ follows from Theorem 10.2. Let Z' be an irreducible component of $V(f_1, \dots, f_{r-1})$. By induction, $\operatorname{codim} Z' \le r-1$. By hypothesis Z is an irreducible component of $Z' \cap V(f_r) \subset V(f_1, \dots, f_r)$. If f_r does not vanish identically on Z', then $\dim Z = \dim Z' - 1$, as claimed. $\square$

Proposition 10.10. *Let X be an affine irreducible variety and let $Z \subset X$ be a closed subvariety of codimension r. Then there exist $f_1, \dots, f_r \in \Gamma(X)$ such that Z is a component of $V(f_1, \dots, f_r)$.*

Proof. See [Mum99, Cor. I.7.4]. $\square$

A subvariety $Z \subset X$ of codimension r which is given by r equations is called a **complete intersection**. Being a complete intersection is a rather strong constraint. For example, if $X = \mathbb{P}^n$ and $Y = V_p(F_1, \dots, F_r$ has codimension r then by Bèzout theorem the **degree** of Y must be

$$r = \deg(Y) = \deg(F_1) \cdot \ldots \cdot \deg(F_r).$$

This degree may often be calculated by other cohomological methods.

Example 10.11. *The degree of the twisted cubic $C := \nu_e(\mathbb{P}^1) \subset \mathbb{P}^3$ equals to 3. Since C is not included in a hyperplane, it follows that $I(C)$ cannot be generated by 2 equations.*

Example 10.12. *Consider the variety of matrices:*

$$M_1 := \{A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} : rk(A) \le 1\} \subset \mathbb{A}^5.$$

One may projectivize M_1, and $\mathbb{P}(M_1)$ is the image of $\mathbb{P}^2 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^5$ under the Segre embedding. In particular, codim $M_1 = 2$. *One may show that $\deg(\mathbb{P}(M_1)) = 3$, which implies that $I_p(\mathbb{P}(M_1))$ cannot be generated by 2 equations. Then $I(M_1)$ (the cone over $\mathbb{P}(M_1)$) cannot be generated bby 2 equations.*

These results show that a codimension 1 irreducible variety may be realized as a component of the zero locus of a single equation. But does it equal to $V(f)$? This is not always possible; the next result gives a sufficient condition for this to hold.

Proposition 10.13. *Let X be an affine irreducible variety such that $\Gamma(X)$ is an UFD. let $Z \subset X$ be closed of pure codimension 1. Then there exists $f \in \Gamma(X)$ such that $Z = V(f)$.*

Proof. We start by observing that since $R := \Gamma(X)$ is an UFD, any minimal nonzero prime must be principal. (Proof: Let $\mathfrak{p} \subset R$ be a minimal prime, and let $f \in \mathfrak{p}$. Since R is an UFD, we may decompose f as a product of prime factors. The fact that $\mathfrak{p}$ is prime implies that it must contain one of the prime factors of f, say f'. But the ideal $\langle f' \rangle$ is prime and $\langle f' \rangle \subset \mathfrak{p}$, thus $\langle f' \rangle = \mathfrak{p}$.)

Now the components of Z correspond to minimal prime ideals in R, generated by some prime elements $f_1, \ldots, f_r$ in R. Then $Z = V(f_1 \cdot \ldots \cdot f_r)$ and we are done. □

10.4. **Projective versions.** All these results have projective versions, involving projective coordinate rings and homogeneous polynomials. For instance, the following are the analogues of Theorem 10.2 and Proposition 10.13.

Theorem 10.4. *Let X be an irreducible projective variety with projective coordinate ring $\Gamma_p(X)$ and $F \in k[x_0, \ldots, x_n]$ be a homogeneous, non-constant polynomial such that F does not vanish identically on X. Then $X \cap V_p(F)$ is nonempty and of pure codimension* 1*, unless* $\dim X = 0$.

Sketch of the proof. Except for the non-emptiness, the claim follows from Theorem 10.2. To prove non-emptiness, assume that $\dim X \geq 1$ and let $X^* \subset \mathbb{A}^{n+1}$ be the cone over X. By definition, the cone is the affine variety given by the homogeneous ideal $I_p(X)$. One may prove that $\dim X^* = \dim X + 1$. (Exercise!) Then $0 \in X^* \cap V(F)$, and by Theorem 10.2 $\dim X^* \cap V(F) \geq 1$. In particular $X^* \cap V(F)$ contains other points except for the origin, and since it is k^*-stable it must contain a full line through the origin. We deduce that $X \cap V_p(F)$ is non-empty. □

Proposition 10.14. *Every closed subset of $\mathbb{P}^n$ of pure codimension* 1 *is a hypersurface, i.e. it is given by the vanishing of a single homogeneous polynomial.*

11. The fibres of a morphism

For the proofs of the results in this section we refer to [Mum99, S 8].

Let X, Y be two varieties and $f : X \to Y$ a morphism. The morphism f is called **dominant** if $f(X)$ is dense in Y, i.e. $\overline{f(X)} = Y$.

The following is a generalization of Proposition 3.21.

Proposition 11.1. *Let X be an algebraic variety, and let $f : X \to Y$ be a dominant morphism. Then $Z := \overline{f(X)}$ is irreducible and and f induces an injective map $f^* : k(Z) \hookrightarrow k(X)$.*

Recall the definition of 'pure dimensional' from Definition 10.3. The main result of this section is the following:

Theorem 11.1. *Let $f : X \to Y$ be a dominant morphism of irreducible algebraic varieties. Then the following hold:*

(a) Let $W \subset Y$ be an irreducible subset and let $Z \subset f^{-1}(W)$ be an irreducible component of $f^{-1}(W)$ that dominates W. Then

$$\mathrm{codim}_X Z \leq \mathrm{codim}_Y W.$$

(b) There exists an open set $U \subset Y$ such that:

- *$U \subset f(X)$;*
- *For any irreducible $W \subset Y$ such that $W \cap U \neq \emptyset$, and for any component $Z \subset f^{-1}(W)$ such that $Z \cap f^{-1}(U) \neq \emptyset$,*

$$\mathrm{codim}_Y W = \mathrm{codim}_X Z.$$

Among the most important consequences is the following corollary:

Corollary 11.2. *Let $f : X \to Y$ be a dominant morphism of irreducible algebraic varieties. Let $r := \dim X - \dim Y$. Then for any $y \in Y$, the fibre $f^{-1}(y)$ either empty, or each irreducible component has dimension $\geq r$.*

Further, there exists an open set $U \subset Y$ such that for any $y \in U$, $f^{-1}(y)$ is pure dimensional of dimension r.

Proof. We apply Theorem 11.1 to the case when $W = \{y\}$. Observe that $\mathrm{codim}_Y W = \dim Y$, therefore the condition $\mathrm{codim}_Y W \geq \mathrm{codim}_X Z$ translates into $\dim Y \geq \dim X - \dim Z$, i.e. $\dim Z \geq \dim X - \dim Y = r$. □

Example 11.3. *Consider the projection $p_2 : X \times Y \to Y$ sending $(x, y) \mapsto y$. Clearly all fibres are isomorphic to X. In particular, this shows that*

$$\dim X \times Y = \dim X + \dim Y.$$

Example 11.4. *Consider the hyperbola $C := V(xy - 1) \subset \mathbb{A}^2$. Since C is closed, The second projection restricted to C is a morphism $f : C \to \mathbb{A}^1$. This is not surjective, but it is dominant (the image is $\mathbb{A}^1 \setminus \{0\}$.) The fibre $f^{-1}(y)$ is a single point unless $y = 0$, when $f^{-1}(0) = \emptyset$. This illustrates Corollary 11.2.*

Example 11.5 (Ramified covers)**.** *Consider $f : \mathbb{P}^2 \to \mathbb{P}^2$ defined by $[x_0 : x_1] \mapsto [x_0^2 : x_1^2]$. By Proposition 8.4 this is a morphism. The fibers over $z \neq [0:1], [1:0]$ are two points, while the fibres over $[0:1], [1:0]$ are a single point. The latter is called the* **ramification locus** *of f.*

It is typical for a surjective map $f : X \to Y$ where X, Y are projective and $\dim X = \dim Y$ to have a ramification locus where the fibre changes behavior.

Example 11.6. *(The blowup at a point.) Consider the morphism $f : \mathbb{A}^2 \to \mathbb{A}^2$ defined by $(x, y) \mapsto (x, xy)$. This is a surjective morphism. The fibre over $(0, 0)$ is*

$$f^{-1}(0,0) = \{(0, y)\} \simeq \mathbb{A}^1$$

(thus positive dimensional), while the fibre over a point in the open set $U = \mathbb{A}^2 \setminus \{(0, 0)\}$ a single point. This gives another example illustrating Corollary 11.2.

Example 11.7. *(Line - plane incidence) Recall that the dual projective plane $(\mathbb{P}^2)^\vee$ parametrizes two-dimensional vector spaces in k^3. Consider the* **incidence***:*

$$X := \{(L \subset P \subset \mathbb{C}^3) : \text{ linear subspaces such that } \dim L = 1, \dim P = 2\} \subset \mathbb{P}^2 \times (\mathbb{P}^2)^*$$

This is also called the **flag variety** *for $\mathbb{C}^3$. There are projections $p_1 : X \to \mathbb{P}^2$ and $p_2 : X \to (\mathbb{P}^2)^*$. The fibres are all isomorphic to $\mathbb{P}^1$:*

$$p_1^{-1}(L) = \mathbb{P}(\mathbb{C}^3/L); \quad p_2^{-1}(P) = \mathbb{P}(P).$$

We claim that X is a projective variety.

To start, observe that we may cover (let's say) $(\mathbb{P}^2)^*$ by affine open sets U_i such that $p_2^{-1}(U_i) \simeq U_i \times \mathbb{P}^1$. For instance, take $U_0 = \{X + aY + bZ = 0\} \subset (\mathbb{P}^2)^*$ where $(a, b) \in \mathbb{A}^2$. The projective line $X + aY + bZ = 0$ corresponds to a two-dimensional vector space $V_{a,b} \subset k^3$, and one can find a basis $v_1, v_2 \subset V_{a.b}$. (Of course this basis depends on a, b.) We identify $V_{a,b}$ with (a, b). The fibre

$$p_2^{-1}(a, b) = \{([u : v], (a, b)) : [u : v] \in \mathbb{P}(V_{a,b})\}$$

The coordinate $([u : v], (a, b))$ may be regarded as the line $uv_1 + vv_2 \subset V_{a,b}$. This gives a bijection

$$p_2^{-1}(U_0) \simeq U_0 \times \mathbb{P}^1.$$

Such sets cover X and they may be glued to give an structure of an algebraic variety on X. This is closed in $\mathbb{P}^2 \times (\mathbb{P}^2)^*$, because it is closed in any of these sets. For instance,

$$p_2^{-1}(U_0) \cap X \simeq U_0 \times \mathbb{P}^1 \subset U_0 \times \mathbb{P}^2 = p_2^{-1}(U_0) \cap (\mathbb{P}^2 \times (\mathbb{P}^2)^*).$$

Either from this description, or from Theorem 11.1, one deduces that $\dim X = 3$.

12. Sheaves of modules

We have seen in Section 5 that sheaves sit at the foundation of the definition of algebraic varieties. In particular, an algebraic variety $(X, \mathcal{O}_X)$ is equipped with a sheaf of regular functions on it. However, a variety carries more structure on it than just the regular functions. For instance:

- There is a correspondence between affine varieties and their coordinate rings. Is there a geometric interpretation of *modules* over coordinate rings ?
- We would like to include homogeneous polynomials on the projective coordinate rings as global sections of a certain sheaf;
- Inspired from differential topology, we would like to associate sheaves to (local sections of) vector bundles;
- We want to be able to talk about extensions of (generalized) functions from a subvariety to an ambient variety.

12.1. **Definitions.**

Definition 12.1. *Let $(X, \mathcal{O}_X)$ be a ringed space. An* $\mathcal{O}_X$**-module** *is a sheaf $\mathcal{F}$ such that for any $U \subset X$ open, $\mathcal{F}(U)$ is a $\mathcal{O}_X(U)$-module, and the restriction homomorphisms are maps of modules, i.e. for any $V \subset U$,*

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "res"] & \mathcal{F}(V) \\ \mathcal{O}_X(U) \arrow[r, "res"] \arrow[u] & \mathcal{O}_X(V) \arrow[u] \end{tikzcd}$$

Here $\mathcal{O}_X(V)$ and $\mathcal{F}(V)$ become $\mathcal{O}_X(U)$-modules via the restriction maps.

A **homomorphism of** $\mathcal{O}_X$**-modules** *$\varphi : \mathcal{F} \to \mathcal{G}$ is given by the data of $\mathcal{O}_X(U)$-module homomorphisms $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ for any open set $U \subset X$, compatible with the restriction maps.*

Definition 12.2. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\varphi : \mathcal{F} \to \mathcal{G}$ a homomorphism of $\mathcal{O}_X$-modules.*

- *The* **kernel** *of φ, denoted by $\ker(\varphi)$, is the sheaf given by the assignment*
$$U \mapsto \ker(\varphi_U).$$
- *The* **image** *of φ, denoted by $Im(\varphi)$, is the sheafification of the presheaf*
$$U \mapsto Im(\varphi_U).$$
- *A sequence of sheaves and sheaf homomorphisms*
$$0 \longrightarrow \mathcal{F} \xrightarrow{u} \mathcal{G} \xrightarrow{v} \mathcal{H} \longrightarrow 0$$
is (short) exact if $\ker(u) = 0$ (the zero sheaf), $Im(v) = \mathcal{H}$ and $Im(u) = \ker(v)$.

Example 12.1 (The exponential sequence)**.** *The following sequence of sheaves of abelian groups is short exact:*

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_{\mathbb{C}} \xrightarrow{e^{2\pi i z}} \mathcal{O}_{\mathbb{C}}^{*} \longrightarrow 0$$

Here $\mathbb{Z}$ be the sheaf given by locally constant $\mathbb{Z}$-valued functions, $\mathcal{O}_{\mathbb{C}}$ is the (additive) sheaf of holomorphic functions on $\mathbb{C}$, and $\mathcal{O}_{\mathbb{C}}^{*}$ is the (multiplicative) sheaf of non-vanishing holomorphic functions. All these are sheaves of abelian groups.

Consider the non-vanishing holomorphic function $f(z) = z$ on $\mathbb{C}^* = \mathbb{C} - 0$. Using that the complex logarithm is defined on any simply connected open subset of $\mathbb{C}^*$, we see that z is in the image of the exponential map over any such domain. However, z is not in the image of any single holomorphic function defined on the whole $\mathbb{C}^*$. (The fundamental group of $\mathbb{C}^*$ is $\mathbb{Z}$.) This shows that sheafification is needed in the definition of the image.

Definition 12.3. *Let $f : X \to Y$ be a morphism of algebraic varieties, and $\mathcal{F}$ a sheaf on X. The* **direct image** *of $\mathcal{F}$ is the sheaf on Y, denoted by $f_*\mathcal{F}$, and defined by*

$$V \mapsto \mathcal{F}(f^{-1}(V)).$$

Example 12.2. *Let $i : Y \hookrightarrow X$ be a closed subvariety and let $\mathcal{F}_Y$ be the sheaf of k-valued functions on Y. There is a morphism of $\mathcal{O}_X$-modules $r : \mathcal{O}_X \to \mathcal{F}_Y$ defined by sending $s \in \mathcal{O}_X(U)$ to its restriction on $U \cap Y$. By definition, the image of this sheaf $Im(r) = \mathcal{O}_Y$, regarded as a $\mathcal{O}_X$-module. (Again recall that the definition of $\mathcal{O}_Y$ required the use of sheafification.)*

The kernel of r is an **ideal sheaf** *i.e. a sheaf $\mathcal{I}_Y$ on X where $\mathcal{I}_Y(U)$ is the ideal of sections on $U \subset X$ vanishing of $Y \cap U$. (In particular this is an $\mathcal{O}_X$-module.) We have another short exact sequence*

$$0 \longrightarrow \mathcal{I}_Y \longrightarrow \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Y \simeq \mathcal{O}_Y \longrightarrow 0 \tag{12.1}$$

Remark 12.1. *As we shall see later, the sequence from Equation* (12.1) *may be also be realized by gluing "local data". if one assumes that Y, X are affine and $Y \subset X$ is closed, this is the sequence associated with the short exact sequence of $\Gamma(X)$-modules $0 \to I_Y \to \Gamma(X) \to \Gamma(Y) \to 0$ where I_Y is an ideal of Y in X.*

Definition 12.4. *Let $\mathcal{F}, \mathcal{G}$ be two $\mathcal{O}_X$-modules. The* **tensor product** *$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is the sheafification of the presheaf*

$$U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U).$$

12.2. **Quasi-coherent sheaves on affine varieties.** Let $X \subset \mathbb{A}^n$ be an affine algebraic variety and let $R := \Gamma(X) = \mathcal{O}_X(X)$ be its coordinate ring. For any sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules there is an associated R-module $M := \mathcal{F}(X)$; we will also use the notation $\Gamma(\mathcal{F}) = \mathcal{F}(X)$. In other words, taking global sections gives a map:

$$\Gamma : \ (\text{ sheaves of } \mathcal{O}_X\text{-modules }) \to (\Gamma(X) - \text{modules}) \tag{12.2}$$

The purpose of this section is to perform the inverse construction: associate a $\mathcal{O}_X$-module for any R-module. Recall that for an R-module M and $f \in R$, the localization

$$M_f = M \otimes_R R_f = \{\frac{m}{f^s} : m \in M, s \in \mathbb{Z}\}/\simeq$$

The equivalence is given by $\frac{m_1}{f^{s_1}} = \frac{m_2}{f^{s_2}}$ if and if only if there exists f^s such that $f^s(f^{s_2}m_1 - f^{s_1}m_2) = 0$ in M.

Definition 12.5. *Let M be an R-module. Define the $\mathcal{O}_X$-module $\widetilde{M}$ as the unique sheaf given by*

$$\widetilde{M}(D_f) = M_f := M \otimes_R R_f.$$

As usual, the fact $\widetilde{M}$ defines uniquely a sheaf a $\mathcal{O}_X$ follows because the distinguished open sets D_f form a basis of open sets for X.

Example 12.3. *If $M = R$, then $\widetilde{M} = \mathcal{O}_X$. More generally, if $I = \sqrt{I} \subset R$ is an ideal, then $\widetilde{I} = \mathcal{I}_Y$, the ideal sheaf of $Y = V(I)$.*

Remark 12.2. *Observe that $\Gamma(\widetilde{M}) = M$, giving an inverse of the map from Equation* (12.2)*.*

This is the main theorem of this section.

Theorem 12.1. *Let X be an affine algebraic variety. The correspondence*

$$(\mathcal{O}_X(X) - \textsf{modules}) \longrightarrow (\textsf{ sheaves of } \mathcal{O}_X\textsf{-modules }); \quad M \mapsto \widetilde{M}$$

is functorial with respect to maps of $\Gamma(X)$-modules, it is exact, and it commutes with direct sums and tensor products.

Proof. As before, set $R := \Gamma(X)$. We start with the proof of *functoriality.* Take $\varphi : M \to N$ a map of R-modules. For any $f \in R$, this induces a map $\varphi_f : M_f \to N_f$ of R_f-modules, defined in an obvious way:

$$\varphi_f(\frac{m}{f^s}) = \frac{\varphi(m)}{f^s}.$$

This determines a map $\widetilde{M}(D_f) \to \widetilde{N}(D_f)$, which extends to the required map $\widetilde{M} \to \widetilde{N}$ of $\mathcal{O}_X$-modules.

We now prove that the assignment preserves *exactness.* Consider a short exact sequence of R-modules

$$0 \longrightarrow M \xrightarrow{u} N \xrightarrow{v} P \longrightarrow 0$$

We need to prove that

$$0 \longrightarrow \widetilde{M} \xrightarrow{u'} \widetilde{N} \xrightarrow{v'} \widetilde{P} \longrightarrow 0$$

is exact, where u', v' are the induced maps. To check exactness, it suffices to show exactness when restricted to distinguished open sets. For this, it suffices to show that the localized sequence

$$0 \longrightarrow M_f \xrightarrow{u_f} N_f \xrightarrow{v_f} P_f \longrightarrow 0$$

is exact.[3] Clearly v_f is surjective, and $v_f \circ u_f = 0$, therefore $\mathrm{Im}(u_f) \subset \ker(v_f)$. We need to prove the converse inclusion, and also that u_f is injective.

Let $\frac{n}{f^s} \in \ker(v_f)$. Then $\frac{v(n)}{f^s} = 0$ in P_f. Then there exists a nonnegative integer k such that $f^k v(n) = v(f^k n) = 0$. It follows that $f^k m \in \ker(v) = \mathrm{Im}(u)$, so there exists $m \in M$ such that $u(m) = f^k n$. Then

$$\frac{n}{f^s} = \frac{nf^k}{f^{k+s}} = \frac{u(m)}{f^{k+s}} \in \mathrm{Im}(u_f).$$

We now prove that u_f is injective. Let $\frac{m}{f^s} \in \ker(u_f)$. Then $f^k u(m) = 0$ for some nonnegative integer k. This implies that $f^k m \in \ker(u)$, i.e. $f^k m = 0$ in M. This implies that $\frac{m}{f^s} = 0$ in M_f, as claimed.

Finally, we need to show that the assignment preserves *direct sums and tensor products*. This follows from the standard facts that

$$(M \oplus M')_f = M_f \oplus M'_f; \quad (M \otimes_R M')_f = M_f \otimes_{R_f} M'_f.$$

□

Example 12.4. *Let $X \subset \mathbb{A}^n$ be an affine variety, and let $Y \subset X$ be a closed subvariety such that $Y = V(I)$. We have an exact sequence of $\Gamma(X)$-modules*

$$0 \longrightarrow I \longrightarrow \Gamma(X) \longrightarrow \Gamma(Y) = \Gamma(X)/I \longrightarrow 0$$

If one takes the associated sequence of $\mathcal{O}_X$-modules one recovers the sequence from Equation (12.1)*.*

Definition 12.6. *Let X be an affine variety. A $\mathcal{O}_X$-module $\mathcal{F}$ is called* **quasi-coherent** *it $\mathcal{F} = \widetilde{M}$ for some $\Gamma(X)$-module M. The $\mathcal{O}_X$-module $\mathcal{F}$ is called* **coherent** *if in addition M is finitely generated over $\Gamma(X)$.*

Example 12.5 (A non quasi-coherent sheaf)**.** *Let X be affine and $p \in X$. Define a presheaf $\mathcal{F}$ on X by*

$$\mathcal{F}(U) = \begin{cases} \mathcal{O}_X(U) & \text{if } a \notin U \\ 0 & \text{if } a \in U. \end{cases}$$

This is actually a sheaf. Observe that $\mathcal{F}(X) = 0$, thus if $\mathcal{F}$ were quasi-coherent then $\mathcal{F}$ is the zero sheaf. However, it is clear that $\mathcal{F}$ has non-zero local sections. (See [Hartshorne], [Perrin].)

[3] Equivalently, R_f is a flat R-module.

Proposition 12.6. *There is an equivalence of categories*

$$\mathsf{Mod}_R \simeq \mathsf{Mod}_{\mathcal{O}_X}$$

where Mod_R is the category of R-modules and R-module homomorphisms and $\mathsf{Mod}_{\mathcal{O}_X}$ is the category of $\mathcal{O}_X$-modules and $\mathcal{O}_X$-module homomorphisms.

Proof. This follows from definitions of the objects involved and Theorem 12.1. $\square$

Definition 12.7. *Let X be an algebraic variety and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$ modules. We say that $\mathcal{F}$ is* **quasi-coherent** *(respectively* **coherent** *if there exists a covering by affine varieties $X = \bigcup U_i$ such that for each i, $\mathcal{O}_X(U_i) = R_i$ and $\mathcal{F}|_{U_i} \simeq \widetilde{M_i}$ for some R_i-module M_i (respectively a finitely generated R_i-module).*

If in addition the modules M_i are free over R_i of the same rank, then $\mathcal{F}$ is called **locally free***.*

Remark 12.3. *One may prove that if X is an affine algebraic variety and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$ modules, then $\mathcal{F}$ is (quasi-)coherent) in the sense of Definition 12.6 if and only if it is (quasi-)coherent in the sense of Definition 12.7.*

Remark 12.4. *One may prove that there is an equivalence between locally free sheaves on X and vector bundles on X.*

12.3. **Quasi-coherent sheaves on projective varieties.** The construction of quasi-coherent sheaves on affine algebraic varieties generalizes to the projective case, provided the grading is taking into account. This is analogous to the construction of the coordinate ring of a projective algebraic varieties.

We recall few things from §7.1. Let $R = \oplus R_d$ be a graded ring, and let $M = \oplus M_d$ be a graded R-module. If $f \in R_d$ is a homogeneous element, then

$$M_{(f)} = \{\frac{m}{f^k} : \ m \text{ homogeneous and } \deg(m) = k\deg(f)\}.$$

This is an additive subgroup of M_f; more importantly, it is a module over $R_{(f)}$, the subring of R_f consisting of fractions of total degree 0.

Let $X \subset \mathbb{P}^n$ be a projective subvariety with projective coordinate ring $R := \Gamma_p(X)$.

Definition 12.8. *Let M be a graded R-module. Define the $\mathcal{O}_X$-module $\widetilde{M}$ as follows: if $f \in R$ is homogeneous of degree > 0, define $\widetilde{M}(D_f^+) := M_{(f)}$.*

Example 12.7. *Of course, $\widetilde{R} = \mathcal{O}_X$.*

Remark 12.5. *The sheaf $\widetilde{M}$ is quasi-coherent; furthermore, if M is finitely generated, it is coherent. Indeed, we may cover X by distinguished open sets D_f^+. Then observe that D_f^+ is affine and that $\widetilde{M}|_{(D_f^+)} = \widetilde{M_{(f)}}$. (Note that $\Gamma(D_f^+) = (\Gamma_p(X))_{(f)}$, and that $\widetilde{M_{(f)}}$ is an $(\Gamma_p(X))_{(f)}$-module.)*

Proposition 12.8. *The correspondence $M \mapsto \widetilde{M}$ is functorial, it preserves exact sequence, and it commutes with direct sums and tensor products.*

Sketch of the proof. Essentially the same as the proof of Theorem 12.1, except for the proof of the statement that it commutes with tensor products. First, if M, N are graded R-modules, the tensor product $M \otimes_R N$ is given the grading

$$(M \otimes_R N)_k = \oplus_{i+j=k} M_i \otimes_k N_j.$$

Then one constructs as isomorphism of $\mathcal{O}_X$-modules $\widetilde{M} \otimes_{\widetilde{R}} \widetilde{N} \to \widetilde{M \otimes_R N}$. □

For further use we record the following lemma. (It may also be used for the exactness statement in the previous proposition.)

