LECTURES ON QUANTUM K THEORY FOR FLAG MANIFOLDS (PARIS, JUNE 2022)

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Contents

Warning: These notes are under construction.

1. Preliminaries

Throughout the notes, $G := \mathrm{GL}_n$ denotes the complex general linear group, B, B^- a pair of opposite Borel subgroups, e.g. the upper/lower triangular matrices. The associated Weyl group is $W = S_n$, the symmetric in n letters. It is generated by simple reflections $s_i = (i, i+1)$ where $1 \le i \le n-1$. Denote by $\ell : W \to \mathbb{N}$ the length function and the longest element by w_0. A sequence $I = (1 \le i_1 < \ldots < i_k \le n-1)$ determines a collection of simple roots s_i, $i \in I$. and a parabolic subgroup $P \subset G$. The **partial flag manifold** is the homogeneous space

$$G/P = \{F_{i_1} \subset F_{i_2} \subset \ldots \subset \mathbb{C}^n : \dim F_{i_s} = i_s\}.$$

Of particular interest will be the **Grassmannians** and the **flag manifolds**:

$$\mathrm{Gr}(k,n) = \{V \subset \mathbb{C}^n : \dim V = k\}$$

(for $I = \{k\}$) and the flag manifold by

$$\mathrm{Fl}(n) = \{F_1 \subset F_2 \subset \ldots \subset \mathbb{C}^n : \dim F_i = i\}$$

(for $I=[1,n-1]$). These are homogeneous spaces for G. (For those who know, you may replace $\mathrm{Fl}(n)$ by any G/B, and Grassmannians by any cominuscule Grassmannians.) For a sequence I, define W_P to be the subgroup generated by the simple reflections $s_i = (i,i+1)$ where $i \notin I$, and set $W^P = W/W_P$, the set of minimal length representatives. One can check that

$$W^P = \{w \in W : w \text{ has descents at most in positions } i_1, \dots, i_s\}.$$

Fix $\{e_1, \dots, e_n\}$ the standard basis of $\mathbb{C}^n$. The maximal torus of diagonal matrices $T \subset G$ acts on G/P and the T-fixed points are coordinate flags $\{E_w : w \in W^P\}$ where

$$E_w := \langle e_{w(1)}, \dots, e_{w(i_1)} \rangle \subset \langle e_{w(1)}, \dots, e_{w(i_2)} \rangle \subset \dots \subset \langle e_{i_1}, \dots, e_{w(i_k)} \rangle \subset \mathbb{C}^n.$$

To each $w \in W^P$ there are two Schubert varieties:

$$X_w = \overline{B.E_w}; \quad X^w = \overline{B^-.E_w},$$

where $B, B^- \subset \mathrm{GL}_n$ are opposite Borel subgroups. The orbits

$$X_w^\circ = B.E_w; \quad X^{w,\circ} = B^-.E_w$$

are called **Schubert cells**. With these conventions, $X_w^\circ \simeq \mathbb{A}^{\ell(w)}$ and $X^{w,\circ} \simeq \mathbb{A}^{\dim \mathrm{Fl}(\mathbf{i}) - \ell(w)}$. In particular,

$$\dim X_w = \mathrm{codim} X^w = \ell(w); \quad X_w \cap X^w = \{E_w\}; \quad X^w = w_0 X_{w_0 w w_\lambda}$$

where $w_P \in W_P$ is the longest element. With these definitions, the Bruhat order on W^P is defined by

$$v < w \text{ in } W^P \quad \Leftrightarrow \quad X_v \subset X_w \quad \Leftrightarrow \quad X^v \supset X^w.$$

The partial flag manifolds have a stratification by Schubert cells:

$$G/P = \bigsqcup_{w \in W^P} X_w^\circ = \bigsqcup_{w \in W^P} X^{w,\circ}.$$

Consider now the K-theory ring $\mathrm{K}(G/P)$; an excellent source for learning about this is Brion's notes [Bri05]. This is a ring under the addition and tensor product of vector bundles. It has a nondegenerate **intersection pairing**

$$\langle E, F \rangle = \int_{\mathrm{Fl}(\mathbf{i})} E \cdot F = \chi(\mathrm{Fl}(\mathbf{i}), E \otimes F).$$

Denote the classes in K-theory of the partial flag manifolds by

$$\mathcal{O}_w = [\mathcal{O}_{X_w}], \quad \mathcal{O}^w = [\mathcal{O}_{X^w}] \in \mathrm{K}(G/P) \quad w \in W^P.$$

These classes form a $\mathbb{Z}$-basis called the **Schubert basis**:

$$\mathrm{K}(G/P) = \oplus \mathbb{Z}\mathcal{O}_w = \oplus \mathbb{Z}\mathcal{O}^w.$$

Plan:

(1) Definitions of QK theory, of curve neighborhoods, and applications;
(2) 'Quantum = classical' , structure theorems, two presentations;
(3) Positivity.