Lemma 12.9. *Let $\varphi : M \to N$ be an R-module homomorphism of degree 0. If $\varphi : M_n \to N_n$ is surjective for $n \gg 0$ then $\widetilde{\varphi} : \widetilde{M} \to \widetilde{N}$ is surjective.*

Proof. Choose a distinguished open set $D_f^+ \subset X$, where f is homogeneous of degree > 0. It suffices to show that $\varphi_f : M_{(f)} \to N_{(f)}$ is surjective. To do this, observe that any element $\frac{y}{f^r} \in N_{(f)}$ may be written as $\frac{yf^n}{f^{n+r}}$ and $\deg(f^n y)$ may be made arbitrarily large. The hypothesis implies that such elements are in the image of φ; the claim follows from this. □

Example 12.10. *As in the affine case, if $Y \subset X$ is closed, the sequence $0 \to I \to \Gamma_p(X) \to \Gamma_p(Y) = \Gamma_p(X)/I \to 0$ gives the natural sequence from Equation* (12.1) *associated to Y.*

We now come to the main difference from the affine case: the ability to re-define the grading on a graded module.

Definition 12.9. *Let R be a graded ring and let $M = \bigoplus_{n\in\mathbb{Z}}$ be a graded R-module.*[4] *Fix $d \in \mathbb{Z}$. The module $M(d)$ is the graded module M, but with the shifted grading*

$$M(d)_n := M_{d+n}.$$

Definition 12.10. *Let $X \subset \mathbb{P}^n$ be a projective algebraic variety and let $R := \Gamma_p(X)$ be the projective coordinate ring. Define the sheaf $\mathcal{O}_X(d) := \widetilde{R(d)}$. If $\mathcal{F}$ is any sheaf of $\mathcal{O}_X$-modules, define*

$$\mathcal{F}(d) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(d).$$

The sheaves $\mathcal{O}_X(d)$ are coherent (in fact, locally free of rank 1), and they depend fundamentally on the grading on $\Gamma_p(X)$. Sometimes these are called the **Serre twisting sheaves**. Observe also that if M is a graded R-module then by commutativity of tensor products

$$\widetilde{M(d)} = \widetilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(d) = \widetilde{M}(d).$$

[4]We are grading over $\mathbb{Z}$, but this does not make a difference.

Proposition 12.11. *Let $R := \Gamma_p(\mathbb{P}^n) = k[x_0, \dots, x_n]$ graded by $\deg x_i = 1$, and let R_d be the degree d part of R. Then*

$$\Gamma(\mathcal{O}_{\mathbb{P}^n}(d)) = \begin{cases} R_d & \textit{if } d \geq 0; \\ 0 & \textit{otherwise} . \end{cases}$$

In particular, $\Gamma(\mathcal{O}_{\mathbb{P}^n}) = k$.

Proof. Let P be a global section of $\mathcal{O}_{\mathbb{P}^n}(d)$. Its restrictions to $U_i := D^+_{x_i}$ are of the form $\frac{P_i}{x_i^{d_i}}$ where $P_i \in R$ is homogeneous and $\deg P_i = d_i + d$. W.l.o.g. we may assume that x_i does not divide P_i. The restrictions must agree on the overlaps $U_i \cap U_j = D^+_{x_i x_j}$, giving equalities of the form

$$\frac{P_i}{x_i^{d_i}} = \frac{P_j}{x_j^{d_j}} \Longleftrightarrow x_j^{d_j} P_i = x_i^{d_i} P_j \quad \in k(x_0, \dots, x_n).$$

Since x_i does not divide P_i, nor x_j, it follows that (all) $d_i = 0$. Therefore $P = P_i$ is a homogeneous polynomial of degree d. This proves the claim. $\square$

A natural question is whether we can recover the graded R-module out of the quasi-coherent sheaf $\mathcal{F} = \widetilde{M}$. The answer is more subtle that in the affine case. First, define

$$\Gamma_*(\mathcal{F}) := \bigoplus_{d \in \mathbb{Z}} \Gamma(\mathcal{F}(d)).$$

This may be given a graded R-module structure by: $s \in R_d = \Gamma(\mathcal{O}_{\mathbb{P}^n}(d))$ and $t \in \Gamma_*(\mathcal{F}(d'))$ then one may multiply: $s.t \in \Gamma(\mathcal{F}(d+d'))$. A geometric fact which currently is outside our reach is the following (cf. [Hartshorne, Ch.5 and Ex. II.5.9]):

Theorem 12.2. *Let $\widetilde{M} \simeq \mathcal{F}$ be a quasi-coherent module. Then the following hold:*

(a) There exists an isomorphism of $\mathcal{O}_X$-modules $\widetilde{\Gamma_*(\mathcal{F})} \simeq \mathcal{F}$

(b) There exists a graded R-module homomorphism $M \to \Gamma_*(\mathcal{F})$ such that for $d \gg 0$, $M_d \simeq \Gamma(\mathcal{F}(d))$ as k-vector spaces.

13. Hilbert polynomials and Bézout's theorem

13.1. **The Hilbert function and the Hilbert polynomial.** Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The global sections of $\mathcal{F}$ may be considered as generalized functions on X, which carry much geometric information. If X is a projective variety, then it may be proved that $\Gamma(\mathcal{F})$ is a finite dimensional vector space. (This is not always the case for affine varieties.) A fundamental problem is to determine $\Gamma(\mathcal{F})$, or at least its dimension. In general this is very difficult, but it turns out that the sections of a large enough twist $\mathcal{F}(d)$ are easier to calculate. The reason for this (with details which are beyond the scope of this course) is that in this case

$$\dim \Gamma(\mathcal{F}(d)) = \chi(X; \mathcal{F}(d)),$$

where the latter is the sheaf cohomology Euler characteristic. Various methods allow for the determination of the Euler characteristic.[5]

Here we pursue a slightly different approach, based on commutative algebra techniques. It leads to explicit calculations, and algorithms implemented in mathematical software such as `Maple, Mathematica, Macaulay2` etc.

We start with the following definition. As usual $R := k[x_0, \dots, x_n]$ which we regard as a graded ring with $\deg x_i = 1$.

Definition 13.1. *Let M be a graded R-module. The* **Hilbert function** *is defined by*

$$h_M(d) := \dim_k M_d.$$

Now let $X \subset \mathbb{P}^n$ be a projective algebraic variety. The **Hilbert function of** X *is defined by*

$$h_X := h_{\Gamma_p(X)}$$

The fundamental theorem about Hilbert functions is that their asymptotics is given by a single (Hilbert) polynomial.

Theorem 13.1. *Let M be a finitely generated graded R-module. Then there exists a (single) polynomial $P_M(x) \in \mathbb{Q}[x]$ such that*

$$P_M(d) = h_M(d) \text{ for } d \gg 0.$$

Furthermore, $\deg P_M = \dim V_p(Ann(M))$ *where $Ann(M)$ is the annihilator*

$$Ann(M) := \{r \in R : r.m = 0, \forall m \in M\}.$$

Proof. TODO in Spring semester. □

Corollary 13.1. *Let $X \subset \mathbb{P}^n$ be a projective algebraic variety with projective coordinate ring $\Gamma_p(X)$. Then there exists a polynomial $P_X \in \mathbb{Q}[d]$ such that for all $d \gg 0$,*

$$P_X(d) = \dim_k \Gamma(\mathcal{O}_X(d)).$$

[5] Among the most popular is the Atiyah-Bott localization theorem. For an equivariant sheaf on a variety with a group action, the Euler characteristic is a sum of 'localized' data coming from the fixed loci of the group.

This polynomial is of the form:

$$P_X(d) = \frac{\deg(X)}{n!} d^n + \text{ lower order terms}$$

where $n = \dim X$*, and* $\deg(X)$ *is an integer called the* **degree** *of* X*.*

Proof. From Theorem 12.2 it follows that $\Gamma(\mathcal{O}_X(d)) \simeq (\Gamma_p(X))_d$ for $d \gg 0$. Since R is a domain, the annihilator of the R-module $\Gamma_p(X) = R/I_p(X)$ is precisely $I_p(X)$. Therefore the Hilbert polynomial of X has degree $\dim V_p(I_p(X)) = \dim X$. □

13.2. **Examples of Hilbert polynomials; arithmetic genus.** Throughout this section, $R := k[x_0, \dots, x_n]$. We start with the following short exact sequence associated to a hypersurface in $\mathbb{P}^n$.

Lemma 13.2. *Let* $F \in k[x_0, \dots, x_n]$ *be a homogeneous polynomial of degree* $d > 0$*. Then there exists a short exact serquence graded* R*-modules:*

$$0 \longrightarrow R(-d) \xrightarrow{\cdot F} R \longrightarrow R/\langle F \rangle \longrightarrow 0 \tag{13.1}$$

If in addition F *has no multiple factors, let* $X = V_p(F)$*. Then we have a short exact sequence of coherent* $\mathcal{O}_{\mathbb{P}^n}$*-modules:*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-d) \xrightarrow{\cdot F} \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_X \longrightarrow 0 \tag{13.2}$$

Proof. The polynomial F gives a degree 0 map of graded modules:

$$R(-d) \xrightarrow{\cdot F} R$$

This map is clearly injective, and its cokernel is $R/\langle F \rangle$. By the Nullstellensatz, $\langle F \rangle = I_p(X)$, thus $R/\langle F \rangle = \Gamma_p(X)$. This gives the first short exact sequence of graded R-modules. Taking the associated short exact sequence of coherent $\mathcal{O}_{\mathbb{P}^n}$-modules proves the statement. □

For an indeterminate z define

$$\binom{z}{r} := \frac{1}{r!} z(z-1) \cdot \ldots \cdot (z - r + 1).$$

Example 13.3. *The Hilbert polynomial of* $\mathbb{P}^n$ *is*

$$P_{\mathbb{P}^n}(z) = \binom{z+n}{n} = \frac{1}{n!} z^n + l.o.t.$$

In particular, $\deg \mathbb{P}^n = 1$.

Example 13.4 (Two points in $\mathbb{P}^1$). *Consider* $I = \langle x_0 x_1 \rangle \subset k[x_0, x_1]$ *and let* $S := k[x_0, x_1]/I$*. Then*

$$h_S(d) = \begin{cases} 1 & d = 0 \\ 2 & d > 0 \end{cases}$$

Therefore $P_S(x) = 2$. We may also consider the ideal $I = \langle x_0^2 \rangle$ (a double point). The Hilbert polynomial is the same. In particular the degree of 2 distinct points equals the degree of one double point.

Example 13.5. *While proving Corollary 13.1 we will also show that if $\dim X = 0$ (i.e. if X is consists of finitely many points) then $h_X(z) \equiv \dim_k \Gamma(X)$. Here we utilize that any such X is affine, and one may show that if $\dim X = 0$ then its coordinate ring $\Gamma(X)$ is a finite dimensional vector space (i.e. $\dim_k \Gamma(X) < \infty$).*

Next we calculate the Hilbert polynomial of a hypersurface.

Proposition 13.6. *Let $F \in k[x_0, \dots, x_n]$ be a homogeneous polynomial and let $M := k[x_0, \dots, x_n]/\langle F \rangle$.*

$$P_M(z) = \binom{z+n}{n} - \binom{z - \deg F + n}{n}.$$

In particular, if F is reduced (i.e., multiplicity free), then $\deg V_p(F) = \deg F$.

Proof. Taking degree d components in Equation (13.1) we obtain the short exact sequence of vector spaces

$$0 \longrightarrow (R)_{d-\deg F} \xrightarrow{\cdot F} R_d \longrightarrow (R/\langle F \rangle)_d \longrightarrow 0$$

Then $h_M(d) = \binom{d+n}{n} - \binom{d-\deg(F)+n}{n}$. The result follows after performing the algebraic calculations. $\square$

Example 13.7 (A conic)**.** *Consider the conic $C := V_p(y^2 - xz) \subset \mathbb{P}^2$. By Proposition 13.6,*

$$p_C(z) = \frac{(z+2)(z+1)}{2} - \frac{z(z-1)}{2} = 2z + 1.$$

Alternatively, one may calculate directly from the definition that

$$h_C(d) = \begin{cases} 1 & d = 0 \\ 3 & d = 1 \\ 5 & d = 3 \end{cases}$$

Again this shows that $h_C(z) = 2z + 1$.

Example 13.8 (A cubic curve in the plane)**.** *Consider a cubic curve $C := V_p(F) \subset \mathbb{P}^2$ where F is a multiplicity free homogeneous polynomial of degree 3. Again by Proposition 13.6,*

$$p_C(z) = \frac{(z+2)(z+1)}{2} - \frac{(z-1)(z-2)}{2} = 3z.$$

Definition 13.2. *Let $Y \subset \mathbb{P}^n$ be an irreducible projective variety. The* **arithmetic genus** *of Y is defined by*

$$p_a(Y) := (-1)^{\dim Y}(P_Y(0) - 1).$$

It may be proved that the arithmetic genus is independent of the embedding.

Example 13.9. *The arithmetic genus of a plane conic* $p_a(C_2) = 0$*; the arithmetic genus of a plane cubic* $p_a(C) = 1$.

13.3. **Bézout's theorem.** Recall that $R = k[x_0, \dots, x_n]$. We start with a generalization of Lemma 13.2.

Lemma 13.10. *Let* $X \subset \mathbb{P}^n$ *be a projective variety and let* $F \in k[x_0, \dots, x_n]$ *such that* $\dim(X \cap V_p(F)) = \dim X - 1$*. Then there exists a short exact sequence of graded* R*-modules:*

$$0 \longrightarrow (R/I_p(X))(-\deg F) \xrightarrow{\cdot F} R/I_p(X) \longrightarrow R/(I_p(X) + \langle F \rangle) \longrightarrow 0$$

Proof. Write $X = X_1 \cup \dots \cup X_r$ the decomposition into irreducible components. Then $I(X) = \bigcap_{i=1}^r I(X_j)$ and each $I(X_j)$ is a prime ideal. By Theorem 10.4, the hypothesis that $\dim(X \cap V_p(F)) = \dim X - 1$ implies that F does not vanish identically on each of the X_i, i.e. that $F \notin I(X_j)$, for $1 \leq j \leq r$. This implies that the map $\Gamma_p(X) \to \Gamma_p(X)$ obtained by multiplication by F is injective. The rest follows from definitions. $\square$

Theorem 13.2 (Bézout)**.** *Let* $X \subset \mathbb{P}^n$ *be a projective variety and let* $F \in k[x_0, \dots, x_n]$ *such that* $\dim(X \cap V_p(F)) = \dim X - 1$ *and the ideal* $I_p(X) + \langle F \rangle$ *(i.e. the ideal generated by* $I_p(X)$ *and* F*) is radical. Then*

$$\deg(X \cap V_p(F)) = \deg(X) \cdot \deg(F).$$

Proof. From Lemma 13.10 we obtain the short exact sequence

$$0 \longrightarrow R/I_p(X) \xrightarrow{\cdot F} R/I_p(X) \longrightarrow R/(I_p(X) + \langle F \rangle) \longrightarrow 0$$

Since J is radical, it follows that $J = I_p(X \cap V_p(F))$ (exercise!) Taking degree d homogeneous components implies that

$$P_{X \cap V_p(F)}(d) = P_X(d) - P_X(d - \deg(F)).$$

Let $r := \dim X$. Write $P_X(d) = \frac{\deg X}{r!} d^r + c_{r-1} d^{r-1} + l.o.t.$ Then

$$\begin{aligned} P_{X \cap V_p(F)}(d) &= P_X(d) - P_X(d - \deg(F)) \\ &= \frac{\deg X}{r!} d^r + c_{r-1} d^{r-1} + l.o.t. - (d - \deg(F))^r - c_{r-1}(d - \deg(F))^{r-1} - l.o.t. \\ &= (c_{r-1} + \frac{\deg(X)}{r!} r \deg(F) - c_{r-1}) d^{r-1} + l.o.t. \\ &= \frac{\deg(X) \deg(F)}{(r-1)!} d^{r-1} + l.o.t. \end{aligned}$$

$\square$

Remark 13.1. *The hypothesis on* J *being radical may be removed, but then we need to consider the* **scheme** $X \cap V_p(F)$*, associated to the ideal* J*.*

Corollary 13.11. *Let $F, G \in k[x,y,z]$ be multiplicity free homogeneous polynomials without common factors. Assume that $\langle F, G\rangle$ is a radical ideal. Then*

$$\deg(V_p(F) \cap V_p(G)) = \deg(F)\deg(G).$$

Proof. The hypotheses imply that $I_p(V_p(F)) = \langle F\rangle$, $I_p(V_p(G)) = \langle G\rangle$ and $I_p(V_p(F) \cap V_p(G)) = \langle F, G\rangle$. Then apply Bézout's theorem. □

Theorem 13.3. *Let $X \subset \mathbb{P}^n$ be an irreducible projective subvariety of codimension r. Then $\deg(X)$ equals the number of points in the intersection $X \cap H_1 \cap \ldots \cap H_r$ where H_i's are general hyperplanes in $\mathbb{P}^n$.*

Here 'general' means that there exists an open set $U \in (\mathbb{P}^n)^* \times \ldots \times (\mathbb{P}^n)^*$ (r components) such that $(H_1, \ldots, H_r) \in U$. To prove this we need two preparatory results.

Lemma 13.12. *Let $X \subset \mathbb{P}^n$ be a closed subvariety. Then there exists a hyperplane $H \subset \mathbb{P}^n$ such that no component of X is included in H.*

Proof. It suffices to show that there exists a nonempty open set $U \subset (\mathbb{P}^n)^*$ such that for all $H \in U$, X is not included in H. We will abuse notation and identify H with the linear form which determines it. Let $I_p(X) = \langle F_1, \ldots, F_r\rangle$ where F_i are homogeneous. The condition that $X \subset H$ amounts to $(F_i)_{|H} \equiv 0$ for all i, i.e. $H|F_i$ for all i. Since each F_i can be divided by only finitely many linear factors, the claim follows. □

We also need the following generalization of Lemma 13.12, which is currently outside our technology.

Theorem 13.4 (Bertini's theorem)**.** *Let $X \subset \mathbb{P}^n$ be a projective variety. Then there exists an open set $U \subset (\mathbb{P}^n)^*$ such that for all $H \in U$, $H \cap X$ has codimension 1 and $I_p(X) + H$ is radical.*

Proof of Theorem 13.3. Choose H as in Bertini's theorem. Then the claim follows by induction from Bézout's theorem, since $\deg(X \cap H) = \deg(X)$. □

13.4. **Degree of the union of two projective varieties.** We now turn to the degree of a union of projective subvarieties.

Lemma 13.13. *Let I_1, I_2 be two homogeneous ideals in $R := k[x_0, \ldots, x_n]$. Then there exists a short exact sequence of graded R-modules*

$$0 \longrightarrow R/(I_1 \cap I_2) \xrightarrow{(f,f)} R/I_1 \oplus R/I_2 \xrightarrow{(f,-g)} R/(I_1 + I_2) \longrightarrow 0$$

where the first morphism is the diagonal, and the second sends $(f, g) \in R/I_1 \oplus R/I_2$ to $f - g \in R/(I_1 + I_2)$.

Proof. Clear from definitions. □

Proposition 13.14. *Let $X_1, X_2 \subset \mathbb{P}^n$ be two closed subvarieties of the same dimension r, and such that* $\dim X_1 \cap X_2 < r$*. Let $I_1 = I_p(X_1)$ and $I_2 = I_p(X_2)$. Assume that $I_1 \cap I_2$ and $I_1 + I_2$ are radical (equivalently, $I_1 \cap I_2 = I_p(X_1 \cup X_2)$ and $I_1 + I_2 = I_p(X_1 \cap X_2)$. Then the following hold:*

$$\deg(X_1 \cup X_2) = \deg X_1 + \deg X_2.$$

Proof. Recall $I_p(X_1 \cup X_2) = I(X_1) \cap I(X_2)$. From Lemma 13.13 we get a short exact sequence

$$0 \longrightarrow \Gamma_p(X_1 \cup X_2) \longrightarrow \Gamma_p(X_1) \oplus \Gamma_p(X_2) \longrightarrow \Gamma_p(X_1 \cap X_2) \longrightarrow 0$$

Then $P_{X_1} + P_{X_2} = P_{X_1 \cap X_2} + P_{X_1 \cup X_2}$, and since $\deg P_{X_1 \cap X_2} < r$, the leading term of $P_{X_1 \cup X_2}$ must be the sum of the leading terms of P_{X_1} and P_{X_2}. $\square$

END OF SEMESTER 1

14. Start of semester 2: goals

The goal in semester 2 is to introduce basic notions in intersection theory, and work out some applications from enumerative geometry. For this, we will need to work with **schemes**, define flat and proper morphisms, Weil and Cartier divisors, line bundles and vector bundles, and Chern classes of vector bundles. We will use the projective spaces, and, more generally, Grassmannians, as the basic models of schemes which will be utilized to set up and solve problems in enumerative geometry. Examples of such problems are:

- Find the numbers of lines meeting 4 given lines in $\mathbb{P}^3$. (Answer: 2.)
- Find number of conics meeting 8 general lines in $\mathbb{P}^3$. (Answer: 92.)
- Find number of lines on a general cubic surface in $\mathbb{P}^3$. (Answer: 27.)

All these may be answered just by analyzing the intersection theory of $\mathrm{Gr}(2,4)$, the Grassmannian of subspaces of dimension 2 in $\mathbb{C}^4$. (This is the same as the space parametrizing lines in $\mathbb{P}^3$.)

Unlike the first semester, where (almost) all statements have been given a complete proof, we will need to take some statements by heart. However, I aim to give rigorous definitions of all notions we are going to utilize.

Throughout the notes from now on, k will denote an algebraically closed field. As usual, all rings are commutative with 1 and all rings are Noetherian.

15. Schemes

Even if one wishes to work with algebraic varieties, the appearance of schemes is unavoidable. For example:

- the intersection of a conic and a line tangent to it is a double point.
- Consider a degree 2 morphism $f : \mathbb{P}^1 \to \mathbb{P}^1$, e.g. $[x : y] \mapsto [x^2 : y^2]$. Then a general fibre has 2 points, but there will be fibres which are double points.

To understand better what we are generalizing, recall that an algebraic variety is (by definition) covered by affine algebraic varieties, each of which being inside of some affine space $\mathbb{A}^n$. Recall the correspondence:

$$\{\text{Affine algebraic varieties } X \subset \mathbb{A}^n\} \leftrightarrow \{ \text{ radical ideals } I \subset k[x_1, \dots, x_n]\}$$

We may generalize this correspondence in several directions:

1. We may allow the ideals I to be non-radical, leading to non-reduced schemes.
2. We do not have to consider ideals in a polynomial ring, leading to schemes which are non-necessarily embedded in an affine space.
3. We may remove the requirement that I (or the polynomial ring) is finitely generated. This leads to the theory of schemes of infinite type. We will not work in this generality.

15.1. Spectrum of a ring.

In this section we largely follow Mumford's treatment from [Mum99].

Definition 15.1. *Let R be a ring. The* **spectrum** *of R is*

$$\mathrm{Spec}(R) = \{\mathfrak{p} \subset R : \mathfrak{p} \textit{ is a prime ideal } \}$$

Example 15.1. *If $R = K$ is a field then $\mathrm{Spec}(K) = (0)$. More generally, $(0) \in \mathrm{Spec}(R)$ whenever R is an integral domain.*

We make $\mathrm{Spec}(R)$ into a topological space by defining its closed sets to be of the form

$$V(I) := \{\mathfrak{p} \in \mathrm{Spec}(R) : \mathfrak{p} \supset I\}$$

where $I \subset R$ is an ideal. Note that $V(I)$ may also be defined by requiring that $\varphi_f(\mathfrak{p}) = 0$ for all $f \in I$. This topology is again called the **Zariski topology**.

Proposition 15.2. *The closed sets form a topology on $\mathrm{Spec}(R)$ such that the corresponding topological space is paracompact. Furthermore, the following hold for any ideals $I, J, I_s \subset R$:*

- $V(\bigcup_s I_s) = \bigcap_s V(I_s)$;
- $V(I \cap J) = V(I) \cup V(J)$

Proof. **Homework!** □

As usual, if $f \in R$, we denote by $V(f) = V(\langle f \rangle)$.

Spec(R) will be the underlying set of the points of an affine scheme. The points in the previous notion of affine variety correspond to the maximal ideals in R, but in this new context we have many more points than before. The interpretation of $\mathfrak{p} \in \mathrm{Spec}(R)$ is that it corresponds to a (closed) subscheme of Spec(R), namely $V(\mathfrak{p})$, and if $\mathfrak{p}$ is smaller, then the subscheme is larger. In this context the following notion is useful:

Definition 15.2. *Let $Z \subset \mathrm{Spec}(R)$ be an irreducible closed subset. A point $z \in Z$ is called a* **generic point** *of Z if $\overline{\{z\}} = Z$, i.e., for any open set $U \subset Z$, $z \in U$.*

Example 15.3. *If R is an integral domain, its only generic point is (0). More generally, if $z \in \mathrm{Spec}(R)$, then (homework!)*

$$\overline{\{z\}} = \{\mathfrak{p} \in \mathrm{Spec}(R) : \mathfrak{p} \supset z\} = V(z).$$

In particular, the only closed points in Spec(R) are the maximal ideals, and, furthermore, Spec(R) is not T_1.

Example 15.4. *TODO. Describe $\mathrm{Spec}(\mathbb{C}[x,y])$ and the generic points of irreducible subsets.*

The following gives an idea of what to expect when two closed sets are equal. Recall that the radical of an ideal is the intersection

$$\sqrt{I} = \bigcap_{\mathfrak{p} \supset I; \mathfrak{p} \text{ is prime}} \mathfrak{p}.$$

Lemma 15.5. *Let I, J be ideals in R. Then $V(I) = V(\sqrt{I})$. If in addition I, J are radical then $V(I) = V(J)$ if and only if $\sqrt{I} = \sqrt{J}$.*

Proof. The equality $V(I) = V(\sqrt{I})$ follows from the definition of the radical. This also implies that $V(I) = V(J)$ for I, J radical. Conversely, if $V(I) = V(J)$ then

$$\sqrt{I} = \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p} = \bigcap_{\mathfrak{p} \in V(J)} \mathfrak{p} = \sqrt{J}.$$

□

Proposition 15.6. *Let $\mathfrak{p} \in \mathrm{Spec}(R)$. Then the following hold:*

(a) $V(\mathfrak{p})$ is irreducible and $\mathfrak{p}$ is its unique generic point.

(b) Every irreducible closed subset of Spec(R) is of the form $V(\mathfrak{p})$ for some $\mathfrak{p} \in \mathrm{Spec}(R)$.

Proof. Write $V(\mathfrak{p}) = Z_1 \cup Z_2$ some closed sets Z_1, Z_2. If $\mathfrak{p} \in Z_i$ then $\overline{\{\mathfrak{p}\}} \subset \overline{Z_i} = Z_i$, thus equality must occur. If there is another generic point $\mathfrak{q}$ then since $\overline{\{\mathfrak{p}\}} = \overline{\{\mathfrak{q}\}}$ it follows that $\mathfrak{p} \subset \mathfrak{q}$ and $\mathfrak{q} \subset \mathfrak{p}$, thus $\mathfrak{p} = \mathfrak{q}$. This proves (a).

Now take $Z = V(I) \subset \mathrm{Spec}(R)$ closed and irreducible, where $I \subset R$ is an ideal. By Lemma 15.5 $V(I) = V(\sqrt{I})$, therefore we may assume that I is radical. We will prove that I is prime. Take $f, g \in R$ such that $fg \in I$, and define $B := I + \langle f \rangle$ and $C := I + \langle g \rangle$.

We claim that $I = B \cap C$. Clearly the inclusion $\subset$ holds. To prove the reverse inclusion, take $h := a_1 + \alpha f = a_2 + \beta g \in B \cap C$ where $a_1, a_2 \in I$ and $\alpha, \beta \in R$. Then

$$h^2 = (a_1 + \alpha f)(a_2 + \beta g) = a_1 a_2 + a_1 \beta g + a_2 \alpha f + \alpha\beta f g \in I.$$

Since I is radical, it follows that $h \in I$, which proves the claim.

From the claim, $V(I) = V(B) \cup V(C)$, and since $V(I)$ is irreducible it follows (w.l.o.g.) that $V(I) = V(B)$. Then

$$B \subset \bigcap_{\mathfrak{p} \supset B} \mathfrak{p} = \bigcap_{\mathfrak{p} \supset I} \mathfrak{p} = \sqrt{I} = I,$$

therefore $B \subset I$, hence thus $f \in I$, which proves that I is prime. $\square$

The **distinguished open set** in $\operatorname{Spec}(R)$ associated to f is defined by

$$D_f := \{\mathfrak{p} \in \operatorname{Spec}(R) : f \notin \mathfrak{p}\} = \operatorname{Spec}(R) \setminus V(f).$$

Then $\operatorname{Spec}(R) \setminus V(I) = \bigcup_{f \in I} D_f$, showing that the distinguished open sets form a basis for the topology of $\operatorname{Spec}(R)$.

We will also need the following:

Lemma 15.7. *(a)* $\operatorname{Spec}(R) = \bigcup_{s \in S} D_{f_s}$ *if and only if the ideal* $\langle f_s : s \in S \rangle = R$.

(b) $\operatorname{Spec}(R)$ *is quasi-compact, i.e. from any cover we may extract a finite subcover.*

Proof. Let J be the ideal generated by f_s. If $J = R$, then for any $\mathfrak{p} \in \operatorname{Spec}(R)$, one of the f_s's cannot be in $\mathfrak{p}$. Conversely, if $\operatorname{Spec}(R) = \bigcup_{s \in S} D_{f_s}$ then each prime ideal must be in some D_{f_s}, i.e. not all the f_s's are contained in any single prime ideal. Then $J = R$.

For part (b), take a cover $\operatorname{Spec}(R) = \bigcup D_{f_s}$. Then f_s generate R, and there must be finitely many of them which still generate; these give the finite subcover. $\square$

In general one may prove that D_f is quasicompact. But arbitrary open sets U may not be.

We also record the following properties of the distinguished open sets:

(a) $D_f \cap D_g = D_{fg}$ (**homework!**).
(b) $D_f \supset D_g$ if and only if $g \in \sqrt{\langle f \rangle}$.

Proof. We have $\sqrt{\langle f \rangle} = \bigcap_{f \in \mathfrak{p}} \mathfrak{p}$, therefore $g \in \sqrt{\langle f \rangle}$ iff $g \in \mathfrak{p}$ for any prime $\mathfrak{p}$ with $f \in \mathfrak{p}$. The statement follows by negating this. $\square$

Observe that the second statement implies that $D_f = D_g$ if and only if $\sqrt{\langle f \rangle} = \sqrt{\langle g \rangle}$.

Remark 15.1. *The inclusion* $D_f \supset D_g$ *gives a partial order* $\leq_1$ *on the elements of* R *by* $f \leq_1 g$ *iff* $D_f \supset D_g$. *This is a* **directed set***, in the sense that for any* $f, g \in R$, $D_{fg} = D_f \cap D_g$, *thus* $f, g \leq_1 fg$. *By (b) above, this is isomorphic (as a directed set) to the partial order on* R *given by* $f \leq_2 g$ *iff* $g = ff'$ *for some* $f' \in R$. *Then* $(R, \leq_1) \simeq (R, \leq_2)$

as partial ordered sets, and we identify the two orderings. The limits over subsets of elements in R will be taken with respect to these limits.

15.2. **The structure sheaf of** $\operatorname{Spec}(R)$**.** In this section we will define the structure sheaf $\mathcal{O}_{\operatorname{Spec}(R)}$. We will be associating $D_f \mapsto R_f$ for each $f \in R$, and we need to check a number of compatibilities.

Lemma 15.8. *Let $f, g, h \in R$.*

(a) If $D_g \subset D_f$ then there is a natural (restriction) map $R_f \to R_g$. Furthermore, if $D_h \subset D_g \subset D_f$, then there is a commutative diagram given by restriction maps

$$\begin{tikzcd} R_f \arrow[r] \arrow[dr] & R_g \arrow[d] \\ & R_h \end{tikzcd}$$

(b) The assignment $D_f \mapsto R_f$ is well defined, i.e., $R_f \simeq R_g$ if $D_f = D_g$.

(c) Let $\mathfrak{p} \in \operatorname{Spec}(R)$. Then the localization $R_{\mathfrak{p}}$ is the direct limit

$$R_{\mathfrak{p}} = \varinjlim_{f \in R \setminus \mathfrak{p}} R_f = \varinjlim_{\mathfrak{p} \in D_f} R_f.$$

Proof. If $D_g \subset D_f$ then $g \in \sqrt{\langle f \rangle}$, i.e. $g^n = hf$ for some positive integer n and $h \in R$. Then the map $R_f \to R_g$ is given by

$$\frac{a}{f} \mapsto \frac{ah}{g^n}.$$

If $D_f = D_g$ then we get maps $R_f \to R_g$ and $R_g \to R_f$ which are inverse to each other. (Check!) We leave part (c) as an exercise. □

We will see that the localization $R_{\mathfrak{p}}$ is the stalk at $\mathfrak{p}$ of the structure sheaf $\mathcal{O}_{\operatorname{Spec} R}$ and part (c) is the analogue of the fact that germs of functions a point $\mathfrak{p}$ have representatives in a neighborhood of $\mathfrak{p}$.

We are ready to define the structure of $\operatorname{Spec}(R)$. We utilize the same idea as in Lemma 5.5. The field K from this lemma is replaced by $\prod_{\mathfrak{p} \in \operatorname{Spec}(R)} R_{\mathfrak{p}}$.

Definition 15.3. *The structure sheaf $\mathcal{O}_{\operatorname{Spec} R}$ is defined as follows. Let $U \subset \operatorname{Spec}(R)$ be open. Then $\mathcal{O}_{\operatorname{Spec} R}(U)$ is the set of tuples $(s_{\mathfrak{p}}) \in \prod_{\mathfrak{p} \in U} R_{\mathfrak{p}}$ such that for each $\mathfrak{q} \in U$ there exists a distinguished open set $D_{f_i} \subset U$ with the property that for all $\mathfrak{p} \in D_{f_i}$, and all $(s_{\mathfrak{p}})_{\mathfrak{p} \in D_{f_i}}$, there exists $s_i \in R_{f_i}$ such that $s_{\mathfrak{p}}$ is obtained by the restriction $R_{f_i} \to R_{\mathfrak{q}}$.*

If $V \subset U$ are open sets, then the restriction map $\rho_{UV} : \mathcal{O}_{\operatorname{Spec} R}(U) \to \mathcal{O}_{\operatorname{Spec} R}(V)$ is the natural projection.

It is easy to check that each $\mathcal{O}_{\operatorname{Spec} R}(U)$ is a ring.

Theorem 15.1. *(a) $\mathcal{O}_{\operatorname{Spec} R}$ is a sheaf.*

(b) $\mathcal{O}_{\operatorname{Spec} R}(D_f) = R_f$.

(c) The stalk $(\mathcal{O}_{\operatorname{Spec} R})_{\mathfrak{p}} = R_{\mathfrak{p}}$.

In order to prove the theorem we will need two lemmas.

Lemma 15.9. *Assume $D_f = \bigcup_i D_{f_i}$ for some f_i. If $g \in R_f$ has image 0 in all R_{f_i} then $g = 0$.*

Proof. Write $g = \frac{b}{f^n}$ and define the annihilator ideal $A = \{c : cb = 0\}$. The following statements are equivalent:

- $g = 0$ in R_f;
- There exists m such that $f^m \cdot b = 0$ in R;
- $f \in \sqrt{A}$;
- If a prime ideal $\mathfrak{p} \supset A$ then $f \in P$.

Assume now that $g \neq 0$. Then we may choose a prime ideal $\mathfrak{p} \supset A$ such that $f \notin \mathfrak{p}$, i.e., $\mathfrak{p} \in D_f$. Since $f \notin \mathfrak{p}$ and $D_f = \bigcup D_{f_i}$ we may choose i such that $\mathfrak{p} \in D_{f_i}$. Consider the commutative diagram

$$\begin{tikzcd} R_f \arrow[r] \arrow[dr] & R_{f_i} \arrow[d] \\ & R_{\mathfrak{p}} \end{tikzcd}$$

The image of g in $R_{\mathfrak{p}}$ factors through R_{f_i}, thus it is equal to $0 \in R_{\mathfrak{p}}$. Then there exists $c \in R \setminus \mathfrak{p}$ such that $cb = 0$, i.e. $c \in A$. But then $c \in A \subset \mathfrak{p}$ by assumption, which is a contradiction. $\square$

Lemma 15.10. *Assume $D_f = \bigcup_i D_{f_i}$ for some f_i, and assume we have elements $g_i \in R_{f_i}$ such that the restriction to $R_{f_i f_j}$ coincide. Then there exists $g \in R_f$ such that the restriction to each R_{f_i} is g_i.*

Proof. Similar to proof of the gluing property in Equation (5.3) above; see [Mum99, p. 70, Lemma 2]. $\square$

We are now ready to prove Theorem 15.1.

Proof of Theorem 15.1. To prove (a), by Lemma 5.5, it suffices to show the restriction and gluing property for the basis of open sets. It is clear that the restriction maps ρ_{UV} satisfy the required compatibilites. The gluing property follows from Lemma 15.10.

To prove (b), there is a natural map

$$R_f \to \mathcal{O}_{\mathrm{Spec} R}(D_f) \subset \prod_{\mathfrak{p} \in D_f} R_{\mathfrak{p}} = \prod_{f \notin \mathfrak{p}} R_{\mathfrak{p}}$$

obtained by projections $R_f \to R_{\mathfrak{p}}$. This map is injective by Lemma 15.9 and surjective by Lemma 15.10.

To calculate the stalk at a point $\mathfrak{p}$, we may consider the elements of the basis of open sets which contain $\mathfrak{p}$. Then

$$(\mathcal{O}_{\mathrm{Spec} R})_{\mathfrak{p}} = \varinjlim_{\mathfrak{p} \in D_f} \mathcal{O}_{\mathrm{Spec} R}(D_f) = \varinjlim_{f \notin \mathfrak{p}} R_f = R_{\mathfrak{p}}.$$

□

Remark 15.2. *Note that the stalks of* $\mathcal{O}_{\mathrm{Spec}R}$ *are* $R_{\mathfrak{p}}$*, and these are* local rings*. Then* $(\mathrm{Spec}R, \mathcal{O}_{\mathrm{Spec}R})$ *is called a* **locally ringed space***.*

Remark 15.3. *The fact that we have stalks at non-closed points gives an additional structure. If* $\mathfrak{p}_1 \subset \mathfrak{p}_2$ *are two prime ideals in* R*, then* $\mathfrak{p}_2 \in \overline{\{\mathfrak{p}_1\}}$*, therefore any neighborhood* U *of* $\mathfrak{p}_2$ *contains* $\mathfrak{p}_1$*. Then*

$$R_{\mathfrak{p}_2} = (\mathcal{O}_{\mathrm{Spec}R})_{\mathfrak{p}_2} = \varinjlim_{\mathfrak{p}_2 \in U} \mathcal{O}_{\mathrm{Spec}R}(U) \to \varinjlim_{\mathfrak{p}_1 \in V} \mathcal{O}_{\mathrm{Spec}R}(V) = (\mathcal{O}_{\mathrm{Spec}R})_{\mathfrak{p}_1} = R_{\mathfrak{p}_1}.$$

In particular, germs of functions at closed points restrict to germs of functions at generic points of larger closed subsets. Algebraically, this corresponds to the natural map $R_{\mathfrak{p}_2} \to R_{\mathfrak{p}_1}$*.*

Compare this to the fact that in the previous world of varieties, we had natural maps between the stalk at a closed point $x_0 \in X$ *and the function field on some irreducible variety* X*. The map was simply* $\frac{f}{g} \mapsto \frac{f}{g}$ *where* $g(x_0) \neq 0$ *for* f, g *regular functions in some neighborhood of* x_0*.*

We may still regard elements in R as functions, except that *not in the same field*. If $\mathfrak{p} \in \mathrm{Spec}(R)$ then define recall that $R_{\mathfrak{p}}$ is a local ring, with maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$. Define the **residue field** $k(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. Since quotient and localization commute [AM69, Cor. 3.4], $k(\mathfrak{p}) = (R/\mathfrak{p})_{\langle 0 \rangle}$ (the fraction field of $R/\mathfrak{p}$).

Then any $r \in R$ may be regarded as a function

$$\varphi_r : \mathrm{Spec}(R) \to \bigcup_{\mathfrak{p} \in \mathrm{Spec}(R)} k(\mathfrak{p}); \quad \mathfrak{p} \mapsto \text{image of } f \text{ under } R \to R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} = k(\mathfrak{p}).$$

Observe that if $R = k[x_1, \ldots, x_n]/I$ and $\mathfrak{p}$ is maximal then (e.g., by Nullstellensatz) $k(\mathfrak{p}) = k$. This recovers our previous point of view that elements in the coordinate ring of an affine variety may be regarded as functions to the ground field. But if R is an integral domain, then $\langle 0 \rangle$ is prime in R, and $k(\langle 0 \rangle) = Frac(R)$, the fraction field of R.

However, assume that an element $r \in R$ satisfies $\varphi_r(\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$. Then $r \in \mathfrak{p}$ for all $\mathfrak{p} \subset R$ prime, thus $r \in \bigcap_{\mathfrak{p} \in \mathrm{Spec}(R)} \mathfrak{p} = Nil(R)$ (the nilradical of R). In other words, r must be nilpotent.

Example 15.11. *Let* $R = k[x]$*. Then*

$$k(\mathfrak{p}) = \begin{cases} k & \text{if } \mathfrak{p} = \langle x - a \rangle, \quad a \in \mathbb{C}; \\ k(x) & \text{if } \mathfrak{p} = 0. \end{cases}$$

Example 15.12. *Consider* $R = k[x]/\langle x^2 \rangle$*. Then* $\mathrm{Spec}(R) = \{\langle x \rangle\}$ *consists of a single point, but* R *is not a field. Observe that the residue field is* $k(\langle x \rangle) = k$*. The functions* 0 *and* x *in* R *have the same value on* $\mathrm{Spec}(R)$*, but they are different as functions.*

The global functions on $\mathrm{Spec}R$ *are just* $R = k[x]/\langle x^2 \rangle$*, i.e. functions* $f \in k[x]$ *modulo functions vanishing on the second order at* $x = 0$*. We will see later that the natural* *projection map* $k[x] \to R$ *gives an embedding* $\mathrm{Spec} R \hookrightarrow \mathbb{A}^1$. *At the level of functions, the restriction of* $f \in k[x]$ *to* R *remembers the information about its value of* f *at* 0 and also its derivative. *From this point, one may interpret embeddings of* $\mathrm{Spec}(R)$ *as a point together with a tangent direction at it.*

Proposition 15.13. *Let* R *be a ring and* $f \in R$. *Then there is an isomorphism of ringed spaces*

$$((\mathrm{Spec} R)_f, \mathcal{O}_{(\mathrm{Spec} R)_f}) \simeq (\mathrm{Spec}(R_f), \mathcal{O}_{\mathrm{Spec} R_f}),$$

where $(\mathrm{Spec}(R))_f$ *is the distinguished open set corresponding to* f.

For the proof, we need the following lemma (cf. [AM69, Prop. 3.11(iv)]:

Lemma 15.14. *Let* R *be a ring an let* $S \subset R$ *be a multiplicative set. Then there is a one to one correspondence between the prime ideals in* R_S *and the prime ideals* $\mathfrak{q} \subset R$ *such that* $S \cap \mathfrak{q} = \emptyset$. *The correspondence is given by extension and contraction of ideals:*

$$\mathfrak{q} \subset R \mapsto S^{-1}\mathfrak{q} \subset S^{-1}R; \quad \widetilde{\mathfrak{q}} \subset S^{-1}R \mapsto i^{-1}(\widetilde{\mathfrak{q}}),$$

where $i : R \to S^{-1}R$ *is the natural morphism.*

Proof of Proposition 15.13. By Lemma 15.14, there is a bijective map between $(\mathrm{Spec} R)_f \to \mathrm{Spec}(R_f)$ sending $\mathfrak{p} \mapsto \mathfrak{p} R_f$. Furthermore, for any $g \in R$, the same lemma implies that there is a bijection

$$(\mathrm{Spec} R)_{fg} \simeq (\mathrm{Spec} R_f)_g.$$

(Check!) This bijection induces a ring isomorphism at the level of sections over these distinguished open sets, because both rings equal to R_{fg}, by Theorem 15.1(b). This implies that the bijection $(\mathrm{Spec} R)_f \to \mathrm{Spec}(R_f)$ is a homeomorphism. □

15.3. **Preschemes and morphisms of preschemes.**

Definition 15.4. *An* **affine scheme** *is a ringed space isomoprhic to* $(\mathrm{Spec} R, \mathcal{O}_{\mathrm{Spec} R})$ *for some ring* R.

A **prescheme** *is a ringed space* $(X, \mathcal{O}_X)$ *such that there exists an open cover* $X = \bigcup X_i$ *such that* $(X_i, (\mathcal{O}_X)|_{X_i})$ *is isomorphic to an affine scheme.*

Note that if X is a prescheme and $U \subset X$ is open then $(U, (\mathcal{O}_X)|_U)$ is also a prescheme. Indeed, cover X by affine schemes X_i. Then $U = \bigcup (U \cap X_i)$ is an open cover, and each $U \cap X_i$ is open in the affine scheme X_i. By Proposition 15.13 it follows that $U \cap X_i$ is covered by some distinguished open sets in X_i, therefore $U \cap X_i$ is a prescheme, and so is U.

Morphisms of preschemes are the same as morphisms of locally ringed spaces. To define these we recall few facts from commutative algebra.

A ring $(R, \mathfrak{m})$ is **local** if it contains a unique maximal ideal $\mathfrak{m}$. The most important example for us is $R_{\mathfrak{p}}$, where $\mathfrak{p} \subset R$ is a prime ideal. A homomorphism of local rings $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ is called a **local ring map** if

(a) $\varphi(\mathfrak{m}) \subset \mathfrak{n}$, or, equivalently,

(b) $\varphi^{-1}(\mathfrak{n}) = \mathfrak{m}$.

(The nontrivial implication $(a) \Rightarrow (b)$ uses that $\varphi^{-1}(\mathfrak{n})$ is a prime ideal containing the maximal ideal $\mathfrak{m}$, thus they must be equal.)

Example 15.15. *Let $\varphi : R \to S$ be a ring homomorphism. Let $\mathfrak{q} \subset S$ be a prime ideal and $\mathfrak{p} := \varphi^{-1}(\mathfrak{q})$. Then the induced map $\varphi_{\mathfrak{p}} : R_{\mathfrak{p}} \to S_{\mathfrak{q}}$ is a local ring map.*

Furthermore, in this case φ also induces a map $k(\mathfrak{p}) \to k(\mathfrak{q})$ between the residue fields.

Definition 15.5. *Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two preschemes. Then a morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of preschemes is a morphism of locally ringed spaces, i.e.:*

- *A continuous map $f : X \to Y$;*
- *A collection of ring homomorphisms for each $U \subset Y$ open*

$$f_U^* : \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U));$$

This data satisfies the following conditions:

- *they are compatible with restrictions, i.e., for any $V \subset U$, the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{O}_Y(U) & \xrightarrow{f_U^*} & \mathcal{O}_X(f^{-1}(U)) \\ {\scriptstyle res_{f^{-1}(U)f^{-1}(V)}}\big\downarrow & & {\scriptstyle res_{UV}}\big\downarrow \\ \mathcal{O}_Y(V) & \xrightarrow{f_V^*} & \mathcal{O}_X(f^{-1}(V)) \end{array}$$

- *Let $f(\mathfrak{p}) = \mathfrak{q}$. By taking limits, the homomorphims f_U^* induce a homomorphism*

$$f_{\mathfrak{p}}^* : \varinjlim_{\mathfrak{q} \in U} \mathcal{O}_Y(U) = \mathcal{O}_{Y,\mathfrak{q}} \to \varinjlim_{\mathfrak{p} \in f^{-1}(U)} \mathcal{O}_X(f^{-1}(U)) \to \mathcal{O}_{X,\mathfrak{p}}.$$

  We require that this is a **local ring map**, *i.e.,*

$$(f_{\mathfrak{p}}^*)^{-1}(\mathfrak{p}\mathcal{O}_{X,\mathfrak{p}}) = \mathfrak{q}\mathcal{O}_{X,\mathfrak{q}} \Longleftrightarrow (f_{\mathfrak{p}})_*(\mathfrak{q}\mathcal{O}_{X,\mathfrak{q}}) \subset \mathfrak{p}\mathcal{O}_{X,\mathfrak{p}}.$$

The definition of morphisms is slightly different from the one for algebraic varieties from the previous semester. A key distinction is that if X is an algebraic variety, the structure sheaf consisted of rational functions $a : U \to k$, where k is a fixed field. For a continuous map $f : X \to Y$ and $U \subset Y$ open, there is a diagram

$$\begin{tikzcd} f^{-1}(U) \arrow[r, "f"] \arrow[dr, dashed, "f^*(a)=a\circ f"'] & U \arrow[d, "a"] \\ & k \end{tikzcd}$$

Therefore one can easily pull back rational functions. However, in the case of preschemes, the pull backs are now part of the data.

Proposition 15.16. *For any rings R, S there is a bijection*

$$\textit{Scheme morphisms } f : (\mathrm{Spec} R, \mathcal{O}_{\mathrm{Spec} R}) \to (\mathrm{Spec} S, \mathcal{O}_{\mathrm{Spec} S}) \longleftrightarrow$$
$$\textit{ring homomorphisms } f^* : S \to R$$

This gives a contravariant equivalence of categories

$$\mathsf{Sch} \to \mathsf{Rings}$$

between the category of affine schemes and the category of (commutative) rings.

Proof. If $f : (\mathrm{Spec} R, \mathcal{O}_{\mathrm{Spec} R}) \to (\mathrm{Spec} S, \mathcal{O}_{\mathrm{Spec} S})$ is a morphism of affine schemes then by definition and Theorem 15.1 there is a ring homomorphism $f^* : \mathcal{O}_{\mathrm{Spec} S}(\mathrm{Spec} S) \simeq S \to \mathcal{O}_{\mathrm{Spec} R}(\mathrm{Spec} R) \simeq R$.

Conversely, assume that we are given a ring homomorphism $\varphi : S \to R$. Then for any prime ideal $\mathfrak{p} \in R$, $\mathfrak{q} := \varphi^{-1}(\mathfrak{p})$ is a prime ideal in S. (Check!) The map φ induces a map $f : \mathrm{Spec} R \to \mathrm{Spec} S$ defined by $\mathfrak{p} \to \varphi^{-1}(\mathfrak{p})$. This map is continuous because

$$f^{-1}(V(J)) = \{\mathfrak{p} : \varphi^{-1}(\mathfrak{p}) \supset J\} = \{\mathfrak{p} : \mathfrak{p} \supset \varphi(J)\} = V(\langle \varphi(J) \rangle).$$

For $\mathfrak{q} = \varphi^{-1}(\mathfrak{p})$, φ also induces a local ring map $\varphi_{\mathfrak{p}} : S_{\mathfrak{q}} \to R_{\mathfrak{p}}$. The sections over open sets $U \subset \mathrm{Spec} S$ and $V \subset \mathrm{Spec} R$ are tuples in the local rings $R_{\mathfrak{p}}$ and $S_{\mathfrak{q}}$, and one may check that the homomorphisms $\varphi_{\mathfrak{p}}$ glue to give the required ring homomorphisms f^*_U.

Finally, one needs to check that the two constructions are inverse to each other - this is left as an exercise. $\square$

Example 15.17. *All morphisms of varieties from the previous semester are morphisms of locally ringed spaces.* **Homework!**

Example 15.18. *Let $R = k[x]_{\langle x-a \rangle}$, i.e the local ring at a closed point in $\mathbb{A}^1_k$. (This is an example of a discrete valuation ring; cf.* [AM69, Ch. 9]*. More generally, R could the local ring at a closed point of a nonsingular curve - in this case the maximal ideal is principal).*

Then $\mathrm{Spec}(R) = \{0, \mathfrak{m}\}$, $\mathfrak{m}$ is closed, and 0 is open and dense in $\mathrm{Spec}(R)$. Let $K = R_{(0)}$ be the fraction field of R. We wish to define a morphism $f : \mathrm{Spec}(K) \to \mathrm{Spec}(R)$:

- *At the level of points, send $f(0) = 0$. This is continuous, as $(0) \in \mathrm{Spec} R$ is open.*
- *We need to define the ring homomorphisms $f^*_U : \mathcal{O}_{\mathrm{Spec}(R)}(U) \to \mathcal{O}_{\mathrm{Spec}(K)}(f^{-1}(U))$ at the level of rings of sections. Observe that there are two open sets:*
  - *$U = \mathrm{Spec}(R)$. Then $\mathcal{O}_{\mathrm{Spec}(R)}(U) = R$ and define*
  $$f^*_{\mathrm{Spec} R} = R \to K; \quad r \mapsto r.$$
  - *$U = (0)$. Then $U = D_r$ is a distinguished open set for any $0 \neq r \in \mathfrak{m}$. In this case*
  $$f^*_{D_r} = R_r \to K; \quad r \mapsto r.$$

  In other words, this is induced by the natural inclusion $R \hookrightarrow K$.

At the level of stalks, we need to check that the map

$$\mathcal{O}_{\mathrm{Spec}(R),0} = R_{\langle 0 \rangle} \to \mathcal{O}_{\mathrm{Spec}(K),0} = K$$

is a local ring map. Observe that since $\mathfrak{m}$ is a principal ideal $R_{(0)} = R_r$ for any $0 \neq r \subset \mathfrak{m}$ (since any element outside $\mathfrak{m}$ is already invertible). Then the map $R_{(0)} \to K$ is the identity map, it sends $0 \mapsto 0$ and the claim follows: this is a morphism of locally ringed spaces.

We may also define the map $f : \mathrm{Spec}(K) \to \mathrm{Spec}(R)$ by requiring that $f(0) = \mathfrak{m}$ and given by the inclusion $R \hookrightarrow K$ at the level of sheaves. This is a morphism of ringed spaces, but **not of locally ringed spaces**. *(Since the inclusion $R_{\mathfrak{m}} \to K$ is not a local ring map.)*

One may also prove a more general version of Proposition 15.16 (cf. [Mum99, §2, Thm.1]:

Theorem 15.2. *Let X be any prescheme, and R any ring. Then there is a bijection*

$$Hom_{Sch}(X, \mathrm{Spec}(R)) \longleftrightarrow Hom_{Rings}(R, \mathcal{O}_X(X)),$$

which associates to any $f : X \to \mathrm{Spec}(R)$ the ring homomorphism $f^* : R \simeq \mathcal{O}_{\mathrm{Spec} R}(\mathrm{Spec}(R)) \to \mathcal{O}_X(X)$.

Applying this theorem for $R = \mathcal{O}_X(X)$ and the identity map $R \to R$, we obtain the following corollary:

Corollary 15.19. *(a) There exists a canonical morphism $X \to \mathrm{Spec}(\mathcal{O}_X(X))$.*

(b) Let S be any ring and fix a morphism $f : X \to \mathrm{Spec}(S)$. This morphism factors through the previous morphism:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & \mathrm{Spec}(S) \\ & \mathrm{Spec}(\mathcal{O}_X(X)) \arrow[ur] & \end{tikzcd}$$

Remark 15.4. *In some situations $\mathcal{O}_X(X)$ is very small. E.g., if X is a projective k-variety, then $\mathcal{O}_X(X) = k$ (This will be shown later.) Therefore it follows that any morphism from a projective scheme to an affine scheme must be constant!*

Corollary 15.20. $\mathrm{Spec}(\mathbb{Z})$ *is a final object in the category of preschemes, i.e. for any prescheme X there is a unique morphism $X \to \mathrm{Spec}(\mathbb{Z})$.*

Corollary 15.21. *Let X be a prescheme and let $x \in X$ be an arbitrary point with residue field $k(x)$. Then there exists a canonical morphism $i_x : \mathrm{Spec}(k(x)) \to X$ sending $0 \mapsto x$.*

Proof. Take $U = \mathrm{Spec}(R)$ to be an open affine neighborhood of x. There is a natural inclusion $U \hookrightarrow X$, and using this we may assume $X = \mathrm{Spec}(R)$ is affine. Let $\mathfrak{p}_x \subset R$ be the prime ideal corresponding to x. Then the morphism i_x is given by the natural map $R \to R_{\mathfrak{p}_x} \to k(x)$. $\square$

To relate preschemes to the algebraic varieties from the previous semester we give the more general definition:

Definition 15.6. *Let R be a ring. Then a prescheme X over R is a morphism $\pi_X : X \to \operatorname{Spec}(R)$. A morphism $f : X \to Y$ is a morphism of preschemes over R if there is a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\pi_X"'] & & Y \arrow[dl, "\pi_Y"] \\ & \operatorname{Spec}(R) & \end{tikzcd}$$

There is a category of preschemes and morphisms over R. There is an analogue of Proposition 15.16 which establishes an equivalence between the category of affine R-preschemes and morphisms to ring homomorphisms of R-algebras. There is a further generalization of Theorem 15.2 to the context of R-preschemes. The most important situation is when $R = k$ and the sheaves of sections satisfy a finite generation condition. In this case, the schemes over k correspond to the algebraic varieties we studied in the prior semester.

We also record the following useful lemma:

Lemma 15.22. *Let X be a prescheme and $Z \subset X$ a non-empty closed irreducible subset. Then there exists a unique point $z \in Z$ such that $\overline{\{z\}} = Z$. (This is called the* **generic point** *of Z.)*

Proof. Let U be any affine open scheme which intersects Z. Then $Z \cap U$ must be dense in Z, and any dense point in Z must be in $Z \cap U$. Observe that the latter is closed and irreducible in the affine scheme U. By Proposition 15.6, there exists a unique point dense in $Z \cap U$ (the generic point), and this finishes the proof. $\square$

Definition 15.7. *Let R be a ring. The n-dimensional* **affine space** *over R is the affine scheme*

$$\mathbb{A}^n_R = \operatorname{Spec}(R[x_1, \ldots, x_n]).$$

16. Products of preschemes

Our next goal is to define schemes; these will be separated preschemes, i.e. preschemes X such that the image of the diagonal morphism $X \to X \times X$ is closed. For this, we first need to discuss products of preschemes.

We adapt the notation and defintions from §9.1 above to the category of preschemes. If $f : X \to Z$ and $g : Y \to Z$ are morphisms of preschemes, a product is a prescheme $X \times_Z Y$ equipped with two morphisms $p_X : X \times_Z Y \to X$ and $p_Y : X \times_Z Y \to Y$ such the following universality property is satisfied:

- For any prescheme W and morphisms $p_X : W \to X$ and $p_Y : W \to Y$ so that $f \circ p_X = g \circ p_Y$ there exists a unique morphism $p : W \to X \times_Z Y$ such that $\pi_X \circ p = p_X$ and $\pi_Y \circ p = p_Y$.

$$\begin{tikzcd}
W \arrow[dr, dotted, "p"] \arrow[drr, bend left, "p_X"] \arrow[ddr, bend right, "p_Y"'] & & \\
& X \times_Z Y \arrow[r, "\pi_X"'] \arrow[d, "\pi_Y"] & X \arrow[d, "f"'] \\
& Y \arrow[r, "g"] & Z
\end{tikzcd}$$

The product $X \times_S Y$, if it exists, is unique up to an isomorphism commuting with the projections. The following is the main result of this section:

Theorem 16.1. *(a) Let R, S be T-algebras. Then* $\mathrm{Spec} R \times_{\mathrm{Spec} T} \mathrm{Spec}(S)$*, together with natural projections, exists in the category of T-preschemes and it is isomorphic to* $\mathrm{Spec}(R \otimes_T S)$.

(b) Given any preschemes $p_X : X \to Z$ and $p_Y : Y \to Z$, a fibre product $X \times_Z Y$ exists.

Proof. In the category of *affine* schemes, part (a) follows from the universality property of tensor products (see Example 9.2), together with the correspondence between rings and affine varieties from Proposition 15.16. The proof of (a) in the larger category of preschemes follows by reducing to the affine case, utilizing that any morphism $X \to \mathrm{Spec}(R \otimes_T S)$ factors through $\mathrm{Spec}(\mathcal{O}_X(X)) \to \mathrm{Spec}(R \otimes_T S)$; see Corollary 15.19.

Part (b) follows by covering X and Y by affine schemes, taking fibre products of these, then proving that they 'glue' to a prescheme $X \times_Z Y$. For details please consult e.g., [Har77, Thm. 2.3.3] or [Mum99, Thm. II.2.3(b)]. ☐

16.1. **Examples of fibre products.** Some important situations of fibre products are the following.

16.1.1. *Fibres of morphisms.*

Definition 16.1. *Let $f : X \to Y$ be a morphism of preschemes, and a (possibly not closed) point $y \in Y$ with residue field $k(y)$ and canonical morphism $i_y : \mathrm{Spec}(k(y)) \to Y$.*

The **fibre** $f^{-1}(y)$ *is the prescheme* $k(y) \times_Y X$ *defined by the fibre diagram below:*

$$\begin{tikzcd} f^{-1}(y) \simeq k(y) \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ k(y) \arrow[r, "i_y"] & Y \end{tikzcd}$$

This is naturally a prescheme over the residue field of y*.*

Example 16.1. *Let* $f : \mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$ *defined by* $f(x) = x^2$*. Let* $a \in \mathbb{A}^1_{\mathbb{C}}$ *be a closed point, corresponding to the maximal ideal* $\mathfrak{m}_a = \langle t - a \rangle \subset \mathbb{C}[t]$*. The residue field is* $k(a) = \mathbb{C}[x]/\mathfrak{m}_a \simeq \mathbb{C}$*. Note that* f *is induced by the* $\mathbb{C}$*-algebra homomorphism* $f^* : \mathbb{C}[t] \to \mathbb{C}[x]$ *defined by* $t \mapsto x^2$*.*

Then

$$f^{-1}(a) \simeq \operatorname{Spec}(\mathbb{C}[x] \otimes_{\mathbb{C}[t]} \mathbb{C}[t]/\mathfrak{m}_a) = \operatorname{Spec}(\mathbb{C}[x]/\langle x^2 - a \rangle)$$

The prescheme $f^{-1}(a)$ *consists of two reduced points if* $a \neq 0$*; if* $a = 0$ *then* $\mathbb{C}[x]/\langle x^2 \rangle$ *is non-reduced, and the fibre* $f^{-1}(0)$ *is a single non-reduced point.*

Consider now the generic point $0 \in \operatorname{Spec}(\mathbb{C}[t])$*. The residue field is* $\mathbb{C}[t]_{(0)} = \mathbb{C}(t)$ *(the fraction field of* $\mathbb{C}[t]$*). Then*

$$f^{-1}(\langle 0 \rangle) = \operatorname{Spec}(\mathbb{C}[x] \otimes_{\mathbb{C}[t]} \mathbb{C}(t)) = \operatorname{Spec}(\mathbb{C}[x]_{\mathbb{C}[x^2]\setminus 0}).$$

(This uses that for a multiplicative set S *in a ring* R *and an* R*-module* M*, we have* $M \otimes_R (S^{-1}R) \simeq S^{-1}M$ *as* $S^{-1}R$*-modules; see* [AM69, Prop. 3.5]*.) Note that* $\mathbb{C}[x]_{\mathbb{C}[x^2]\setminus 0}$ *is a* $\mathbb{C}(t)$*-vector space of dimension* 2*, with basis* $1, x$*.*

16.1.2. *Base change.* More generally, consider the fibre diagram

$$\begin{tikzcd} Y' \times_Y X \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

This is called a **base change**, or **base extension** (from Y to Y'). It is the natural algebraic analogue of the pull-back of a vector bundle from topology. It also leads to the notion of extending the field of a definition. For example, for any ring R,

$$\mathbb{A}^n_R = \mathbb{A}^n_{\mathbb{Z}} \times_{\operatorname{Spec}(\mathbb{Z})} \operatorname{Spec}(R).$$

This follows because $R \otimes_{\mathbb{Z}} \mathbb{Z}[x_1, \dots, x_n] = R[x_1, \dots, x_n]$. In other words we're base changing from coefficients in $\mathbb{Z}$ to coefficients in an arbitrary ring.

One important property of the base extension is that f, f' have the same fibres, and have the same geometric properties (preserves flatness, properness, etc.)

Remark 16.1. *The fibre product* $X \times_Z Y$ *depends on the base prescheme* Z*, and in general there is no* injective *map to the set theoretic product. For instance,* $\operatorname{Spec}(\mathbb{C} \times_{\mathbb{C}}$

$\mathbb{C}) = \mathrm{Spec}(\mathbb{C})$ *is a point. However,*

$$\mathrm{Spec}(\mathbb{C} \times \mathbb{C}) = \mathrm{Spec}(\mathbb{C} \times_{\mathbb{R}} \mathbb{C}) = \mathrm{Spec}(\mathbb{C}) \bigsqcup \mathrm{Spec}(\mathbb{C})$$

is a disjoint union of two points. (The two points are given by the prime ideals generated by $(1,0)$ *and* $(0,1)$*; note also the idempotent* $\frac{1\otimes 1 + i \otimes i}{2}$ *in the tensor product, showing that* $\mathrm{Spec}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})$ *must be disconnected.) Thus the base extension may not preserve connectedness.*

However, the set theoretic and fibre product will coincide for the reduced, irreducible preschemes of finite type over algebraically closed fields. (See §16.2 below or [Mum99, Ch. II, §3, Prop. 5]*.)*

16.1.3. *The diagonal map and intersections.* Assume we are given a morphism $f : X \to Y$. The **diagonal** $\Delta : X \to X \times_Y X$ is the unique morphism given by the universality property of fibre products:

(16.1)
$$\begin{tikzcd}
X \arrow[dr, dotted, "\Delta"] \arrow[drr, bend left, "id"] \arrow[ddr, bend right, "id"'] & & \\
& X \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}$$

Now take $Y = \mathrm{Spec}(k)$ (k a field) and consider $X_1, X_2 \subset X$ closed sub(pre)schemes with inclusions $i_1 : X_1 \hookrightarrow X$ and $i_2 : X_2 \hookrightarrow X$. (Technically, we did not talk about how these inclusions are defined, but we have some expectations from the 1st semester; this will be done later.) The **intersection** $X_1 \cap X_2$ is the prescheme given by the fibre product:

$$\begin{tikzcd}
X_1 \cap X_2 \arrow[r] \arrow[d] & X_1 \times_{\mathrm{Spec}(k)} X_2 \arrow[d, "i_1 \times i_2"] \\
X \arrow[r, "\Delta"] & X \times X
\end{tikzcd}$$

Equivalently, one may also define it from the fibre diagram

$$\begin{tikzcd}
X_1 \cap X_2 \arrow[r] \arrow[d] & X_1 \arrow[d, "i_1"] \\
X_2 \arrow[r, "i_2"] & X
\end{tikzcd}$$

Of course, the intersection $X_1 \cap X_2$ may not have good properties (e.g. expected dimension, singularities etc.)

16.2. **Preschemes and algebraic varieties.**

Definition 16.2. *A prescheme X over R is of* **finite type over** *R if X is quasi-compact and for any $U \subset X$ open, affine, $\mathcal{O}_X(U)$ is a finitely generated R-algebra.*

Warning: In general, this does not imply that $\mathcal{O}_X(X)$ is a finitely generated R-algebra.

Definition 16.3. *A prescheme X if* **reduced** *if for any $U \subset X$ open, $\mathcal{O}_X(U)$ has no nilpotent sections. Equivalently (check!) the stalks $\mathcal{O}_{X,\mathfrak{p}}$ have no nilpotents, or there exists a cover $X = \bigcup U_i$ with $U_i \simeq \mathrm{Spec}(R_I)$ open affine and R_i are reduced rings.*

The notion of a *variety* in the following theorem is the same as the one from Definition 6.2. In particular, at this time we do not require that an algebraic variety is separated.

Theorem 16.2. *Let k be an algebraically closed field. Then there is an equivalence of categories between:*

(a) Category of reduced, irreducible, preschemes of finite type over k, and k-morphisms;

(b) Category of irreducible algebraic varieties over k and morphisms of these.

Furthermore, if X, Y are preschemes over k as above, then the fibre product $X \times_{\mathrm{Spec}(k)} Y$ in the category of preschemes coincides with the product $X \times Y$ in the category of algebraic varieties.

Rough idea of proof - see [Har77, Prop. II.2.6] . One needs to define functors in both directions.

$(a) \Rightarrow (b)$: If $\mathcal{X}$ is a prescheme, take X to be the set of closed points of $\mathcal{X}$. Then one may prove that if $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of k-varieties, it takes closed points to closed points. (**Homework!**) Therefore it will also define a map $X \to Y$, which one may check it will be a morphism of varieties.

$(b) \Rightarrow (a)$ Take X be a variety. We need to define the points of a scheme $\mathcal{X}$. We take $\mathcal{X}$ to be the points in X union the symbols $[Z]$ for any $Z \subset X$ which is closed and irreducible. If $f : X \to Y$ is a morphism, then we send $x \in X$ to $f(x) \in Y$, and $[Z]$ to $[\overline{f(Z)}]$. Again one checks that this satisfies the required properties. $\square$

17. Proj(R) and projective schemes

17.1. Proj(R) and its structure sheaf.

Definition 17.1. *Let $R = \bigoplus_{d\geq 0} R^d$ be a graded ring, and denote by $R_+ = \bigoplus_{d>0} R^d$ the ideal of elements of positive degree. We define by*

$$\mathrm{Proj}(R) = \{\mathfrak{p} \subset \mathrm{Spec}(R) : \mathfrak{p} \text{ is homogeneous and } R_+ \not\subseteq \mathfrak{p}\}.$$

For $I \subset R$ homogeneous ideal, define

$$V_p(I) := \{\mathfrak{p} \in \mathrm{Proj}(R) : \mathfrak{p} \supset I\}.$$

The next lemma is the projective analogue of Proposition 15.2 with almost the same proof.

Lemma 17.1. *The following hold for any homogeneous ideals $I, J, I_s \subset \mathrm{Proj}(R)$:*

- $V_p(\bigcup_s I_s) = \bigcap_s V_p(I_s)$;
- $V_p(I \cap J) = V_p(I) \cup V_p(J)$.

In particular, the sets $V_p(I)$ form the closed sets on a topology of Proj(R), called again the Zariski topology. Next we define a sheaf of functions on Proj(R), by adapting the definition of $\mathcal{O}_{\mathrm{Spec}(R)}$ to the projective situation, in analogy to the definition of the structure sheaves on projective spaces from the first semester. First, for the graded ring $R = \oplus R_d$ and $\mathfrak{p} \in \mathrm{Proj}(R)$, define

$$R_{(\mathfrak{p})} = \{\frac{f}{g} : f, g \in R^d \text{ some } d, g \notin \mathfrak{p}\} \subset R_{\mathfrak{p}}.$$

This is a subring of $R_{\mathfrak{p}}$. Next, let $U \subset \mathrm{Proj}(R)$ be an open set. Define $\mathcal{O}_X(U)$ by

$$\mathcal{O}_X(U) = \{s = (s_{\mathfrak{p}})_{\mathfrak{p}\in U} \in \prod_{\mathfrak{p}\in U} R_{(\mathfrak{p})} :$$

$$\text{s. t. } s \text{ is locally given by } \frac{f}{g}, f, g \in R^d\}$$

In other words, for any $\mathfrak{p} \in U$ there exists $V \subset U$ open and $f, g \in R^d$ such that $s_{\mathfrak{q}} = \frac{f}{g} \in R_{(\mathfrak{q})}$ for all $\mathfrak{q} \in V$.

Proposition 17.2. *(a) $\mathcal{O}_{\mathrm{Proj}(R)}$ is a sheaf.*

(b) For any $\mathfrak{p} \in \mathrm{Proj}(R)$, the stalk $\mathcal{O}_{\mathrm{Proj}(R),\mathfrak{p}} = R_{(\mathfrak{p})}$.

(c) Let $f \in R_+ = \oplus_{d>0} R^d$ be a homogeneous element and consider the open sets

$$D_f^+ = \mathrm{Proj}(R) \setminus V_p(f) = \{\mathfrak{p} \in \mathrm{Proj}(R) : f \notin \mathfrak{p}\}.$$

Then the following hold:

- *The sets D_f^+ cover Proj(R);*
- *Each D_f^+ is an affine scheme. More precisely, $D_f^+ \simeq \mathrm{Spec} R_{(f)}$, where*

$$R_{(f)} = \{\frac{a}{f^n} : a \text{ is homogeneous and } \deg a = n \deg f\} \subset R_{\mathfrak{p}}.$$

In particular, $\mathrm{Proj}(R)$ *is a prescheme.*

Proof. **Homework!** $\square$

For any graded ring R, we denote by $\mathbb{P}^n_R = \mathrm{Proj}(R[x_0, \ldots, x_n])$.

17.2. **Projective subschemes.** From now on we will take $R = k[x_0, \ldots, x_n]$, for k algebraically closed. As in the case of projective algebraic varieties (from the first semester), we would like to have a correspondence between the closed subschemes of $\mathbb{P}^n$ and ideals of R.

Definition 17.2. *Let I be a homogeneous ideal in $k[x_0, \ldots, x_n]$. The* **saturation** *of I is*

$$\overline{I} = \{P \in k[x_0, \ldots, x_n] : \exists m \text{ s.t. } x_i^m P \in I \ \forall 0 \leq i \leq n\}.$$

We say that I is **saturated** *if $I = \overline{I}$.*

Example 17.3. $\overline{\langle fx_0, fx_1, \ldots, fx_n\rangle} = \langle f\rangle$*. In other words, the saturation removes the part coming from the 'irrelevant ideal' $\langle x_0, \ldots, x_n\rangle$.*

Proposition 17.4. *Let $I, J \subset k[x_0, \ldots, x_n]$ be homogeneous ideals. Then the following hold:*

(a) The saturation $\overline{I}$ is a homogeneous ideal;

(b) $\mathrm{Proj}(k[x_0, \ldots, x_n]/I) = \mathrm{Proj}(k[x_0, \ldots, x_n]/\overline{I})$*;*

(c) $\mathrm{Proj}(k[x_0, \ldots, x_n]/I) = \mathrm{Proj}(k[x_0, \ldots, x_n]/J)$ *if and only if $\overline{I} = \overline{J}$.*

(d) $I^d = \overline{I}^d$ for $d \gg 0$, i.e., an ideal and its saturation are the same for degree d large enough.

Proof. To prove (a), let $P \in \overline{I}$. Then $x_i^m P \in I$ for some m and each $0 \leq i \leq n$. Then the graded pieces

$$(x_i^m P)^{(d)} = x_i^m P^{(d-m)} \in I,$$

therefore by definition $P^{(d-m)} \in I$.

For (b), the inclusion $\mathrm{Proj}(k[x_0, \ldots, x_n]/I) \supset \mathrm{Proj}(k[x_0, \ldots, x_n]/\overline{I})$ is obvious from definition. Let $\mathfrak{p} \in \mathrm{Proj}(k[x_0, \ldots, x_n]/I)$. Its preimage is a prime ideal $\mathfrak{q} \subset k[x_0, \ldots, x_n]$ which contains I. If $\mathfrak{q}$ does not contain $\overline{I}$ then there exists $P \in \overline{I} \setminus \mathfrak{q}$. By definition, there exists a positive integer m such that $x_i^m P \in I \subset \mathfrak{q}$. Since $P \notin \mathfrak{q}$ it follows that $\langle x_0, \ldots, x_n\rangle \subset \mathfrak{q}$, which is a contradiction.

To prove (c), we need to recover the ideal I from $\mathrm{Proj}(k[x_0, \ldots, x_n]/I)$. For this, consider the cover by the open affine sets $D^+_{x_j}$. By Proposition 17.2,

$$D^+_{x_j} \cap \mathrm{Proj}(k[x_0, \ldots, x_n]/I) = \mathrm{Spec}(k[\frac{x_0}{x_j}, \frac{x_1}{x_j}, \ldots, \frac{x_n}{x_j}]/I_j),$$

where I_i is an ideal. One may recover I_j as the kernel of the morphism $k[\frac{x_0}{x_j}, \frac{x_1}{x_j}, \ldots, \frac{x_n}{x_j}] \to k[\frac{x_0}{x_j}, \frac{x_1}{x_j}, \ldots, \frac{x_n}{x_j}]/I_j$. Then (**homework!**),

$$I = \langle\{P \in k[x_0, \ldots, x_n] \text{ homogeneous } : \frac{P}{x_j^{\deg P}} \in I_j, 0 \leq j \leq n\}\rangle.$$

For (d), it suffices to show that $\overline{I}^d \subset I^d$ for $d \gg 0$. Since $\overline{I}$ is finitely generated, take $P_1, \ldots, P_s$ be a set of homogeneous generators. Take D_1 large enough such that $x_i^d P_j \in I$ for $d \geq D_1$ and all $0 \leq i \leq n$, $1 \leq j \leq s$. Take D_2 the maximum degrees of P_j. Now let $d \geq D_2 + (n+1)D_1$ and let P be any element in $\overline{I}^d$, and write

$$P = \sum a_j P_j$$

for some homogeneous a_j. Then $\deg a_j \geq d - D_2 = (n+1)D_1$. This shows that each term of a_j contains a factor which is a multiple of $x_i^{D_1}$ for some i. Then $a_j P_j \in I$ and we are done. □

Obviously $\mathrm{Proj}(k[x_0, \ldots, x_n]/I) = V_p(I)$ is closed subset of $\mathbb{P}^n_k$; this is called a **projective subscheme**. (We will see in the next section the precise meaning of the word 'subscheme'; roughly, this means that we have a locally ringed space with an induced topology, and surjections at the level of stalks, induced from stalks of the structure sheaf of the ambient space.)

Corollary 17.5. *There is a one-to-one correspondence between projective subschemes of $\mathbb{P}^n$ and saturated ideals in $k[x_0, \ldots, x_n]$. The correspondence associates the saturated homogeneous ideal I to* $\mathrm{Proj}(k[x_, \ldots, x_n]/I)$.

18. More properties of schemes

In this section we gather several results we'll need later. Most of these are not proved in these notes; a good reference is [Har77, Ch. 3 , Ch. 4].

18.1. Integral schemes; open and closed embeddings.

We start with two definitions:

Definition 18.1. *Let X be a prescheme.*

- *X is* **connected** *if X is connected as a topological space.*
- *X is* **integral** *if for any open set $U \subset X$, $\mathcal{O}_X(U)$ is an integral domain.*

Example 18.1. *Let $X = \mathrm{Spec}(R)$ be an affine scheme. Then (***homework!***):*

- *X is reduced iff $Nil(R) = 0$;*
- *X is irreducible iff $Nil(R)$ is prime.*
- *X is integral iff R is an integral domain.*

More generally, one may prove the following:

Proposition 18.2. *A prescheme is integral if and only if it is irreducible and reduced.*

Definition 18.2. *A prescheme X is* **locally Noetherian** *if it can be covered by affine open sets $\mathrm{Spec}(R_i)$ where R_i is a Noetherian ring; it is* **Noetherian** *if it is locally Noetherian and quasicompact. (Equivalently, X is Noetherian if it may be covered by finitely many open affines $\mathrm{Spec}(R_i)$ with R_i Noetherian.*

Remark 18.1. *One can show that if $\mathrm{Spec}(R)$ is a Noetherian prescheme then $\mathrm{Spec}(R)$ is Noetherian as a topological space (i.e., every descending chain of closed subsets stabilizes). However, the converse is not true. For example, take*

$$R = k[x_1, x_2, \dots]/\langle x_1^2, x_2^2 \dots, \rangle.$$

The only prime ideal is the (maximal) ideal $\mathfrak{m} = \langle x_1, x_2, \dots, \rangle$. Therefore $\mathrm{Spec}(R)$ is a Noetherian topological space. But R is not Noetherian, since $\mathfrak{m}$ is not finitely generated.

The following shows that the property of being (locally) Noetherian is independent of the choice of the cover.

Proposition 18.3. *A prescheme X is locally Noetherian if and only if for every open affine $\mathrm{Spec}(R_i) \subset X$, the ring R_i is Noetherian. In particular $\mathrm{Spec}(R)$ is Noetherian if and only if R is Noetherian.*

Proof. See [Har77, Prop. II.3.2]. □

Definition 18.3. *An* **open sub(pre)scheme** *of a prescheme X is an open prescheme $U \subset X$ equipped with the restriction structure sheaf $(\mathcal{O}_X)_{|U}$. An* **open embedding** *if a morphism $i : Y \to X$ such that $i(Y) \subset X$ is open and the restriction $Y \to i(Y)$ is an isomorphism.*

Definition 18.4. *A* **closed embedding** *is a morphism of preschemes $f : Y \to X$ such that $f(Y)$ is a closed subspace of X, f induces a homeomorphism $Y \to f(Y)$, and the induced map $f^* : \mathcal{O}_X \to f_*(\mathcal{O}_Y)$ of sheaves is surjective. (Equivalently, if $f(\mathfrak{p}) = \mathfrak{q}$, then the induced map of stalks $f^*_{\mathfrak{p}} : \mathcal{O}_{X,\mathfrak{q}} \to \mathcal{O}_{Y,\mathfrak{p}}$ is surjective.)*

A **closed subprescheme** *of a prescheme X is an equivalence class of closed embeddings, where $f : Y \to X$ and $f' : Y' \to X$ are equivalent if there is a diagram*

$$\begin{tikzcd} Y' \arrow[rr, "i"] \arrow[dr, "f'"'] & & Y \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

where $i : Y' \to Y$ is an isomorphism.

Example 18.4. *(Closed subschemes of affine schemes.) Let $X = \mathrm{Spec}(R)$ and $I \subset R$ be an ideal. The projection $i^* : R \to R/I$ induces an injective morphism $i : \mathrm{Spec}(R/I) \to \mathrm{Spec}(R)$. This is a closed sub(pre)scheme, because the map i^* induces surjective maps after localization.*

In particular, observe that for any closed subset $V(J) \subset \mathrm{Spec}(R)$ there are many closed subscheme structures *corresponding to ideals I such that $V(I) = V(J)$. The smallest such subscheme corresponds to the largest such ideal, which is necessarily $\sqrt{J}$. This gives $V(\sqrt{J})$ the* **reduced (induced) scheme structure***.*

Definition 18.5 (Dimension)**.** *If X is a prescheme, $\dim X$ is the dimension when regarded as a (locally) ringed space. In other words, if X is irreducible, then $\dim X = n$ if there is a maximal chain of irreducible closed sets*

$$\emptyset \subsetneq X_0 \subsetneq X_1 \subsetneq \ldots \subsetneq X_n = X.$$

If X is not irreducible, then $\dim X$ is the supremum of dimensions of the irreducible components of X.

Example 18.5. *If $\mathrm{Spec}(R)$ is irreducible, then $\dim \mathrm{Spec}(R) = \dim R$ (the Krull dimension of R).*

18.2. **Separated morphisms; schemes.**

Definition 18.6. *Let $f : X \to Y$ be a morphism of preschemes. The* **diagonal morphism** *is the unique morphism $\Delta_f : X \to X \times_Y X$ induced by universality property of the fibre product $X \times_Y X$; see Eq.* (16.1).

The morphism f is **separated** *if Δ_f is a closed embedding. In this case X is said to be* **separated** *over Y. A prescheme is called a* **scheme** *if it is separated over* $\mathrm{Spec}(\mathbb{Z})$.

Remark 18.2. *Most often we will work with schemes separated over a field k. It follows from Theorem 18.2 below that if X is separated, then X is separated over k.*

Example 18.6. *A prescheme which is not separated is the affine line with two origins; see §9.3 above.*

Proposition 18.7. *Any morphism of affine schemes $f : \operatorname{Spec}(S) \to \operatorname{Spec}(R)$ is separated. In particular, any affine scheme is separated.*

Proof. By Proposition 15.16, the diagonal Δ_f is induced by the ring homomorphism $\Delta_f^* : S \otimes_R S \to S$ obtained by multiplication: $a \otimes a' \mapsto aa'$. This is obviously surjective, therefore the induced map Δ_f is a closed embedding. $\square$

Corollary 18.8. *A morphism $f : X \to Y$ is separated if and only if the image of the diagonal map $\Delta_f : X \to X \times_Y X$ is closed.*

Proof. If f is separated the the image of Δ_f is closed by definition. To prove the converse, we need to show that Δ_f is a closed embedding, i.e. that $\Delta_f : X \to \Delta_f(X)$ is a homeomorphism and that $\mathcal{O}_{X\times_Y X} \to \Delta^*(\mathcal{O}_X)$ is surjective. Note that by definition

$$pr_1 \circ \Delta_f = pr_2 \circ \Delta_f = id_X,$$

therefore the restrictions of the projections pr_1, pr_2 to $\Delta_f(X)$ agree. We denote by $pr : \Delta_f(X) \to X$ the common restriction. To show that Δ_f is a homeomorphism, we show that $pr : \Delta_f(X) \to X$ is a continuous inverse. Consider the diagram

$$\begin{tikzcd}
\Delta_f(X) \arrow[dr, dotted, "\Delta_f\circ pr"] \arrow[drr, bend left, "pr"] \arrow[ddr, bend right, "pr"'] & & \\
& \Delta_f(X) \subset X \times_Y X \arrow[r, "pr"'] \arrow[d, "pr"] & X \arrow[d, "f"'] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}$$

Since $pr \circ \Delta_f = id_X$, it follows that all diagrams commute. Then by the universality property of the fibre product it follows that $\Delta_f \circ pr = id_{\Delta_f(X)}$, thus Δ_f is a homeomorphism.

The surjectivity of sheaves is a local question. We need to show that there is a surjective map of sheaves $\mathcal{O}_{X\times_Y X} \to (\Delta_f)_*(\mathcal{O}_X)$. This amounts to showing surjectivity at the level of stalks. Further, observe that the stalk $(\Delta_f)_*(\mathcal{O}_X)_{\Delta_f(x)} = \mathcal{O}_{X,x}$ and that the induced map is the same as $(\Delta_f)^*_x : \mathcal{O}_{X\times_Y X, \Delta_f(x)} \to \mathcal{O}_{X,x}$. Take $x \in X$ and let U be an open affine neighborhood. We may take U to be small enough so that $f(U) \subset V$ is included in some open affine subscheme. Then $U \times_V U$ is an open affine neighborhood of $\Delta_f(x)$, and by Proposition 18.7 $\Delta_f : U \to U \times_V U$ is a closed embedding. Then the pull-back map at the level of sheaves is surjective. $\square$

Here is a list of properties which guarantee separatedness; cf. [Har77, Cor. 4.6].

Theorem 18.1. *Assume all preschemes are Noetherian. Then the following hold:*

(a) Open and closed embeddings are separated.
(b) Composition of separated morphisms is separated.
(c) Separated morphisms are stable under base extension.

(d) A morphism $f : X \to Y$ is separated if and only if Y may be covered by open subsets V_i such that $f^{-1}(V_i) \to V_i$ is separated for each i.

18.3. **Proper morphisms.**

Definition 18.7. *A morphism $f : X \to Y$ of preschemes is* **locally of finite type** *if there exists a covering of Y by open affines $V_i = \mathrm{Spec}(S_i)$ such that for each i, $f^{-1}(V_i)$ may be covered by open affines $U_{ij} = \mathrm{Spec}(R_{ij})$ where each R_{ij} is a finitely generated S_i-algebra.*

The morphism f is **of finite type** *if the covering of each $f^{-1}(V_i)$ may be taken to be finite.*

The morphism f is **finite** *if in addition each $f^{-1}(V_i) = \mathrm{Spec}(R_i)$ is affine and R_i is a finitely generated S_i-module.*

Definition 18.8. *A morphism $f : X \to Y$ is* **proper** *if it is separated, of finite type, and universally closed.*

Here a morphism is **closed** *if for any $Z \subset X$ closed, $f(Z) \subset Y$ is again closed; it is* **universally closed** *if it is closed and for any morphism $Y' \to Y$, the base extension $f' : X \times_Y Y' \to Y$ is a closed morphism.*

Example 18.9. *The line $\mathbb{A}^1_k$ over a field k is separated, but not proper. Indeed, consider the base extension $\pi : \mathbb{A}^1_k \times_{\mathrm{Spec}(k)} \mathbb{A}^1_k \to \mathbb{A}^1_k$. It is induced by the k-algebra homomorphism $k[x] \hookrightarrow k[x, y]$, and at the level of closed points, this is simply the first projection. This is not a closed map: for instance the image of the hyperbola $xy - 1 = 0$ is given by $x \neq 0$, which is not closed.*

Theorem 18.2. *Assume all preschemes are Noetherian. Then the following hold:*
(a) Closed embeddings are proper.
(b) Composition of proper morphisms is proper.
(c) Proper morphisms are stable under base extension.
(d) If $f : X \to Y$ and $g : Y \to Z$ are two morphisms such that $g \circ f$ is proper and g separated, then f is proper.

Proof. See [Har77, Cor.4.8]. □

Theorem 18.3. *Let $f : X \to Y$ be a morphism of Noetherian schemes. Then the following are equivalent:*
(a) f is finite;
(b) f is proper with finite fibres.

Proof. See Lemma 30.21.1 in The Stacks Project. □

Definition 18.9. *If Y is a scheme, define the* **projective n-space** *over Y as $\mathbb{P}^n_Y := \mathbb{P}^n \times_{\mathrm{Spec}\mathbb{Z}} Y$. A morphism of schemes $f : X \to Y$ is* **projective** *if it factors as*

$$X \hookrightarrow \mathbb{P}^n_Y \to Y$$

into a closed embedding followed by the projection.

The morphism f is **quasi-projective** *if it factors into an open embedding $X \to X'$ followed by a projective morphism $g : X' \to Y$.*

Example 18.10. *Let A be a ring, R be a graded ring with $R_0 = A$ and which as an A-algebra is finitely generated by R_1. Then $\mathrm{Proj}(R) \to \mathrm{Spec}(A)$ is a projective morphism.*

Indeed, since R is a finitely generated over R_1, it means there is a surjective A-algebra homomorphism $A[x_0, \dots, x_n] \to R$. This induces a closed embedding $\mathrm{Proj}(R) \hookrightarrow \mathbb{P}^n_A$ and the morphism $\mathrm{Proj}(R) \to \mathrm{Spec}(A)$ factors through $\mathbb{P}^n_A$.

Example 18.11. *Let $i : X \subset \mathbb{P}^n_k$ be a projective subscheme of some projective spaces and $f : X \to Y$ any morphism. Then any morphism $f : X \to Y$ is projective. Indeed, the morphism f factors as:*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "{(i,f)}"'] & & Y \\ & \mathbb{P}^n_k \times_{\mathrm{Spec}(\mathbb{Z})} Y \arrow[ur, "pr_2"'] & \end{tikzcd}$$

Definition 18.10. *A scheme X over a field k is called* **complete** *if the morphism $X \to \mathrm{Spec}(k)$ is proper. The scheme is called* **projective** *if $X \to \mathrm{Spec}(k)$ is a projective morphism; equivalently, there is a closed embedding $X \hookrightarrow \mathbb{P}^n_k$.*

A key property of projective morphisms is that they are proper:

Theorem 18.4. *A projective morphism if Noetherian schemes is proper. A quasi-projective morphism of Noetherian schemes is of finite type and separated.*

Remark 18.3. *A consequence of the theorem is that any projective morphism is closed. This is an instance of the* **elimination theory***: assume we have a system of equations*

$$\begin{cases} f_1(x_0, \dots, x_n; y_1, \dots, y_p) = 0 \\ f_2(x_0, \dots, x_n; y_1, \dots, y_p) = 0 \\ \vdots \\ f_s(x_0, \dots, x_n; y_1, \dots, y_p) = 0 \end{cases}$$

where f_i's are polynomials in x's and y' over some field k, and such that they are homogeneous in variables x. Then there exist polynomials $g_1(y_1, \dots, y_p), \dots g_\ell(y_1, \dots, y_p)$ such that

$$g_1(y_1, \dots, y_p) = \ldots = g_\ell(y_1, \dots, y_p) = 0.$$

In other words, one may 'eliminate' the variables x from the system above. This follows from the fact that $pr_2 : \mathbb{P}^n \times_{\mathrm{Spec}(k)} \mathbb{A}^p \to \mathbb{A}^p$ is projective, thus it is proper, and thus the image of any closed subset under pr_2 is closed.

If k is algebraically closed, it turns out that (quasi-)projective algebraic varieties from the first semester correspond to (quasi-)projective schemes over k, in the sense above. This is proved in [Har77, Prop. II.4.10]. **From now on, a variety will mean a separated, integral scheme of finite type over an algebraically closed field k.**

19. Relative differentials

19.1. **(Quasi-)coherent sheaves.** Let $X = \mathrm{Spec}(R)$ be an affine scheme, and let M be an R-module. Then one may associate a sheaf of $\mathcal{O}_X$-modules $\widetilde{M}$ in a similar way to the definition of $\mathcal{O}_X$: if $U \subset X$ is open then $\widetilde{M}(U)$ is the module over $\mathcal{O}_X(U)$ given by tuples $(m_\mathfrak{p}) \in \prod_{\mathfrak{p} \in U} M_\mathfrak{p}$ which are 'locally given' by elements of the form $m_i/f_i^{n_i} \in M_{f_i}$. The proof of the following lemma follows closely the proof for properties of $\mathcal{O}_X$:

Lemma 19.1. *$\widetilde{M}$ is a sheaf of $\mathcal{O}_X$-modules with the following properties:*

(a) The sections $\widetilde{M}(X_f) = M_f$;

(b) The stalk $\widetilde{M}_\mathfrak{p} = M_\mathfrak{p}$.

in particular, one may recover M from $\widetilde{M}$ by taking global sections: $M = \widetilde{M}(X)$.

Furthermore, the assignment $M \to \widetilde{M}$ gives an equivalence of categories:

Proposition 19.2. *The assignment $M \to \widetilde{M}$ gives an equivalence between the category of R-modules and the category of $\mathcal{O}_X$-modules of the form $\widetilde{M}$.*

This equivalence preserves exactness, in the sense that the sequence of R-modules $0 \to M \to N \to P \to 0$ is exact if and only if the corresponding sequence of $\mathcal{O}_X$-modules $0 \to \widetilde{M} \to \widetilde{N} \to \widetilde{P} \to 0$ is exact.

Furthermore, the equivalence is compatible with taking direct sums, tensor products, duals.

Proof. **Homework!** □

Definition 19.1. *Let X be a (pre)scheme and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules. The sheaf $\mathcal{F}$ is called* **quasi-coherent** *if X may be covered by open affine schemes $U_i = \mathrm{Spec}(R_i)$ such that $\mathcal{F}|_{U_i} \simeq \widetilde{M_i}$ for some R_i-module M_i.*

If in addition each M_i can be taken to be a finitely generated R_i-module then $\mathcal{F}$ is called **coherent**.

As usual, the check of (quasi-)coherence may be done for one/any affine cover of a (pre)scheme X. More precisely, the following holds:

Theorem 19.1. *Let X be a scheme and $\mathcal{F}$ an $\mathcal{O}_X$-module. The following are equivalent:*

(1) For any $U \subset X$ open, affine, $\mathcal{F}|_U \simeq \widetilde{M}$ for some $\mathcal{O}_X(U)$-module M.

(2) There exists an open affine cover U_i of X such that $\mathcal{F}|_{U_i} \simeq \widetilde{M_i}$ for $\mathcal{O}_X(U_i)$-modules M_i.

(3) For all $x \in X$, there exists a neighborhood U of x and an exact sequence of $(\mathcal{O}_X)|_U$-modules

$$(\mathcal{O}_X)^{\oplus I}|_U \to (\mathcal{O}_X)^{\oplus J}|_U \to \mathcal{F}|_U \to 0.$$

(4) For all $V \subset U$ affine, the canonical map

$$\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V) \to \mathcal{F}(V)$$

is an isomorphism.

Proof. See [Mum99, §III.1]. □

All this admits a 'projective' version, generalizing in the obvious way the content from §12.3 above.

Definition 19.2. *A sheaf $\mathcal{F}$ is called* **locally free of rank** r *if there is an open cover* $\{U_i\}$ *of X such that* $\mathcal{F}|_{U_i} \simeq (\mathcal{O}_X)^{\oplus r}|_{U_i}$. *A locally free sheaf of rank* 1 *is called* **invertible**.

19.2. **The module of relative differentials.** In this section we define a quasi-coherent sheaf to any morphism $f : X \to Y$.

Fix $\varphi : A \to B$ be a homomorphism making B an A-algebra. Define the B-module **of relative differentials** $\underline{\Omega_{B/A}}$ as the quotient of the free B-module with symbols $\{d\beta : \beta \in B\}$ modulo the following relations:

1. $d(\beta_1 + \beta_2) = d\beta_1 + d\beta_2$;
2. (Leibniz rule) For $\alpha, \beta \in B$, $d(\alpha\beta) = \alpha d\beta + \beta d\alpha$;
3. $d(\varphi(a)) = 0$ for any $a \in A$.

For any B-module M, an A-**derivation** is a $B-$module map $D : B \to M$ which satisfies the 3 conditions above.

Lemma 19.3. *There is a one-to-one correspondence* $Hom_B(\underline{\Omega_{B/A}}, M)$ *to the module of* A*-derivations* $D : B \to M$.

Proof. If $\tau \in Hom_B(\underline{\Omega_{B/A}}, M)$ then an A-derivation is defined by $D_\tau(\beta) = \tau(d\beta)$. Conversely, if $D : B \to M$ is an A-derivation, define τ_D by

$$\tau_D(\sum \alpha_i d\beta_i) = \sum a_i D(\beta_i).$$

One checks that the two operations are inverse to each other. □

Remark 19.1. *An important situation is when $A = M = k$ with k algebraically closed, and B is a finitely generated k-algebra. Consider $X = \mathrm{Spec}(B)$ and a closed point $x = \mathfrak{m}_x \in X$ with $k = k(x)$ a structure of a B-module given via the natural map $B \to B/\mathfrak{m}_x \simeq k$. Then*

$$\underline{\Omega_{B/k(x)}}^* = Hom_B(\underline{\Omega_{B/k(x)}}, k)$$

is the same as the vector space of k-derivations $D : B \to k$. The localization at x will be the stalk of the tangent sheaf of X at x.

The module of relative differentials behaves well with respect to localization: if $S \subset B$ is a multiplicative system, then

$$S^{-1}\underline{\Omega_{B/A}} = \underline{\Omega_{S^{-1}B/A}}.$$

We give next two explicit realizations for $\underline{\Omega_{B/A}}$, first in the 'global' case, then in the 'local' case. Consider the ring homomorphism

$$\delta : B \otimes_A B \to B; \quad \beta_1 \otimes \beta_2 \mapsto \beta_1\beta_2. \tag{19.1}$$

Define $I := \ker(\delta)$; this is a $B \otimes_A B$-module. Then I/I^2 is a $(B \otimes_A B)/I$-module, and observe that for any $\kappa := \sum a_i \otimes b_j \in I$, and any $\beta \in B$, the element $(1 \otimes \beta)\kappa - (\beta \otimes 1)\kappa \in I^2$. This gives I/I^2 a structure of a B-module by

$$\beta.\kappa = (1 \otimes \beta).\kappa \quad .$$

Proposition 19.4. *I/I^2 is isomorphic to $\underline{\Omega_{B/A}}$.*

Proof. Define a module map $\Phi : \underline{\Omega_{B/A}} \to I/I^2$ by sending $d\beta \mapsto \beta \otimes 1 - 1 \otimes \beta$. One needs to check that this is well defined. Clearly condition (1) (additivity) holds, and since $1 \otimes_A a = a \otimes_A 1$ for any $a \in A$, condition (3) holds. For condition (2), we have

$$\begin{aligned} \alpha\beta \otimes 1 - 1 \otimes \alpha\beta &= \alpha\beta \otimes 1 - \alpha \otimes \beta + \alpha \otimes \beta - 1 \otimes \alpha\beta \\ &= \alpha \otimes 1(\beta \otimes 1 - 1 \otimes \beta) + 1 \otimes \beta(\alpha \otimes 1 - 1 \otimes \alpha) \end{aligned}$$

This shows that $\Phi(d(\alpha\beta)) = \alpha\Phi(d\beta) + \beta\Phi(d\alpha)$. One can prove that Φ is an isomorphism - see [Mum99, §III.1]. □

Example 19.5. *Take $A = k$ and $B = k[x_1, \ldots, x_n]$. Then $\underline{\Omega_{B/A}}$ is a free B-module with basis $dx_1, \ldots, dx_n$ and*

$$dP = \sum_{i=1}^{n} \frac{\partial P}{\partial x_i} dx_i.$$

More generally, consider $B = k[x_1, \ldots, x_n]/\langle f_1, \ldots f_s\rangle$. Then $\underline{\Omega_{B/A}}$ is generated (as a B-module by $dx_1, \ldots, dx_n$ subject to relations

$$df_j = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} dx_i = 0.$$

Theorem 19.2. *Let K be a finitely generated field over some field k. Then $\dim_K \underline{\Omega_{K/k}} \geq trdeg_k K$ and equality holds if and only if K is separably generated over k.*

Theorem 19.3. *Let $(B, \mathfrak{m})$ be a local ring which contains a field k isomorphic to the residue field $B/\mathfrak{m}$. Consider the map*

$$\delta' : \mathfrak{m}/\mathfrak{m}^2 \to \underline{\Omega_{B/k}} \otimes_B k; \quad m \mapsto dm \otimes 1.$$

Then δ' is a vector space isomorphism.

Let k be algebraically closed and **perfect**, i.e. either $\mathrm{char}(k) = 0$ or if $\mathrm{char}(k) = p$ then $x \mapsto x^p$ is a field automorphism. A Noetherian local ring $(B, \mathfrak{m})$ is a **regular local ring** if the minimum number of generators of $\mathfrak{m}$ equals the Krull dimension of B.

Theorem 19.4. *Let $(B, \mathfrak{m})$ be a local ring which contains a perfect field k isomorphic to the residue field $B/\mathfrak{m}$. Assume further that B is the localization of a finitely generated k-algebra. Then $\underline{\Omega_{B/k}}$ is a free B-module of rank $\dim B$ if and only if B is a regular local ring.*

19.3. **The sheaf of relative differentials.** Consider a separated morphism $f : X \to Y$ and the associated closed embedding $\Delta : X \hookrightarrow X \times_Y X$. Recall that if $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$, then Δ is determined by the morphism δ from Equation (19.1) above. Let $\mathcal{I}$ be the ideal sheaf of $\Delta(X)$ inside $X \times_Y X$.

Definition 19.3. *$\Omega_{X/Y}$ is the quasi-coherent $\mathcal{O}_X$-module obtained by carrying the $\mathcal{O}_{X\times_Y X}/\mathcal{I}$-module $\mathcal{I}/\mathcal{I}^2$ from $X \times_Y X$ to X via Δ.*

If M is an R module and $I \subset R$ is an ideal then $M \otimes_R R/I$ is an R/I-module. The projection $R \to R/I$ induces the closed embedding $Y := \mathrm{Spec}(R/I) \to X := \mathrm{Spec}(R)$. If $\mathcal{F} = \widetilde{M}$ is the quasi-coherent $\mathcal{O}_X$-module determined by M, then the restriction of $\widetilde{M}$ to Y is simply

$$(\widetilde{M})|_Y = M \widetilde{\otimes_R (R/I)}. \tag{19.2}$$

Recall that varieties are separated integral schemes of finite type over an integrally closed field k.

Definition 19.4. *A variety X is called* **non-singular at** *x if the local ring $\mathcal{O}_{X,x}$ is a regular local ring. It is* **non-singular** *if X is non-singular at every point.*

One may show that being non-singular is an open condition, i.e. if X is non-singular at x then there exists an open set U containing x such that U is non-singular.

The main theorem of this section is the following:

Theorem 19.5. *Let X be a variety. Then X is non-singular if and only if the sheaf of regular differential $\Omega_{X/k}$ is a locally free sheaf of rank $n = \dim X$.*

Proof. See [Har77, Thm. II.8.15]. $\square$

Definition 19.5. *Let X be a non-singular variety of dimension n. The* **tangent sheaf** *is defined as the dual $\mathcal{T}_X := \Omega^*_{X/k}$. The* **canonical sheaf** *ω_X is defined as $\omega_X := \wedge^n \Omega_X$. Each is a locally free sheaf, of rank n, respectively, rank 1.*

The **geometric genus** *of X is defined as $p_g := \dim_k \omega_X(X)$.*

Key tools to understand the sheaves of differentials are the following two theorems. (We prove the second one.)

Theorem 19.6. *Let X be a non-singular variety, and let $Y \subset X$ a closed subvariety defined by the sheaf of ideals $\mathcal{I}$. (I.e., $Y \to X$ is a closed embedding.) Then Y is non-singular if and only if:*

(a) $\Omega_{X/Y}$ is locally free, and,

(b) There is a short exact sequence of $\mathcal{O}_Y$-modules:

$$0 \to \mathcal{I}/\mathcal{I}^2 \to (\Omega_{X/k})|_Y \to \Omega_{Y/k} \to 0. \tag{19.3}$$

Here the first map is induced by the natural map $\mathcal{I}/\mathcal{I}^2 \simeq \Omega_{X/Y} \to \Omega_{X/k}$, and the second is the restriction. (Both maps may be defined locally.)

If this is the case then $\mathcal{I}$ is locally generated by $c := \mathrm{codim}(Y, X)$ elements, and $\mathcal{I}/\mathcal{I}^2$ is locally free sheaf of rank c.

The locally free sheaf $\mathcal{I}/\mathcal{I}^2$ is called the **conormal sheaf** of Y in X.

Remark 19.2. *We will prove later on that there is a one-to-one correspondence between locally free sheaves and vector bundles. Then the sequence above is the dual of the usual sequence defining the normal bundle of a submanifold $Z \subset M$:*

$$0 \to T_Z \to (T_M)|_Z \to N_M Z \to 0.$$

Corollary 19.6 (Adjunction formula)**.** *Let $i : Y \subset X$ be a closed embedding of codimension c of non-singular varieties. Then there is an isomorphism of locally free sheaves of rank 1 on Y:*

$$(\omega_X)|_Y \simeq \omega_Y \otimes \wedge^c(\mathcal{I}/\mathcal{I}^2).$$

Proof. If $0 \to V_1 \to V_2 \to V_3 \to 0$ is a short exact sequence of free R-modules of ranks a, b, c respectively, then there is a canonical isomorphism

$$\wedge^a V_1 \otimes \wedge^c V_3 \to \wedge^b V_2; \quad (e_1 \wedge \dots e_a) \otimes (f_1 \wedge \dots \wedge f_c) \mapsto e_1 \wedge \dots e_a \wedge f_1' \wedge \dots \wedge f_c',$$

where $f_i' \in V_2$ are any elements mapping to f_i. This induces a global map of locally free $\mathcal{O}_Y$-modules $\omega_Y \otimes \mathcal{I}/\mathcal{I}^2 \to \omega_X$ which is a local isomorphism, therefore it is also a global isomorphism. □

The next result shows how the sheaf of differentials on $\mathbb{P}^n = \mathbb{P}^n_k$ is related to standard sheaves on the projective space. In what follows we will use the notation from §12.3 about quasi-coherent sheaves on projective varieties associated to graded modules.

Proposition 19.7 (Euler sequence)**.** *There is an exact sequence of locally free sheaves on $\mathbb{P}^n$:*

$$0 \to \Omega_{\mathbb{P}^n/k} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)} \to \mathcal{O}_{\mathbb{P}^n} \to 0.$$

Taking duals, one obtains the **Euler sequence***:*

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} \to \mathcal{T}_{\mathbb{P}^n} \to 0.$$

Proof. Let $S := k[x_0, \dots, x_n]$ regarded as a graded module with $\deg x_i = 1$. Let $E := S(-1)^{n+1}$ be the free S module of rank $n+1$ with a basis $e_0, \dots, e_n$ in degree 1. (I.e., this is simply the S-module S^{n+1}, but with basis shifted to degree 1.) Define a degree 0 module homomorphism

$$\varphi : E \to S; \quad e_i \mapsto x_i \quad (0 \le i \le n).$$

Let $M := \ker(\varphi)$; this is a graded module, and it fits into an exact sequence of graded modules

$$0 \to M \to E \to S$$

Note that the last map is *not* surjective, as $S_0 = k$ is not in the image. However, $E_n \to S_n$ is surjective for $n \ge 1$. Therefore the induced map on sheaves is surjective, giving a short exact sequence

$$0 \to \widetilde{M} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)} \to \mathcal{O}_{\mathbb{P}^n} \to 0.$$

We need to show that $\widetilde{M} \simeq \Omega_{\mathbb{P}^n/k}$. For that, we define local isomorphisms, and we need to check that these glue to a global isomorphism. Cover $\mathbb{P}^n$ by the standard affine open sets $U_i = \mathrm{Spec}[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}] \simeq \mathbb{A}^n$ for $0 \le i \le n$. Observe that the induced homomorphism of free S_{x_i}-modules $E_{x_i} \to S_{x_i}$ is surjective, with kernel the free module of rank n generated by $e_j - \frac{x_j}{x_i} e_i$. Then the restriction of $\widetilde{M}|_{U_i}$ is a free module of rank n generated by sections

$$\frac{e_j}{x_i} - \frac{x_j}{x_i^2} e_i$$

(we need the extra factor of x_i in the denominator to get elements of degree 0; see §12.3 above). Recall from Example 19.5 that $\Omega_{\mathbb{A}^n/k}$ is locally free of rank n with basis given by sections dx_j for $1 \le j \le n$. Then define

$$\psi_i : \Omega_{U_i/k} \to \widetilde{M}|_{U_i} \quad d(\frac{x_j}{x_i}) = \frac{e_j}{x_i} - \frac{x_j}{x_i^2} e_i.$$

Since both are free modules of the same rank, this is an isomorphism. We need to prove these isomorphisms glue over the intersections $U_i \cap U_j$, i.e.,

$$\psi_j(d(\frac{x_k}{x_i})) = \psi_i(d(\frac{x_k}{x_i})).$$

To this aim, observe that in $\mathcal{O}_{\mathbb{P}^n}(U_i \cap U_j)$,

$$\frac{x_k}{x_i} = \frac{x_k}{x_j} \cdot \frac{x_j}{x_i}.$$

Then

$$\begin{aligned}
\psi_j(d(\frac{x_k}{x_i})) &= \psi_j(d(\frac{x_k}{x_j} \cdot \frac{x_j}{x_i})) \\
&= \psi_j(\frac{x_k}{x_j} \cdot d(\frac{x_j}{x_i}) + \frac{x_j}{x_i} d(\frac{x_k}{x_j})) \\
&= \frac{x_k}{x_j} \cdot \psi_j(d(\frac{x_j}{x_i})) + \frac{x_j}{x_i} \psi_j(d(\frac{x_k}{x_j})).
\end{aligned}$$

Observe also that $\frac{x_i}{x_j} \cdot \frac{x_j}{x_i} = 1$, therefore on $U_i \cap U_j$,

$$\frac{x_i}{x_j} d(\frac{x_j}{x_i}) + \frac{x_j}{x_i} d(\frac{x_i}{x_j}) = 0 \Longrightarrow d(\frac{x_j}{x_i}) = -\frac{x_j^2}{x_i^2} d(\frac{x_i}{x_j}).$$

Then

$$\begin{aligned}\psi_j(d(\frac{x_k}{x_i})) &= \frac{x_k}{x_j}\cdot\psi_j(d(\frac{x_j}{x_i})) + \frac{x_j}{x_i}\psi_j(d(\frac{x_k}{x_j}))\\ &= -\frac{x_j^2}{x_i^2}\cdot\frac{x_k}{x_j}\psi_j(d(\frac{x_i}{x_j})) + \frac{x_j}{x_i}\psi_j(d(\frac{x_k}{x_j}))\\ &= -\frac{x_jx_k}{x_i^2}(\frac{e_i}{x_j} - \frac{x_i}{x_j^2}e_j) + \frac{x_j}{x_i}(\frac{e_k}{x_j} - \frac{x_k}{x_j^2}e_j)\\ &= -\frac{x_k}{x_i^2}e_i + \frac{e_k}{x_i}\\ &= \psi_i(d(\frac{x_k}{x_i})).\end{aligned}$$

One deduces that the morphisms ψ_i, ψ_j coincide on a set of module generators of $\Omega_{\mathbb{P}^n/k}$ over $U_i \cap U_j$. This finishes the proof. □

Lemma 19.8. *Let $I \subset R$ be an ideal in a ring R. Then there is a canonical isomorphism of R/I-modules*

$$I/I^2 \to I \otimes_R R/I; \quad a + I^2 \mapsto a \otimes 1. \tag{19.4}$$

Proof. Consider the short exact sequence of R-modules $0 \to I \to R \to R/I \to 0$. By [AM69, Ex. 2.2], tensoring with the R-module I preserves exactness, giving

$$0 \to I^2 \simeq I \otimes_R I \to I \simeq I \otimes_R R \to I \otimes_R R/I \to 0.$$

The statement follows from the first isomorphism theorem applied to the second map. □

This isomorphism is key to study the important special case when $X = \mathbb{P}^n$ and $Y = V_p(F) \subset \mathbb{P}^n$ is a hypersurface given by a homogeneous polynomial F of degree d. Denote by $i : Y \hookrightarrow \mathbb{P}^n$ the inclusion. The ideal sheaf $\widetilde{I} = \widetilde{\langle F\rangle}$ induces the standard exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{\mathbb{P}^n} \to i_*\mathcal{O}_Y \to 0. \tag{19.5}$$

On the other side, let $S := k[x_0, \dots x_n]$ and consider the short exact sequence of graded S-modules

$$0 \longrightarrow S(-d) \xrightarrow{\cdot F} S \longrightarrow S/\langle F\rangle \longrightarrow 0$$

giving the sequence of $\mathcal{O}_{\mathbb{P}^n}$-modules

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-d) \to \mathcal{O}_{\mathbb{P}^n} \to i_*\mathcal{O}_Y \to 0. \tag{19.6}$$

Proposition 19.9. *(a) There is an isomorphism of locally free $\mathcal{O}_{\mathbb{P}^n}$-modules*

$$\mathcal{I} \simeq \mathcal{O}_{\mathbb{P}^n}(-d).$$

(b) Assume in addition that $Y = V_p(F)$ is non-singular in $\mathbb{P}^n$. Then there is an isomorphism of locally free $\mathcal{O}_Y$-modules of rank 1,

$$\mathcal{I}/\mathcal{I}^2 \simeq (\mathcal{O}_{\mathbb{P}^n}(-d))|_Y.$$

Proof. The first isomorphism follows from comparing the sequences (19.5) and (19.6). The second part follows because the restriction corresponds to tensoring by $\otimes_{\mathcal{O}_{\mathbb{P}^n}}\mathcal{O}_Y$, then using the isomorphism (19.4). □

The following are the first explicit calculations of canonical sheaves.

Corollary 19.10. *The following holds:*

$$\omega_{\mathbb{P}^n} \simeq \mathcal{O}_{\mathbb{P}^n}(-n-1).$$

More generally, if $Y = V_p(F)$ is a non-singular hypersurface in $\mathbb{P}^n$ of degree d, then

$$\omega_Y = \mathcal{O}_Y(d-n-1) = (\mathcal{O}_{\mathbb{P}^n}(d-n-1))|_Y.$$

Proof. The first sequence follows by taking top exterior powers in the short exact sequence from Proposition 19.7. The second follows from the adjunction formula (Corollary 19.6) combined with part (b) of Proposition 19.9. □

Recall the definition of the arithmetic genus p_a from Definition 13.2. One may show that if Y is non-singular, then $p_a(Y) = p_g(Y)$ (the geometric genus).

Example 19.11. *Let $C \subset \mathbb{P}^2$ be a non-singular plane curve given by a polynomial of degree d. Then*

$$\omega_C \simeq \mathcal{O}_C(d-3).$$

We have the following special cases:

- *If C is a line ($d = 1$) then $\omega_C = \mathcal{O}_C(-2)$ (the restriction of $\mathcal{O}_{\mathbb{P}^2}(-2)$ to C).*
- *If C is a conic ($d = 2$) then $\omega_C = \mathcal{O}_C(-1)$. Recall that $C \simeq \mathbb{P}^1$, but the isomorphism is given by the (restriction of) Veronese embedding $\nu_2 : \mathbb{P}^1 \to \mathbb{P}^2$, of degree 2. By taking the corresponding map of graded modules, one may prove that*

  $$\nu_2^*(\mathcal{O}_C(-1)) = \mathcal{O}_{\mathbb{P}^1}(-2),$$

  in agreement to the line case.
- *If C is an elliptic curve ($d = 3$), then $\omega_C \simeq \mathcal{O}_C$ is trivial.*

These calculations eventually lead to a classification of non-singular curves. For example, the knowledge of ω_C leads to knowing what is the topological Euler characteristic:

$$\chi_{top}(C) = 2 - 2g,$$

where g is the genus of C; see Definition 13.2. For example, if C is a non-singular curve of degree d, one may use the Hilbert polynomial techniques to show that $g = \binom{d-1}{2}$, giving

$$\chi_{top}(C) = 2 - (d-1)(d-2).$$

Therefore a line and a conic have Euler characteristic 2, a cubic has Euler characteristic 0 etc. This is (of course!) consistent with the 'real picture' of these curves: complex lines and conics are homeomorphic to a (real!) sphere S^2; an elliptic curve is homeomorphic to a torus $S^1 \times S^1$ etc.

20. Locally free sheaves and vector bundles

20.1. **Definition and equivalence to locally free sheaves and vector bundles.** We follow the definition from [Mum99, III.2].

Definition 20.1. *Let X be a scheme. A* **vector bundle** $\pi : E \to X$ *of rank r is a scheme E together with the 'projection' morphism π which satisfy the following properties:*

There exists an open covering ('trivialization') U_i and isomorphisms of schemes $\psi_i^{-1}(U_i) \simeq U_i \times \mathbb{A}^r$ over U_i:

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[rr, "\psi_i"] \arrow[dr, "\pi"'] & & U_i \times \mathbb{A}^r \arrow[dl, "pr_1"] \\ & U_i & \end{tikzcd}$$

such that for all i, j, the restriction

$$\psi_i \circ \psi_j^{-1} : (U_i \cap U_j) \times \mathbb{A}^r \to (U_i \cap U_j) \times \mathbb{A}^r$$

of morphisms over $U_i \cap U_j$

$$\begin{tikzcd} (U_i \cap U_j) \times \mathbb{A}^r \arrow[dr, "pr_1"'] & \pi^{-1}(U_i) \cap \pi^{-1}(U_j) \arrow[l, "\psi_i"'] \arrow[r, "\psi_j"] \arrow[d, "\pi"] & (U_i \cap U_j) \times \mathbb{A}^r \arrow[dl, "pr_1"] \\ & U_i \cap U_j & \end{tikzcd}$$

is of the form

$$\psi_i \circ \psi_j^{-1} = id \times \psi_{i,j}; \quad \psi_{i,j} : \mathbb{A}^r \times (U_i \cap U_j) \to \mathbb{A}^r \times (U_i \cap U_j)$$

and $\psi_{i,j}$ is determined by a **linear** *isomorphism (with coefficients in $\mathcal{O}_{U_i \cap U_j}(U_i \cap U_j)$):*

$$\psi_{i,j}^* : \mathcal{O}_{U_i \cap U_j}(U_i \cap U_j) \otimes_k k[x_1, \dots, x_n] \to \mathcal{O}_{U_i \cap U_j}(U_i \cap U_j) \otimes_k k[x_1, \dots, x_n];$$

$$\psi_{i,j}^*(x_k) = \sum_{k=1}^{r} a_{k,\ell}^{(i,j)}(u) x_\ell.$$

In other words, $\psi_{i,j}^*$ is an element of $\mathrm{GL}_r(\mathcal{O}_{U_i \cap U_j}(U_i \cap U_j))$.

Proposition 20.1. *There is a one-to-one correspondence between vector bundles of rank r and locally free sheaves of rank r.*

We explain this correspondence next.

20.1.1. *From vector bundles to locally free sheaves.* Let $\pi : E \to X$ be a vector bundle of rank r. We define a sheaf $\mathcal{E}$ on X by taking local sections of E: for any open set $U \subset X$,

$$\mathcal{E}(U) = \{s : U \to E : \pi \circ s = id_U\}$$

We need to make $\mathcal{E}$ into an $\mathcal{O}_X$-module. The sections over the trivializations $E|_{U_i} \simeq U_i \times \mathbb{A}^r$ are given by morphisms $s : U_i \to \mathbb{A}^r$, i.e. by k-algebra homomorphisms

$$s^* : k[x_1, \dots, x_r] \to \mathcal{O}_{U_i}(U_i)); x_k \mapsto f_k \in \mathcal{O}_{U_i}(U_i).$$

(See also Theorem 15.2.) The fibres $E_x = \pi^{-1}(x)$ over points $x \in X$ are k-vector spaces of dimension r, and because of linearity of the transition maps, the vector space structures arising from different trivializations are the same. Therefore the space of sections over U has a $\mathcal{O}_X(U)$-algebra structure. This may be enlarged to an $\mathcal{O}_X$-module structure by taking trivializations over U_i's (i.e., the trivializations ensure that sections may be multiplied by regular elements in $\mathcal{O}_X(U_i)$). Furthermore, there are $\mathcal{O}_X(U_i)$-module isomorphisms

$$\mathcal{E}(U_i) \simeq \mathcal{O}_X(U_i)^{\oplus r}; \quad s \mapsto (s^*(x_1), \dots, s^*(x_r)),$$

which are again compatible under restriction to intersections $U_i \cap U_j$. This makes $\mathcal{E}$ a locally free sheaf of rank r.

20.1.2. *From locally free sheaves to vector bundles.* Conversely, if one is given a locally free sheaf $\mathcal{E}$, then it must have trivializations

$$\mathcal{E}(U_i) \simeq \mathcal{O}_X(U_i)^{\oplus r}; \quad f \mapsto (f_1, \dots, f_r),$$

where $f_i \in \mathcal{O}_X(U_i)$. These are maps of $\mathcal{O}_X(U_i)$-*modules*, and determine sections $s_i : U_i \to \mathbb{A}^r$ obtained from $s_i^*(x_j) \mapsto f_j$. We obtain 'trivial vector bundles' $U_i \times \mathbb{A}^r \to U_i$, and one needs to prove that these glue over the intersections $U_i \cap U_j$. The key fact is that over $U_i \cap U_j$ the two structures of $\mathcal{O}_X$-modules are given by $\mathcal{O}_X(U_i \cap U_j)$-module isomorphisms, i.e. precisely by matrices

$$(g_{i,j}) \in \mathrm{GL}_r(\mathcal{O}_X(U_i \cap U_j)).$$

This is the **same** data which determines trivializations of vector bundles.

Finally, we note that if $\mathcal{L}$ is a locally free sheaf of rank 1, then the evaluation map gives an isomorphism of locally free modules $\mathcal{L}^* \otimes_{\mathcal{O}_X} \mathcal{L} \simeq \mathcal{O}_X$. (At the level of transition functions, $(g_{i,j}(\mathcal{L}^*) = g_{i,j}(\mathcal{L})^{-1}$.) Because of this, line bundles are sometimes called **invertible sheaves**.

Generally speaking, if $\varphi_i : \mathcal{E}(U_i) \to U_i \times \mathbb{A}^r$ and

$$g_{i,j} = \varphi_j \circ \varphi_i^{-1}$$

then local sections $f_i \in \mathcal{O}(U_i \times \mathbb{A}^r)$ have to transform under the law

$$\frac{f_j}{f_i} = g_{i,j} \quad \Longrightarrow \quad f_j = g_{i,j} f_i.$$

In what follows we will need the following notion. Let X be a scheme and $U = \mathrm{Spec}(R) \subset X$ open affine. Define $K(U)$ to be the localization R_S, where S is the multiplicative set of non-zero divisors. These form a presheaf of rings on X, and we denote by $\mathcal{K}_X$ its sheafification. It is called the **sheaf of total quotient rings**.

20.2. **Example: Line bundles on $\mathbb{P}^n$.** As usual, let $S = k[x_0, \dots, x_n]$ be the polynomial ring, graded as usual, and let U_i be the standard affine cover. The graded modules $S(d)$ determine locally free sheaves of rank 1. Locally,

$$\widetilde{S(d)}(U_i) = \{\frac{P}{x_i^s} : \deg P = s + d \text{ homogeneous } \}.$$

Let $S_{(x_i)} = k[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}]$ be the submodule of S_{x_i} consisting of degree 0 elements. Then $\widetilde{S(d)}(U_i)$ is a free, rank 1, $S_{(x_i)}$-module with basis x_i^d (regarded as a fraction).

For example, if $d < 0$, then if

$$P = \sum a_{p_0,\dots,p_n} x_0^{p_0} \cdot \dots \cdot x_i^{p_i} \cdot \dots \cdot x_n^{p_n} \in k[x_0, \dots, x_n]$$

is a homogeneous polynomial of degree s, then

$$\frac{P}{x_i^{d+s}} = \sum a_{p_0,\dots,p_n} \frac{x_0^{p_0}}{x_i^{p_0}} \cdot \dots \cdot \frac{1}{x_i^d} \cdot \dots \cdot \frac{x_n^{p_n}}{x_n^{p_n}},$$

and the right hand side is an element in $\frac{1}{x_i^d} S_{(x_i)}$. To find transition maps, write isomorphisms

$$(S_{(x_i)})_{(x_j)} = S_{(x_i x_j)} \xleftarrow{\phi_i^*} S(d)_{(x_i x_j)} \xrightarrow{\phi_j^*} (S_{(x_j)})_{(x_i)} = S_{(x_i x_j)}$$

To describe the transition maps, we distinguish between the cases $d \geq 0$ and $d < 0$:

- If $d \geq 0$ then

$$(\phi_j^* \circ (\phi_i^*)^{-1})(1) = \phi_j^*(x_i^d) = \phi_i^*(x_j^d \frac{x_i^d}{x_j^d}) = \frac{x_i^d}{x_j^d}.$$

- If $d < 0$ then

$$(\phi_j^* \circ (\phi_i^*)^{-1})(1) = \phi_j^*(\frac{1}{x_i^{-d}}) = \phi_j^*(\frac{1}{x_j^{-d}} \frac{x_j^{-d}}{x_i^{-d}}) = \frac{x_j^{-d}}{x_i^{-d}}.$$

In other words, we proved that for any d the transition matrix is

$$g_{i,j}(\mathcal{O}_{\mathbb{P}^n}(d)) = \phi_j^* \circ (\phi_i^*)^{-1} = \frac{x_i^d}{x_j^d}. \tag{20.1}$$

In particular, since transition functions are inverses to each other,

$$\mathcal{O}_{\mathbb{P}^n}(d) \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n}(-d) \simeq \mathcal{O}_{\mathbb{P}^n}.$$

20.3. **Example: Line bundles from hypersurfaces in $\mathbb{P}^n$ (Cartier divisors I).** Assume now that $d > 0$ and consider $F \in S_d$ a homogeneous polynomial of degree d. The multiplication by F gives an inclusion of sheaves

$$\mathcal{O}_{\mathbb{P}^n}(-d) \xrightarrow{\cdot F} \mathcal{O}_{\mathbb{P}^n} .$$

We analyze this inclusion when restricted to standard open sets. The sections

$$\mathcal{O}_{\mathbb{P}^n}(-d)|_{U_i} = \{\frac{P}{x_i^{s+d}} : \deg P = s\}$$

The inclusion is given by

$$\mathcal{O}_{\mathbb{P}^n}(-d)|_{U_i} \hookrightarrow \mathcal{O}_{\mathbb{P}^n}; \quad \frac{P}{x_i^{s+d}} \mapsto \frac{FP}{x_i^{s+d}}.$$

Observe that the image is the ideal in $S_{(x_i)}$ generated by $F_i := \frac{F}{x_i^d}$, the dehomogenization of F. Utilizing this, we can identify $\mathcal{O}_{\mathbb{P}^n}(-d)|_{U_i}$ with the ideal sheaf $\mathcal{I}_F$, the ideal sheaf of $V_p(F)$. In other words, we proved

$$\mathcal{I}_{V_p(F)} \simeq \mathcal{O}_{\mathbb{P}^n}(-d).$$

The collection of dehomogenizations of F gives a collection $D := \{(U_i, F_i)\}$ called a **Cartier divisor**. Such a collection D determines an invertible sheaf $\mathcal{L}_D$ *together with a global section* $s \in \mathcal{L}_D(X)$, as follows: let $\mathcal{L}_D(U_i)$ given by the

$$\mathcal{L}_D(U_i) = \frac{1}{F_i}\mathcal{O}_{\mathbb{P}^n} \subset \mathcal{K}(U_i). \tag{20.2}$$

In other words, $\mathcal{L}_D$ is the invertible subsheaf of the sheaf of total quotient rings given by the isomorphism

$$\mathcal{O}_{\mathbb{P}^n}(U_i) \xrightarrow{\phi_i^{-1}} \mathcal{L}_D(U_i) , \quad 1 \mapsto \frac{1}{F_i}.$$

One calculates the transition maps by

$$g_{i,j} = \phi_j \circ \phi_i^{-1}(1) = \phi_j\left(\frac{1}{F_i}\right) = \phi_j\left(\frac{F_j}{F_i}\frac{1}{F_j}\right) = \frac{F_j}{F_i} = \frac{\frac{F}{x_j^d}}{\frac{F}{x_i^d}} = \frac{x_i^d}{x_j^d}.$$

Comparing to the transition functions from Equation (20.1) this shows that we have an isomorphism of invertible sheaves

$$\mathcal{O}_{\mathbb{P}^n}(d) \simeq \mathcal{L}_D$$

As a 'reality check', note that the 'local sections' $F_i \in \mathcal{O}_{\mathbb{P}^n}(U_i)$ (corresponding to $1 \in \mathcal{L}_D(U_i)$) satisfy

$$F_j = g_{i,j}F_i$$

therefore they glue to a global section $s = F$.

21. Cartier divisors

Let X be a scheme. Recall that we defined the sheaf of total quotients $\mathcal{K}_X$ by $\mathcal{K}_X(U) = R_P$, where $U = \mathrm{Spec}(R) \subset X$ is open affine and $P \subset R$ is the subset of non-zero divisors in R.

Example 21.1. *The following are* **homework** *exercises:*

- *If X is integral then $\mathcal{K}_X$ is a constant sheaf: for any $U \subset X$ open, $\mathcal{K}_X(U) = K$, where K is the fraction field of any open affine subset of X.*
- *$\mathcal{K}_{\mathbb{A}^n} = k(x_1, \dots, x_n)$;*
- *$\mathcal{K}_{\mathbb{P}^n} = \{\frac{P(x_0,\dots,x_n)}{Q(x_0,\dots,x_n)}\}$ where P, Q are homogeneous polynomials and* $\deg P = \deg Q$.

In what follows we will need the following notion. Let X be a scheme and $U = \mathrm{Spec}(R) \subset X$ open affine. Define $K(U)$ to be the localization R_S, where S is the multiplicative set of non-zero divisors. These form a presheaf of rings on X, and we denote by $\mathcal{K}_X$ its sheafification. It is called the **sheaf of total quotient rings**.

denotes the sheaf of total quotient rings. We denote by $\mathcal{O}_X^*$, respectively $\mathcal{K}_X^*$, the sheaves of multiplicative groups given by the invertible elements in $\mathcal{O}_X$ and $\mathcal{K}_X$.

Definition 21.1.

- *A* **Cartier divisor** *on X is a global section of $\mathcal{K}_X^*/\mathcal{O}_X^*$. In other words, a Cartier divisor is given by a collection (U_i, f_i) where U_i cover X, $f_i \in \mathcal{K}(U_i)^*$, and $f_i/f_j \in \mathcal{O}_X(U_i \cap U_j)^*$.*
- *We say that a Cartier divisor is* **principal** *if there exists a global section $f \in \mathcal{K}_X(X)^*$ such that each f_i is the restriction of f. (Since the restrictions have to agree in $\mathcal{K}^*/\mathcal{O}^*$, this means that $f|_{U_i} = cf_i$ where $c \in \mathcal{O}^*(U_i)$.)*
- *Two Cartier divisors (U_i, f_i) and (U_i, g_i) are* **linearly equivalent** *if $(U_i, f_i/g_i)$ is principal.*

By convention, we use additive notation for operations on Cartier divisors. For example, if D_1, D_2 are given by (U_i, f_i) and (U_i, g_i), then $D_1 + D_2$ is given by $(U_i, f_i \cdot g_i)$, while $D_1 - D_2$ is given by $(U_i, f_i/g_i)$.

Example 21.2. *The following are* **homework** *exercises:*

- *Any Cartier divisor in $\mathbb{A}^n$ is principal.*
- *If $F, G \in k[x_0, \dots, x_n]$ are homogeneous polynomials such that* $\deg P = \deg Q$ *then the dehomogenization of F/G over each of the standard affines induces a principal divisor.*
- *If $F \in k[x_0, \dots, x_n]$ is a homogeneous polynomial, then its dehomogenization over each standard affine is a Cartier divisor, but it is not principal unless $F \equiv c$ (a constant).*

Consider now a Cartier divisor $D = (U_i, f_i)$. To it we associate an invertible subsheaf $\mathcal{L}_D \hookrightarrow \mathcal{K}_X$ by

$$\mathcal{L}_F(U_i) = \frac{1}{f_i}\mathcal{O}_X(U_i) \subset \mathcal{K}_X(U_i).$$

(I.e., this is the rank 1 module over $\mathcal{O}_X$ generated by $1/f_i$.) As before the transition function is given by

$$g_{i,j} = \phi_j \circ (\phi_i^*)^{-1}(1) = \phi_j\left(\frac{1}{f_i}\right) = \phi_j\left(\frac{f_j}{f_i}\cdot\frac{1}{f_j}\right) = \frac{f_j}{f_i}.$$

We need the following lemma.

Lemma 21.3. *Let R be a ring and $T \subset R$ the subset of non-zero divisors. Let M be a free rank 1 R-module generated by an element m_0. Assume that there is an injective R-module homomorphism $\varphi : M \to R_T$ and let $\varphi(m_0) = \frac{a}{t}$. Then $\varphi(m_0)$ is invertible.*

Proof. It suffices to show that $a \in T$. If not, take $r \in R \setminus 0$ such that $ra = 0$. Then $\varphi(rm_0) = 0$, contradicting injectivity. □

Proposition 21.4. *Let X be a scheme.*

(a) There is a one-to-one correspondence $D \leftrightarrow \mathcal{L}_D$ between Cartier divisors and invertible $\mathcal{O}_X$-modules which are subsheaves of $\mathcal{K}_X$.

(b) $\mathcal{L}_{D_1-D_2} \simeq \mathcal{L}_{D_1} \otimes \mathcal{L}_{D_2}^{-1}$.

(c) D_1-D_2 is a principal Cartier divisor if and only if $\mathcal{L}_{D_1} \simeq \mathcal{L}_{D_2}$ as abstract invertible sheaves, i.e., not as subsheaves of $\mathcal{K}_X$.

Proof. To prove (a), we need to show that we can recover the Cartier divisor D out of the invertible sheaf $\mathcal{L}_D$ together with its embedding in $\mathcal{K}_X$. Take f_i to be the inverse of a local generator of $\mathcal{L}_D(U_i) \subset \mathcal{K}_X(U_i)$ as a $\mathcal{O}_X(U_i)$-module. Two such choices must generate the same module, therefore they must differ by an element in $\mathcal{O}_X(U_i)^*$. As such, they give the same element of $\mathcal{K}_X(U_i)^*/\mathcal{O}_X(U_i)^*$, and thus the same Cartier divisor.

Part (b) follows from the formula for the transition functions. Finally, to prove (c), we utilize part (b). Then it suffices to show that D is principal if and only if $\mathcal{L}_D \simeq \mathcal{O}_X$. If D is principal, then it is given by a global section $f \in \mathcal{K}_X(X)^*$. This global section generates $\mathcal{L}_D$, in the sense that its restriction to each U_i generated $\mathcal{L}_D(U_i)$. Then we obtain a global isomorphism $\mathcal{O}_X \to \mathcal{L}_D$ by sending $1 \mapsto f^{-1}$. Conversely, assume that we are given an isomorphism $\mathcal{O}_X \to \mathcal{L}_D$ and consider the image of $1 \in \mathcal{O}_X(X)$. This must be a global section f in $\mathcal{L}_D$, Since $\mathcal{L}_D$ is a subsheaf of $\mathcal{K}_X$, f must be an invertible element in $\mathcal{K}_X(X)^*$, and this element determines a principal divisor on X. □

There are many common situations when every invertible sheaf on a scheme X is a subsheaf of $\mathcal{K}_X$. Some examples are:

- X is integral (see [Har77, Prop. II.6.15]);
- X is projective over a field.

In all these situations, the (additive) group of Cartier divisors on X modulo equivalence is isomorphic to the (multiplicative) group of invertible sheaves. This is called the **Picard group** of X, denoted by $\mathrm{Pic}(X)$.

One can prove that $\mathrm{Pic}(\mathbb{P}^n) = \mathbb{Z}$ (generated by $\mathcal{O}_{\mathbb{P}^n}(1)$), and that $\mathrm{Pic}(\mathbb{A}^n) = 0$.

Definition 21.2. *Let $D = (U_i, f_i)$ be a Cartier divisor. We say that D is* **effective** *if the 'local equations' are regular functions, i.e., $f_i \in \mathcal{O}_X(U_i)$. For an effective Cartier divisor D one may associate a codimension 1 subscheme $Y \subset X$ given locally by the equations $f_i = 0$. Sometimes we denote by $\mathcal{O}_X(D)$, respectively by $\mathcal{O}_X(-D)$, the invertible sheaf corresponding to an effective divisor D, respectively $-D$ (the inverse of D).*

If D is an effective divisor, then $\mathcal{O}_X(D)$ has a canonical global section s_D, which under the isomorphism $\mathcal{O}_X(D)|_{U_i} = \frac{1}{f_i}\mathcal{O}_X(U_i)$ is defined locally by the equations $(s_D)_{U_i} = f_i \in \mathcal{O}_X(U_i)$. Indeed,

$$f_j = \frac{f_j}{f_i} f_i = g_{i,j} f_i$$

showing that this satisfies the required glueing properties.

Example 21.5. *If $X = \mathbb{P}^n$ and F is a homogeneous polynomial of degree d, it determines a Cartier divisor $D = (U_i, \frac{F}{x_i^d})$. The corresponding line bundle $\mathcal{L}_D$ has a section $s_D = F$, with vanishing locus $V_p(F) \subset \mathbb{P}^n$.*

The following follows from definitions:

Corollary 21.6. *Let D be an effective Cartier divisor on a scheme X which determines the codimension 1 subscheme Y. Then $\mathcal{O}_X(-D) \simeq \mathcal{I}_Y$.*

Definition 21.3. *Let $D = (U_i, f_i)$ be a Cartier divisor on a scheme X. The* **support** *of D, denoted by* $\mathrm{supp}(D)$, *consists of those points $x \in X$ with the property that the image of f_i in the local ring $(\mathcal{O}_{X,x}, \mathfrak{m}_x)$ is in $\mathfrak{m}_x$.*

One can show that the support of a closed subset of X. If $U = X \setminus \mathrm{supp}(D)$, then $\mathcal{O}_X(D)$ has a section $s_D \in \mathcal{O}_X(D)(U)$ which does not vanish anywhere. This section extends to a section of $\mathcal{O}_X(D)$ which is 'meromorphic', with poles on the locus where D is not effective.

Often one can write $f_i \in \mathcal{K}_X(U_i)^*$ as $f_i = \frac{a_i}{b_i}$ where $a_i, b_i \in \mathcal{O}_X(U_i)^*$ and such that $A = (U_i, a_i)$ and $B = (U_i, b_i)$ are Cartier divisors. Then the Cartier divisor $D = (U_i, f_i)$ may be written as a difference of two effective divisors: $D = A - B$. This will allow us to define the first Chern class

$$c_1(\mathcal{L}_D) = c_1(\mathcal{L}_A) - c_1(\mathcal{L}_B) = [V(s_A)] - [V(s_B)]$$

where s_A, s_B are the canonical sections.

Definition 21.4. *Let D be a Cartier divisor on a scheme X. A* **meromorphic section** *of $\mathcal{O}_X(D)$ is any global section of $\mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{K}_X^*$. In other words, a meromorphic section is not required to have local equations as regular functions in $\mathcal{O}_X(U_i)$, but in a localization of it.*

From definition, it follows that $s_D = (f_i)$ is a meromorphic section of $\mathcal{O}_X(D)$. The divisor D is effective precisely when s_D is a ('holomorphic') section of $\mathcal{O}_X(D)$.

22. Rational equivalence and the Chow group

Recall that we use *variety* to mean a reduced and irreducible scheme.

22.1. **Rational equivalence.** Let X be a variety and $V \subset X$ a codimension 1 subvariety. Consider $A := \mathcal{O}_{X,V}$, the localization at the generic point of V. There is a well defined group homomorphism

$$\mathrm{ord}_V : \mathcal{K}_X(X)^* \to \mathbb{Z}$$

called **the order of vanishing**. It satisfies $\mathrm{ord}(rs) = \mathrm{ord}(r) + \mathrm{ord}(s)$. If we write $r \in \mathcal{K}_X(X)^*$ as $r = \frac{a}{b}$ with $a, b \in A$, then

$$\mathrm{ord}\left(\frac{a}{b}\right) = \mathrm{ord}(a) - \mathrm{ord}(b).$$

An important example is when X is smooth at V, i.e. if $(\mathcal{O}_{X,V}, \mathfrak{m}_V)$ is the local ring, then $\dim \mathfrak{m}_V/\mathfrak{m}_V^2 = 1$. In this case $\mathcal{O}_{X,V}$ is a discrete valuation ring (DVR) and $\mathfrak{m}_V$ is principal. Then for any $r \in \mathcal{K}_X(X)^*$, we can write (its image) as $r = at^m$ where $a \in \mathcal{O}_{X,V}^*$. In this case

$$\mathrm{ord}_V(r) = m.$$

More generally the order function may be defined as the length $\ell_A(A/\langle r\rangle)$ of the A-module $A/\langle r\rangle$; see [Ful84, App. A].

Lemma 22.1. *For a fixed $r \in \mathcal{K}_X(X)^*$, there are only finitely many V's such that* $\mathrm{ord}_V(r) \neq 0$.

Proof. See [Ful84, App. B]. □

Example 22.2. *Assume X is a smooth curve. Then $\mathcal{O}_{X,V} = \mathcal{O}_{X,x}$ are local rings at the points in X, and the image of r in $\mathcal{O}_{X,x}$ corresponds to the evaluation at x. For all but finitely many points, this evaluation is non-zero.*

Example 22.3. *Let $X = \mathbb{P}^2$ and $r = \frac{x_1}{x_0}$. Let $H_i = V_p(x_i) \subset \mathbb{P}^2$ be the standard hyperplanes. Then $\mathcal{O}_{X,H_i} = \{\frac{P(x_0,x_1,x_2)}{Q(x_0,x_1,x_2)} : \deg P = \deg Q, Q \notin \langle x_i\rangle\}$ and the maximal ideal is generated by x_i. It follows that*

$$\mathrm{ord}_{H_0}(r) = -1; \quad \mathrm{ord}_{H_1}(r) = 1; \quad \mathrm{ord}_{H_2}(r) = 0.$$

We now define the analogues of the fundamental classes from topology.

Definition 22.1. *Let X be a scheme. A k-cycle is a finite formal combination $\sum n_i[V_i]$ where $V_i \subset X$ are irreducible subvarieties of dimension k. A k-cycle is* **effective** *if each coefficient $n_i \geq 0$.*

We denote this by $\sum n_i[V_i] \geq 0$. The (additive) group of k-cycles is denoted by $Z_k(X)$.

Let $r \in \mathcal{K}_X(X)^*$ and $W \subset X$ a $k+1$ dimensional subvariety. There is an associated cycle in $Z_k(X)$ defined by

$$\mathrm{div}(r) = \sum_{V \subset W} \mathrm{ord}_V(r)[V],$$

where the sum is over subvarieties V of dimension k.

Example 22.4. *An important example of this construction is when X is smooth in codimension 1, i.e. every codimension 1 subscheme is locally given by a single equation. (In technical terms, 'every Weyl divisor is Cartier'.) Then every $r \in \mathcal{K}_X(X)^*$ determines a Cartier divisor*

$$\operatorname{div}(r) = \sum_{Y \subset X} \operatorname{ord}_Y(r)[Y] \in \operatorname{Pic}(X),$$

where the sum is over codimension 1 subschemes Y. For a Cartier divisor D one defines:

$$H^0(X, \mathcal{O}_X(D)) = \{f \in \mathcal{K}_X(X)^* : \operatorname{div}(f) + D \geq 0\},$$

i.e. the vector space of meromorphic sections with 'poles of order at most the order of D'. If D is an effective divisor, then $H^0(X, \mathcal{O}_X(D)) = \mathcal{O}_X(D)(X)$ (the global sections of $\mathcal{O}_X(D)$). Studying this vector space is closely related to the Grothendieck-Riemann-Roch theorem.

Definition 22.2. *A k-cycle α is* **rationally equivalent to** 0 *if there exist finitely many subvarieties W_i with $\dim W_i = k+1$ and elements $r_i \in \mathcal{K}_{W_i}(W_i)^*$ such that*

$$\alpha = \sum [\operatorname{div}(r_i)].$$

The (k-th) **Chow group** *is defined as*

$$A_k(X) := Z_k(X)/B_k(X)$$

where $B_k(X)$ is the subgroup of cycles rationally equivalent to 0. The Chow group is defined to be

$$A_*(X) = \oplus_{k \geq 0} A_k(X).$$

Example 22.5. *A scheme X and its induced reduced structure X_{red} have the same subvarieties, therefore $A_*(X) = A_*(X_{red})$.*

Example 22.6. *Assume $X = \bigcup X_i$ is a union of irreducible schemes. Then $Z_k(X) = \oplus Z_k(X_i)$ and $A_k(X) = \oplus A_k(X_i)$. (***homework!***)*

Example 22.7. *Assume $\dim X = n$. Then $A_n(X) = Z_n(X)$ is the free abelian group on the n-dimensional irreducible components of X.*

23. Proper push-forward and flat pull-back

23.1. **Proper push-forward.** Recall from Definition 18.8 the definition of a proper morphism. Examples of proper morphisms are closed embeddings, morphisms of projective varieties, and base changes of proper morphisms.

If $f : X \to Y$ is a proper morphism of schemes, then for any (closed) subvariety $V \subset X$, the image $f(V) \subset Y$ is closed. Since the restriction $f : V \to W := f(V)$ is a dominant map of irreducible schemes, it induces an injective map $f^* : \mathcal{K}_W \hookrightarrow \mathcal{K}_V$.

Lemma 23.1. *The field extension $\mathcal{K}_W \hookrightarrow \mathcal{K}_V$ is finite if and only if $\dim V = \dim W$. If this is the case, then $f : V \to W$ is generically finite.*

If, furthermore, k is algebraically closed, then the general fibre is a zero dimensional scheme with $\deg(\mathcal{K}_V : \mathcal{K}_W)$ reduced points.

Proof. Refer to [Ful84, App. B2]. □

Define

$$\deg(V/W) = \begin{cases} \deg(\mathcal{K}_V : \mathcal{K}_W) & \text{if } \dim V = \dim W; \\ 0 & \text{if } \dim W < \dim V. \end{cases}$$

Define $f_* : Z_k(X) \to Z_k(Y)$ by

$$f_*[V] = \deg(V/f(V))[f(V)], \tag{23.1}$$

extended by linearity. A fundamental result is that rational equivalence is preserved by taking proper push-forward.

Theorem 23.1. *Let $f : X \to Y$ be a proper morphism of schemes, and let $\alpha \in A_k(X)$ such that $\alpha \simeq 0$. Then $f_*(\alpha) \simeq 0$.*

In particular, Equation (23.1) gives a well-defined push-forward (group homomorphism) $f : A_k(X) \to A_k(Y)$.

A key particular case is when X is a complete scheme over a field k. Then by definition the morphism $p : X \to \operatorname{Spec}(k)$ is proper, and there is a **degree map**

$$\deg = \int_X : A_0(X) \to \mathbb{Z}; \quad \alpha \mapsto p_*(\alpha).$$

If k is algebraically closed and $\alpha = \sum n_P[P] \in A_0(X)$ then $p_*[P] = [p(P)]$; in general $p_*[P] = \int_X [P] = \deg(K(P) : k)[p(P)]$.

One may extend the degree morphism to $\int_X : A_*(X) \to \mathbb{Z}$ by sending $\alpha \mapsto 0$ for $\alpha \in A_i(X)$, $i > 0$. Observe also that if $f : X \to Y$ is a proper morphism and X, Y are proper, then

$$\int_X \alpha = \int_Y f_*(\alpha).$$

(This is a special case of functoriality, and an analogue of the chain rule from calculus.)

Example 23.2. *Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ define by $[x_0 : x_1] \mapsto [x_0^2 : x_1^2]$. Then*

$$f_*[\mathbb{P}^1] = 2[\mathbb{P}^1]; \quad f_*[pt] = [pt].$$

Example 23.3 (Gauss-Bonnet; assumes knowledge of Chern classes)**.** *Let X be a complex, projective manifold, with tangent bundle T_X. Then*

$$\int_X c(T_X) = \chi(X),$$

the Euler characteristic of X. Here $c(T_X)$ is interpreted as a non-homogeneous class in $A_*(X)$.

Example 23.4 (Enumerative invariants; assumes knowledge of the intersection product)**.** *Let X be a complex projective manifold, and let $M_d(X)$ be the 'moduli space of stable maps' which compactifies the (scheme!) of morphisms $f : \mathbb{P}^1 \to X$ of some fixed class $f_*[\mathbb{P}^1] = d \in A_1(X)$. Let $\mathrm{ev}_i : M_d(X) \to X$ be the evaluation maps, obtained by evaluating at $0, 1, \infty$. For subvarieties $\Omega_1, \Omega_2, \Omega_3 \subset X$ define*

$$\langle \Omega_1, \Omega_2, \Omega_3 \rangle_d = \int_X [\mathrm{ev}_1^{-1}\Omega_1] \cdot [\mathrm{ev}_2^{-1}\Omega_2] \cdot [\mathrm{ev}_3^{-1}\Omega_3].$$

These count the (virtual) number of curves $f : \mathbb{P}^1 \to X$ of degree d that pass through subvarieties $\Omega_1, \Omega_2, \Omega_3$; **Gromov-Witten invariants** *are examples.*

23.2. **Fundamental class of a subscheme.** We defined the group $Z_k(X)$ of cycles on a scheme X as formal combinations of classes of irreducible subvarities. However, we we need to defined analogues of fundamental classes for possibly reducible, possibly non-reduced schemes. The construction below provides this construction.

Let X be a pure dimensional scheme with irreducible components $X_1, \dots, X_r$. The local rings $\mathcal{O}_{X,X_i}$ have dimension 0. Define the **geometric multiplicity** of X_i is defined by $m_i = \ell_{\mathcal{O}_{X,X_i}}(\mathcal{O}_{X,X_i})$. If k is algebraically closed, then

$$m_i = \dim_k \mathcal{O}_{X,X_i}.$$

The cycle associated to X, or the **fundamental class of** X is

$$[X] = \sum m_i[X_i] \in A_{\dim X}(X).$$

More generally, if $X \subset Y$ is closed in some larger scheme Y, then we regard $[X]$ as a class in $A_*(Y)$, via the natural closed embedding.

23.3. **Flat pull-back.** It is known from commutative algebra (cf. e.g., [AM69, Prop.2.18]) that if $M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence of R-modules, then for any R-module N, $M_1 \otimes N \to M_2 \otimes N \to M_3 \otimes N \to 0$ remains exact.

Definition 23.1. *An R-module N is* **flat** *if for any short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$, the sequence*

$$0 \to M_1 \otimes N \to M_2 \otimes N \to M_3 \otimes N \to 0$$

remains exact.

Definition 23.2. *Let $f : X \to Y$ be a morphism of pure-dimensional schemes. The morphism f is* **flat** *if for any open affine sets $U \subset X, V \subset Y$ such that $f(U) \subset V$, $\mathcal{O}_X(U)$ is a flat $\mathcal{O}_Y(V)$-module.*

We say that f has **relative dimension** *n if for any subvariety $Z \subset Y$, $f^{-1}(Z)$ is pure dimensional of dimension $\dim Z + n$.*

Theorem 23.2. *Assume $f : X \to Y$ is flat and that Y is irreducible and X is pure dimensional. Then $f : X \to Y$ has relative dimension $n := \dim X - \dim Y$, and all base extensions $X \times_Y Y' \to Y'$ are flat of relative dimension n.*

Proof. See [Har77, III.9], especially Cor. 9.6. □

Example 23.5. *Here are some examples of flat morphisms:*

- *An open embedding $i : U \hookrightarrow X$ where $U \subset X$ is open.*
- *The projection from a vector bundle $\pi : E \to X$, or from the projectivization of a vector bundle $\mathbb{P}(E) \to X$.*
- *The projection $X \times Y \to X$.*
- *Any dominant morphism $X \to C$ from an irreducible subvariety of dimension $\dim X \geq 1$ to a non-singular curve.*

Let $f : X \to Y$ be a flat morphism of relative dimension n. Define a pull back

$$f^* : Z_k(Y) \to Z_{k+n}(X), \quad [W] \mapsto [f^{-1}(W)].$$

We extend this definition by linearity.

For any closed subscheme, let $f^{-1}(Z)$ denote its inverse image, defined as the fibre product

$$\begin{tikzcd} X \times_Y Z \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Z \arrow[r] & Y \end{tikzcd}$$

Theorem 23.3. *Let $Z \subset Y$ be any closed subscheme. Then there is a well defined pull-back homomorphism $f^* : A_k(Y) \to A_{k+n}(X)$ defined by:*

$$f^*[Z] = [f^{-1}(Z)] \in A_*(X).$$

Proof. See [Ful84, Theorem 1.7]. □

Proposition 23.6. *Consider a fibre diagram*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

where g is flat and f is proper. Then g' is flat, f' is proper and for any $\alpha \in Z_*(X)$,

$$f'_*(g')^*(\alpha) = g^* f_*(\alpha) \in Z_*(Y').$$

23.4. **An exact sequence.**

Proposition 23.7. *Let Y be a closed subscheme of a scheme X and let $U := X \setminus Y$. Denote by $i : Y \to X$ and $j : U \to X$ the inclusions. Then we have a short exact sequence*

$$A_k(Y) \xrightarrow{i_*} A_k(X) \xrightarrow{j^*} A_k(U) \longrightarrow 0 .$$

Proof. The analogous sequence of cycles is exact:

$$Z_k(Y) \xrightarrow{i_*} Z_k(X) \xrightarrow{j^*} Z_k(U) \longrightarrow 0 ,$$

showing surjectivity. It is also clear from definitions that $j^*i_*(\beta) = 0$. It remains to show that any $\alpha \in \ker(j^*)$ is in the image of i_*. Write $j^*(\alpha) = \sum div(r_i)$ for some $W_i \subset U$ subvarieties. By taking closures of W_i, and observing that $K(W_i) = K(\overline{W_i})$, we obtain

$$j^*(\alpha - \sum div(\overline{r}_i)) = 0 \quad \in Z_k(U),$$

where $\overline{r}_i$ are extensions of r_i to $\overline{W_i}$. Then $\alpha - \sum div(\overline{r}_i) = i_*(\beta)$ for some $\beta \in Z_k(X)$, and we are done. ∎

24. Chern classes of line bundles

In this section we will define the first Chern class of a line bundle $\mathcal{L}$ over a scheme X, as an operator $c_1(\mathcal{L}) : A_k(X) \to A_{k-1}(X)$. The idea is the following. Take $V \subset X$ be a k-dimensional subvariety. The restriction of $\mathcal{L}$ to V determines a Cartier divisor $D_V = \{(U_i, f_i)\}$, up to linear equivalence. Then

$$c_1(\mathcal{L}) \cap [V] = [D_V].$$

Of course, one needs to prove that this construction is independent of choices of representatives in a linear equivalence class of Cartier divisors.

One can actually give a stronger statement in the case $\mathcal{L}$ is given by a Cartier divisor D on X. In this case, the class is supported on $A_{k-1}(V \cap |D|)$ where $|D|$ is the support of D. A key technical point is that occasionally $V \subset |D|$. To deal with the situation when $\mathcal{L}$ restricted to V one needs to

24.1. **Cartier divisors, Weil divisors, pseudodivisors.**

Assume for now that X is a *variety* of dimension n. (Later it will be a scheme.) In particular, there is a well defined function field $K(X)$. A **Weil divisor** is an element of $Z_{n-1}(X)$.

If $D = \{(U_i.f_i)\}$ is a Cartier divisor, and $V \subset X$ is a subvariety then one defines

$$\mathrm{ord}_V(D) := \mathrm{ord}_V(f_i).$$

This is well defined since f_i/f_j is invertible on the overlaps. Using this, the Cartier divisor defines a Weyl divisor

$$[D] = \sum_V \mathrm{ord}_V(D)[V],$$

where the sum is over all codimension 1 subvarieties $V \subset X$. Observe that if two Cartier divisors $D \sim D'$ are linearly equivalent then $D - D'$ is a principal divisor, i.e., $D' - D = \mathrm{div}(f)$ for some $f \in K(X)$. But then the associated Weyl divisors $[D']$ and $[D]$ are rationally equivalent (by definition). This implies that we have a group homomorphism

$$CaCl(X) \to A_{n-1}(X); \quad D \mapsto [D].$$

Recall that if $D = \{(U_i, f_i)\}$ is a Cartier divisor, its **support** is defined to be

$$|D| = \{x \in X : \text{ some } f_i \text{ is not a unit}\}$$

(Observe that f_i is in general a rational function, i.e. an element of $\mathcal{K}(X)^*$. The invertible elements $\mathcal{O}^*_{X,x} \subset \mathcal{K}(X)^*$ are a subset of this. Therefore it makes sense to ask whether f_i is in the smaller subset, although f_i may not have a well defined image in $\mathcal{O}_{X,x}$.) The support is a closed subset of X.

Also recall that D is **effective** if each $f_i \in \mathcal{O}_X(U_i)$ and it is not a zero divisor.

Definition 24.1 (Pseudovisors)**.** *Let X be a scheme. A* **pseudodivisor** *on X is a triple (L, Z, s) where L is a line bundle, $Z \subset X$ is closed and s is a section of L over $X \setminus Z$*

which does not vanish. Equivalently, s provides an isomorphism (i.e., a trivialization) $L|_{X\setminus Z} \simeq (\mathcal{O}_X)|_{X\setminus Z}$.

The data (L', Z', s') defines the same pseudodivisor if $Z = Z'$ and there is an isomorphism $\sigma : L \simeq L'$ such that the restriction $\sigma_{|X\setminus Z}$ sends s to s'.

We call L, Z, s the line bundle, support, *respectively the* section *associated to (L, Z, s).*

According to this definition, a pseudodivisor (L, X, s) with support X is simply an equivalence class of line bundles.

The pseudodivisors form an abelian group with the following operations:

$$(L, Z, s) + (L', Z', s') = (L \otimes L', Z \cup Z', s \otimes s'); \quad -(L, Z, s) = (L^{-1}, Z, 1/s).$$

If $f : X' \to X$ is a morphism, then one can pull-back a pseudodivisor by

$$f^*(L, Z, s) = (f^*(L), f^{-1}(Z), f^*(s)).$$

These operations agree with those involving Cartier divisors, whenever objects are defined.

An important source of pseudodivisors comes from Cartier divisors. Recall (cf. Proposition 21.4 above) that a Cartier divisor D determines a line bundle $\mathcal{L}_D$. Then D determines a pseudodivisor $(\mathcal{L}_D, |D|, s_D)$ where $|D|$ is the support of D, and $s_D = (f_i)$ is the canonical *meromorphic* section of D; cf. Definition 21.4 and after. The section s_D is nowhere vanishing outside the support $|D|$.

Definition 24.2. *A Cartier divisor D* **represents** *a pseudodivisor (L, Z, s) if $|D| \subset Z$ and there is an isomorphism $L \simeq \mathcal{L}_D$ such that, outside Z, sends $s \mapsto s_D$.*

For example, a Cartier divisor D represents any pseudodivisor $(\mathcal{L}, X, \emptyset)$ where $\mathcal{L} \simeq \mathcal{L}_D$.

The main result about pseudodivisors is the following:

Theorem 24.1. *Let X be a variety. Then any pseudodivisor (L, Z, s) is represented by a Cartier divisor. Furthermore,*

- *If $Z \neq X$, then D is uniquely determined.*
- *If $Z = X$, then D is determined only up to linear equivalence.*

Proof. Later, if there is time; see §2.2, [Ful84]. ☐

The following is the main definition underlying the construction of Chern classes via intersection products.

Definition 24.3. *Let $\tilde{D} = (L, |D|, s)$ be a pseudodivisor on an n-dimensional variety X. The* **associated Weyl divisor class**

$$[D] \in A_{n-1}(X)$$

is defined as follows. Take any Cartier divisor D which represents $\tilde{D}$.

- *If $|D| \neq X$ then by Theorem 24.1 D is uniquely defined, and $[D]$ is the associated Weyl divisor, regarded in $A_{n-1}(|D|)$.*
- *If $|D| = X$ then D is only defined up to linear equivalence, but the associated Weyl divisor $[D] \in A_{n-1}(X)$ is well defined.*

24.2. **Intersections by divisors and the first Chern class.** We are now ready to define intersections by divisors and the first Chern class of a line bundle. Assume X is a scheme, $\tilde{D} = (L, |D|, s)$ a pseudivisor on X, and $V \subset X$ a k-dimensional subvariety with inclusion $j :\hookrightarrow X$. Then define an **intersection product**

$$\tilde{D}.[V] := [j^*\tilde{D}] \in A_{k-1}(V)$$

In other words:

- If $V \not\subset |D|$ then D restrict to a unique Cartier divisor D_V on V, and $D.[V] = [D_V]$;
- If $V \subset |D|$, then restrict $\mathcal{L}_D$ to V, and take any Cartier divisor D_V corresponding to this restriction. Then $D.[V] = [D_V]$.

One extends this product by linearity to define an intersection product

$$D \cdot \alpha \in A_{k-1}(X)$$

for any $\alpha \in A_k(X)$.

Definition 24.4 (First Chern class)**.** *Let $L \to X$ be a line bundle over a scheme X. Define the group homomorphism $c_1(L) \cap \_ : A_k(X) \to A_{k-1}(X)$ by*

$$c_1(L) \cap [V] = [C],$$

where C is the Weil divisor associated to the line bundle $L|_C$.

If $L = \mathcal{L}_D$ is the line bundle associated to a pseudo-divisor D, then $c_(L) \cap [V] = D \cdot [V]$. We will list formal properties of the Chern classes later ADDREF, once we define Chern classes of any order; those would be essentially the same as those defined in topology.

Example 24.1. *A homogeneous polynomial $F \in k[x_0, \dots, x_n]$ of degree d determines a Cartier divisor with corresponding line bundle $\mathcal{O}_{\mathbb{P}^n}(d)$. Since the Cartier divisor associated to F is linearly equivalent to that for any x_i^d, it follows that*

$$c_1(\mathcal{O}_{\mathbb{P}^n}(d)) = dc_1(\mathcal{O}_{\mathbb{P}^n}(1)).$$

Furthermore, if $V \subset \mathbb{P}^n$ is a subvariety such that $F|_V \not\equiv 0$,

$$c_1(\mathcal{O}_{\mathbb{P}^n}(d)) \cap [V] = [V \cap V_p(F)]. \tag{24.1}$$

Example 24.2 (Chow group of $\mathbb{P}^n$)**.** *One may use Proposition 23.7 to show that $A_k(\mathbb{P}^n)$ is generated by $[L_k]$, the fundamental class of a k-dimensional plane. We prove that $[L_k] \neq 0$ in $A_k(\mathbb{P}^n)$. From* (24.1) *it follows that*

$$c_1(\mathcal{O}_{\mathbb{P}^n}(1))^k \cap [L_k] = 1.[pt] \in A_0(\mathbb{P}^n).$$

One can use again the sequence from Proposition 23.7 to get that $[pt] \neq 0$. (Remove hyperplanes to reduce to $A_0(pt) = \mathbb{Z}$.) This implies that $[L_k] \neq 0$.

Example 24.3 (Degrees)**.** *Let $\alpha \in A_k(\mathbb{P}^n) \simeq \mathbb{Z}$. The* **degree** *of α is*

$$\deg(\alpha) = \int_{\mathbb{P}^n} c_1(\mathcal{O}_{\mathbb{P}^n}(1))^k \cap \alpha.$$

Let D be an **effective** Cartier divisor. We abuse notation and denote by $D \subset X$ the associated codimension 1 subscheme (given by the local equations of the Cartier divisor). Let $i : D \hookrightarrow X$ be the (closed) embedding.

Definition 24.5 (Gysin maps for divisors)**.** *Define*

$$i^* : A_k(X) \to A_{k-1}(D); \quad [V] \mapsto D \cdot \alpha;$$

(intersection product with the divisor D).

Note that i^* is arrow reversing, although i is not flat. These types of (arrow reversing) morphisms in a 'homology theory' are called **Gysin morphisms** in topology.

We list some useful properties of this pull-back, including a version of the self-intersection formula.

Proposition 24.4. *The morphism i^* satisfies the following properties:*

(a) For any $\alpha \in A_k(X)$,

$$i_* i^*(\alpha) = c_1(\mathcal{L}_D) \cap \alpha.$$

(b) (The self-intersection formula for divisors) If $\alpha \in A_k(D)$, then

$$i^* i_*(\alpha) = c_1(i^*\mathcal{L}_D) \cap \alpha.$$

(Note that, at least if D, X are smooth, then $i^*\mathcal{L}_D$ is the normal bundle of D in X; cf. Proposition 19.9.)

(c) If X is pure-dimensional, then $i^*[X] = [D]$.

(d) For any line bundle $L \to X$,

$$i^*(c_1(L) \cap \alpha) = c_1(i^*L) \cap i^*(\alpha).$$

Example 24.5 (Bèzout)**.** *Let $Z_1 = V_p(F_1), Z_2 = V_p(F_2) \subset \mathbb{P}^2$ be plane curves given by homogeneous polynomials of degree d_1 respectively d_2. Both Z_1 and Z_2 are Cartier divisors. We can utilize the intersection with divisors product to calculate $[Z_1] \cdot [Z_2]$. Since $[Z_1] = d_1[H]$ and $[Z_2] = d_2[H]$ (H any hyperplane), it follows that*

$$\deg([Z_1] \cdot [Z_2]) = d_1 d_2 \int_{\mathbb{P}^2} [H]^2 = d_1 d_2.$$

25. Chern classes of vector bundles

We are now ready to define higher Chern classes. Let $E \to X$ be a vector bundle of rank $e+1$ over a scheme X. To it, we associate the **projective bundle** (or the **projectivization**) $\mathbb{P}(E) \to X$. This is defined as follows. If $E \simeq X \to \mathbb{A}^{e+1}$ is trivial, then $\mathbb{P}(E) = X \times \mathbb{P}^e$. In general, one glues these local schemes.

The projectivization is endowed with a projection $\pi : \mathbb{P}(E) \to X$, with the property that $\pi^{-1}(X) = \mathbb{P}(E_x)$, where E_x is the fibre of $E \to X$ over x. Of course, π is flat (this is a local property, so it suffices to check it locally). It is also proper, since $\mathbb{P}^n$ is proper, and locally it is a base change from $\mathbb{P}^n \to pt$.

Recall that over $\mathbb{P}^n$ there is a tautological sequence of vector bundles:

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1) \to \mathcal{O}_{\mathbb{P}^n}^{\oplus n+1} \to \mathcal{Q} \to 0.$$

The fibre of $\mathcal{O}_{\mathbb{P}^n}(-1)$ over a line $\ell \subset k^{n+1}$ is simply the line ℓ. By pulling back, then gluing, this gives a tautological sequence of vector bundles on $\mathbb{P}(E)$:

$$0 \to \mathcal{O}_E(-1) \to \pi^* E \to \mathcal{Q} \to 0.$$

(We abuse notation and denote the quotient bundle again by $\mathcal{Q}$.) The fibre of $\mathcal{O}_E(-1)$ over a point $[\ell_x] \in \mathbb{P}(E)$ corresponding to a line $\ell_x \subset E_x$ is ℓ_x.

Definition 25.1 (Segre classes of vector bundles)**.** *Let $i \geq 0$. The i-th* **Segre class** *is the group homomorphism $s_i(E) \cap \_ : A_k(X) \to A_{k-i}(X)$ defined by*

$$s_i(E) \cap \alpha = \pi_*(c_1(\mathcal{O}_E(1))^{e+i} \cap \pi^*(\alpha)).$$

Consider the formal power series

$$s_t(E) = \sum_{i \geq 0} s_i(E) t^i.$$

The **Chern polynomial** $c_t(E) = \sum_{i \geq 0} c_i(E) t^i$ is the (formal) inverse of this power series:

$$c_t(E) = s_t(E)^{-1}.$$

Explicitly,

$$c_0(E) = 1; \quad c_1(E) = -s_1(E); \quad c_2(E) = s_1(E)^2 - s_2(E), \ldots$$

Here are the formal properties of Chern classes (cf. Thm. 3.2 in [Fulton, IT] ADDREF.)

Theorem 25.1. *Let $E \to X$ be a vector bundle of rank e over a scheme X. The Chern classes satisfy the following properties.*

1. *(Vanishing) $c_i(E) = 0$ for $i > e$;*
2. *(Commutativity) $c_i(E) \cap (c_j(E) \cap \alpha) = c_j(E) \cap (c_i(E) \cap \alpha)$;*
3. *(Projection formula) For any proper morphism $f : Y \to X$,*

$$f_*(f^*(c_i(E)) \cap \alpha) = c_i(E) \cap f_*(\alpha).$$

(4) *(Pull-back) For any flat morphism* $f : Y \to X$,

$$c_i(f^*(E) \cap f^*(\alpha) = f^*(c_i(E) \cap \alpha).$$

(5) *(Whitney formula) Consider a short exact sequence* $0 \to F \to E \to G \to 0$ *be a short exact sequence of vector bundles on* X*. Then*

$$c_t(E) = c_t(F)c_t(G).$$

(6) *(Normalization) If* E *is a line bundle such that* $E \simeq \mathcal{O}_X(D)$ *for a Cartier divisor* D *on* X*, then*

$$c_1(E) \cap [X] = [D].$$

More generally, if E *has sections and* $s \in H^0(X; E)$ *is a general section with zero locus* $Z(s) \subset X$*, then*

$$c_e(E) \cap [X] = [Z(s)]. \tag{25.1}$$

The class

$$c(E) = 1 + c_1(E) + \ldots + c_e(E)$$

is called the *total Chern class* of E. It can also be regarded as the Chern polynomial $c_t(E)$ evaluated at $t = 1$.

25.1. **Chern roots and the splitting principle.** One can actually show that the properties above determine the Chern classes. The idea is as follows. The *splitting principle* states that if $E \to X$ is a vector bundle, then there exists a flat map $f : X' \to X$ such that:

- The pull back $f^* : H^*(X) \to H^*(X')$ is injective;
- The pull back bundle f^*E has a filtration

$$0 \subset E_1 \subset E_2 \subset \ldots \subset E_e = f^*E$$

  such that $E_i \to X'$ is a vector bundle on X' of rank i.

By the Whitney formula, it follows that

$$c(E_e) = c(E_1) \cdot c(E_2/E_1) \cdot \ldots \cdot c(E_e/E_{e-1}).$$

Since each quotient E_i/E_{i-1} is a line bundle, the normalization formula (applied only for line bundles) calculates each class $c(E_i/E_{i-1})$. Therefore the linear operator $c(f^*E)$ is calculated. Now by injectivity the class $c(E) \cap \alpha$ may be identified to $c(f^*E) \cap f^*(\alpha)$. (A subtlety: in the topological category one needs to ensure that $f^*(\alpha)$ is well defined. For instance this happens if the Poincaré dual of α exists, e.g. if X is non-singular. This is not needed if one works in the algebraic category, and uses Chow groups; in this case *flatness* ensures that pull-backs exist.) To construct $f : X' \to X$ with the required properties, one may take $X' = \mathrm{Fl}(E)$. This is the variety parametrizing complete flags of vector bundles $E_1 \subset E_2 \subset \ldots \subset E_e = E$, equipped with its natural projection to X. For instance, if $X = pt$, then E is just a vector space, and this is the complete flag variety $\mathrm{Fl}(E)$. For details, we refer to [Ful84, §3.2].

The splitting principle allows one to formally define the *Chern roots* of a vector bundle. If $rank(E) = e$, these are formal indeterminates $x_1, \ldots, x_e$ such that

$$c(E) = (1 + x_1)(1 + x_2) \cdot \ldots \cdot (1 + x_e) = \sum_{i \geq 0} e_i(x_1, \ldots, x_e).$$

Then the Chern class $c_i(E) = e_i(x_1, \ldots, x_e)$. The idea is that x_i's are only formal, but their symmetrizations are actual classes. By the splitting principle, one may actually identify $x_i = c_1(E_i/E_{i-1})$.

The Chern roots are very useful tools relating Chern classes of vector bundles. I will list several properties below - all follow from the judicious use of the splitting principle.

Lemma 25.1. *In all statements $x_1, \ldots, x_e$ are the Chern roots of E. Then the following hold:*

- *(Dual bundles) The Chern roots of the dual bundle $E^\vee$ are $-x_1, \ldots, -x_e$. In particular,*
$$c_i(E^\vee) = (-1)^i c_i(E).$$
- *(Tensor products) Let $F \to X$ be a vector bundle with Chern roots $y_1, \ldots, y_f$. Then the Chern roots of $E \otimes F$ are $x_i + y_j$, for $1 \leq i \leq e, 1 \leq j \leq f$.*
- *(Symmetric powers) The Chern roots of $Sym^p E$ are $x_{i_1} + x_{i_2} + \ldots + x_{i_p}$ where $1 \leq i_1 \leq i_2 \leq \ldots \leq i_p \leq e$.*
- *(Exterior powers) The Chern roots of $\bigwedge^p E$ are $x_{i_1} + x_{i_2} + \ldots + x_{i_p}$ where $1 \leq i_1 < i_2 < \ldots < i_p \leq e$.*

We illustrate this by some examples.

Example 25.2 (The Chern classes of the tangent bundle of $\mathbb{P}^n$)**.** *We use the Euler sequence*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} \longrightarrow T_{\mathbb{P}^n} \longrightarrow 0 \ .$$

to calculate the Chern classes of the tangent bundle on $\mathbb{P}^n$. Let $H = c_1(\mathcal{O}_{\mathbb{P}^n}(1)$. By the normalization property this is the class of a hyperplane in $\mathbb{P}^n$. From the Whitney formula we obtain that the total Chern class

$$c(\mathcal{O})c(T_{\mathbb{P}^n}) = c(\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} = c(\mathcal{O}_{\mathbb{P}^n}(1))^{n+1} = (1 + H)^{n+1}.$$

Since $c(\mathcal{O}) = 1$, it follows that

$$c(T_{\mathbb{P}^n}) = (1 + H)^{n+1} = \sum_{k=0}^{n} \binom{n+1}{k} H^k.$$

For instance, $c_n(T_{\mathbb{P}^n}) = (n + 1)[pt]$, reflecting the fact that the topological Euler characteristic of $\mathbb{P}^n$ is $n + 1$. (This is an instance of the Gauss-Bonnet theorem.)

Example 25.3 (The degree of $\mathrm{Gr}(2,4)$.)**.** *Consider the Grassmannian* $\mathrm{Gr}(2,4)$ *parametrizing linear subspaces of dimension* 2 *in* $\mathbb{C}^4$*. We have seen in a previous problem (Fall final exam!) that this is a quadric hypersurface in* $\mathbb{P}^5$*. More precisely, if*

$$A = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{pmatrix}$$

and if we denote by $p_{i,j}$ *the determinant of the* 2×2 *minor in columns* i, j*, then*

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0.$$

Let $D := \{V \in \mathrm{Gr}(2,4) : \dim V \cap F_2 \geq 1\}$ *where* $F_2 = \langle e_1, e_2 \rangle$*. You also proved that this variety is given by the equation* $p_{34} = 0$ *(even scheme theoretically). This says that* D *is in fact the intersection of the hyperplane divisor* $p_{34} = 0$ *in* $\mathbb{P}^5$ *by the quadric* $\mathrm{Gr}(2,4)$*. Then*

$$\deg(\mathrm{Gr}(2,4)) = \int_{\mathbb{P}^5} c_1(\mathcal{O}_{\mathbb{P}^5}(1))^4 \cap [\mathrm{Gr}(2,4)] = \int_{\mathbb{P}^5} c_1(\mathcal{O}_{\mathbb{P}^5}(1))^4 \cap 2c_1(\mathcal{O}_{\mathbb{P}^5}(1)) = 2.$$

This has the following enumerative interpretation. The product $c_1(\mathcal{O}_{\mathbb{P}^5}(1))^4$ *represents the intersection* $[D]^4$*. Since* D *is the variety of projective lines in* $\mathbb{P}^3$ *meeting the line* $\mathbb{P}(F_2)$*. To calculate* $[D]^4$*, we are allowed to take divisors which are linearly equivalent to* D*, then intersect. In particular, we may move the 'reference line'* $\mathbb{P}(F_2)$ *and define* 4 *linear equivalent Cartier divisors* D_i *by taking* 4 *lines 'sufficiently general'. Then we are saying that there are* $\mathbf{2}$ *lines in* $\mathbb{P}^3$ *meeting meeting* 4 *given lines in general position.*

Example 25.4 (Lines on cubic surfaces)**.** *Let* $\Sigma \subset \mathbb{P}^3$ *be a general surface, i.e.* $\Sigma = Z(F)$ *where* F *is a general homogeneous polynomial of degree* 3 *in variables* x_0, x_1, x_2, x_3*. We are interested in how many lines are included in* $\mathbb{P}^3$*. The space of lines in* $\mathbb{P}^3$ *is the same as the Grassmannian* $\mathrm{Gr}(2,4)$*. Let* S *be the tautological subbundle. The polynomial* F *is a section of* $Sym^3(\mathbb{C}^4)^*$*, and the condition that a line* $\mathbb{P}(V) \in \mathrm{Gr}(2,4)$ *is included in* $Z(F)$ *means that* $F|_V \equiv 0$*. In other words,* F *gives a (general) section*

$$s \in H^0(\mathrm{Gr}(2,4), Sym^3(S^*)), \quad V \mapsto F|_V.$$

We are interested in the zero locus of this section. By the normalization property (25.1)

$$[\Sigma] = [Z(s)] = c_r(Sym^3(S^*)) \cap [\mathrm{Gr}(2,4)],$$

where $r = rank(Sym^3 S^*) = 4$*. Let* x_1, x_2 *be the Chern roots of* S^**. Then by Lemma Lemma 25.1 the Chern roots of* $Sym^3 S^*$ *are* $3x_1, 2x_1 + x_2, x_1 + 2x_2, 3x_2$*. It follows that*

$$\begin{aligned} c_4(Sym^3 S^*) &= 9x_1x_2(x_1 + 2x_2)(2x_1 + x_2) \\ &= 9x_1x_2(2x_1^2 + 5x_1x_2 + 2x_2^2) \\ &= 9x_1x_2(2(x_1 + x_2)^2 + x_1x_2) \\ &= 18x_1x_2(x_1 + x_2)^2 + 9(x_1x_2)^2 \\ &= 18c_2(S^*)c_1(S^*)^2 + 9c_2(S^*)^2. \end{aligned}$$

Using the intersection theory of $\mathrm{Gr}(2,4)$*, one may prove that*

$$c_2(S^*)c_1(S^*)^2 \cap [\mathrm{Gr}(2,4)] = c_2(S^*)^2 \cap [\mathrm{Gr}(2,4)] = [pt].$$

Therefore

$$c_4(Sym^3 S^*) \cap [\mathrm{Gr}(2,4)] = 18[pt] + 9[pt] = 27[pt].$$

The enumerative geometry interpretation of this is that a general cubic surface will contain 27 *lines.*

25.2. **Some answers to enumerative geometry questions.** You might enjoy thinking about these numbers:

- 2: number of lines passing through 4 given lines in $\mathbb{P}^3$;
- 92: number of conics in $\mathbb{P}^3$ meeting 8 lines;
- 1: non-singular conics tangent to 5 given lines;
- 3264: number of conics tangent to 5 given conics;
- 4407296: number of conics tagent to 8 general quadrics surfaces;
- 2875: number of lines on a quintic threefold in $\mathbb{P}^4$;
- 5819539783680: number of twisted cubics tangent to 12 quadrics in $\mathbb{P}^3$. (This was found by **H. Schubert** in 1870's.)

THAT'S ALL FOLKS!

26. Appendix: Results from commutative algebra

Theorem 26.1 (Hilbert basis theorem). *Let k be a field. Then any ideal I in the polynomial ring $k[x_1, \ldots, x_n]$ is finitely generated, i.e. there exist polynomials $P_1, \ldots, P_s \in k[x_1, \ldots, x_n]$ such that*

$$I = \langle P_1, \ldots, P_s \rangle.$$

In particular, from any subset $S \subset I$ such that $\langle S \rangle = I$, one may extract a finite subset $S' \subset S$ such that $\langle S' \rangle = I$.

Proof. See e.g. [CLO15, §2.5]. □

Theorem 26.2 (Noether normalization lemma). *Let R be an integral domain, finitely generated over a field k. Assume that the transcendence degree of R over k equals to n. Then there exists elements $y_1, \ldots, y_n \in R$ such that:*

- *$y_1, \ldots, y_n$ are algebraically independent over k;*
- *R is integrally dependent on the subring $k[y_1, \ldots, y_n]$.*

Proof. See a proof in [Mum99, §.1] (attributed to Nagata). □

Lemma 26.1 (A consequence of Cohen-Seidenberg 'Going Up'). *Let R be a field and $S \subset R$ a subring such that R is integrally dependent on S. Then S is a field.*

Proof. Let $0 \neq a \in S$. Then $1/a$ is a root of a monic polynomial with coefficients in S, meaning that

$$(1/a)^n + c_{n-1}(1/a)^{n-1} + \ldots + c_0 = 0$$

in R, with $c_i \in S$. After multiplying by a^{n-1} we obtain that

$$\frac{1}{a} = -c_{n-1} - c_{n-2}a - \ldots - c_0 a^{n-1} \in S.$$

□

26.1. Rings and modules of fractions; localization.

Let R be a commutative ring with 1, and M an R-module. A set $S \subset R$ is called —bf multiplicative if $1 \in S$ and for any $a, b \in S$, the product $ab \in S$.

There are two important examples of multiplicative sets.

Example 26.2. *Let $f \in R$. Take $S = \{f^n : n \in \mathbb{Z}_{\geq 0}\}$.*

Example 26.3. *Let $\mathfrak{p} \subset R$ be a prime ideal. Then $S := R \setminus \mathfrak{p}$ is multiplicative.*

For a multiplicative set $S \subset R$ and an R-module M one may define an equivalence relation on $M \times S$ by

$$(m_1, s_1) \simeq (m_2, s_2) \Longleftrightarrow \exists s \in S \quad s(s_1 m_2 - s_2 m_1) = 0.$$

The equivalence class of (m, s) is denoted by $\frac{m}{s}$. Denote by

$$S^{-1}M := \{\frac{m}{s} : m \in M, s \in S\}.$$

Then $S^{-1}R$ is a ring and $S^{-1}M$ is a $S^{-1}R$-module. There is a natural ring homomorphism

$$R \to S^{-1}R; \quad r \mapsto \frac{r}{1}.$$

This may *not* be injective. (E.g., if f is nilpotent, then $R_f = 0$.)

There is a similar module homomorphism $M \to S^{-1}M$.

Lemma 26.4. *Let R be a ring and $0 \neq f \in R$. If R has no nilpotents then R_f has no nilpotents.*

Proof. Let $\frac{r}{f^a} \in R_f$ such that

$$\left(\frac{r}{f^a}\right)^n = \frac{r^n}{f^{an}} = 0.$$

Then $f^p r^n = 0$ in R for some p, implying that $(f^p r)^n = 0$. Since R has no nilpotents, this implies that $f^p r = 0$, i.e. $\frac{r}{f^a} = 0$ in R_f. This proves the claim. □

26.2. **Primary and irreducible ideals.** The main reference for this section is [AM69, Ch.4 and Ch. 7].

Definition 26.1. *Let R be an arbitrary ring and $I \neq R$ an ideal. The ideal I is* **primary** *if*

$$xy \in I \Longrightarrow x \in I \text{ or } y^n \in I.$$

The ideal I is called **irreducible** *if*

$$I = I_1 \cap I_2 \Longrightarrow I = I_1 \text{ or } I = I_2.$$

Proposition 26.5. *Let I be a primary ideal. Then the radical $\sqrt{I}$ is prime and it is the smallest prime containing I.*

Proof. See [AM69, Prop. 4.1]. □

Proposition 26.6. *Let R be a Noetherian ring.*

(a) Any ideal I is a finite intersection of irreducible ideals.

(b) Any irreducible ideal is primary.

Proof. See Lemmas 7.11 and 7.12 in [AM69]. □

26.3. **Topology.** Let X be a topological space. A basis for the topology of X is a collection $\mathcal{U}$ of open sets such that every open set in X is a union of basis elements in $\mathcal{U}$.

26.4. **Limits, stalks, and localization.** TODO.

References

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