2. Definition of quantum K theory and first properties

2.1. **The moduli space.** Let X be a projective manifold - very soon $X = G/P$. For an effective degree $d \in H_2(X;\mathbb{Z})$, denote by $\overline{\mathcal{M}}_{0,n}(X,d)$ the Kontsevich moduli space of (genus 0, n pointed) stable maps of degree d. This is a projective scheme, with points stable maps:

$$f : (C, p_1, \dots p_n) \to X; \quad f_*[C] = d.$$

Here C is a tree of $\mathbb{P}^1$'s, and f satisfies a stability condition: if C' is a component such that $f(C') = cst$, then C' must have at least three marked points. A marked point is either a node or a marking p_i. There is a natural equivalence relation on this data. The moduli space comes equipped with evaluation maps $\mathrm{ev}_i : \overline{\mathcal{M}}_{0,n}(X,d) \to X$, sending $f \mapsto f(p_i)$. If $n \geq 3$ and $d = 0$, then $\overline{\mathcal{M}}_{0.n}(X,0) = X \times \overline{\mathcal{M}}_{0,n}$, the product of X with the Mumford moduli space of stable curves. The evaluation maps are all equal to the projection to X.

More generally, for a sequence of effective degrees $d_1, \dots, d_r \in H_2(X)$, we can consider the fibre product

$$\overline{\mathcal{M}}_{0,n_1+\dots n_r}(X,(d_1,\dots,d_r)) := \overline{\mathcal{M}}_{0,n_1+1}(X,d_1) \times_X \dots \times_X \overline{\mathcal{M}}_{0,n_r+1}(X,d_r)$$

This may be identified with a boundary component inside $\overline{\mathcal{M}}_{0,n_1+\dots n_r)}(X,(d_1+\dots+d_r))$. We list some important properties of the Kontsevich moduli space.

Theorem 2.1. *Let $X = G/P$ be a flag manifold. Then the following hold:*

- *$\overline{\mathcal{M}}_{0,n}(X,d)$ has finite quotient singularities, hence rational singularities - this follows from construction, see e.g.* [FP97]
- *$\overline{\mathcal{M}}_{0,n}(G/P,d)$ is a connected, thus irreducible variety (Thomsen* [Tho98]*);*
- *$\overline{\mathcal{M}}_{0,n}(X,d)$ is a rational variety (Kim and Pandharipande ADDREF)*

2.2. **Definition of quantum K theory (after Givental and Lee** [Giv00, Lee04]**).** From now on we will take $X = G/P$ to be any partial flag manifold, or any homogeneous space. This results in fewer technicalities, such as the replacement of the 'virtual fundamental sheaves' of Kontsevich moduli spaces by structure sheaves. For the general construction, consult [Lee04].

We define next the **K-theoretic Gromov-Witten invariants** (KGW). Let $a_1, \dots, a_n \in \mathrm{K}(X)$ and $d \in H_2(X)$. The KGW invariant is

$$\langle a_1, \dots, a_n \rangle_d = \int_{\overline{\mathcal{M}}_{0,n}(X,d)} \mathrm{ev}_1^*(a_1) \cdot \ldots \cdot \mathrm{ev}_n^*(a_n). \tag{2.1}$$

In general the moduli space is not smooth, but since X is, one may write each of the classes a_i as a finite alternating sum of classes of vector bundles. Then (2.1) may be written as a finite alternating sum of sheaf Euler characteristics of vector bundles. In the latter case, the product $\cdot$ is the tensor product $\otimes$.

Example 2.1. *Consider $X = G/P$ a partial flag manifold. Then*

$$\langle 1, \ldots, 1 \rangle_d = 1$$

for any degree d. Indeed, from Theorem 2.1 we deduce that $H^i(\mathcal{O}_{\overline{\mathcal{M}}_{0,n}}(X,d)) = 0$ for $i > 0$, hence $\chi(\mathcal{O}_{\overline{\mathcal{M}}_{0,n}}(X,d)) = 1$.

Recall that $H_2(X)$ has a basis of effective curve classes, say $[C_1], \ldots, [C_r]$. Consider the sequence of Novikov variables $q = (q_1, \ldots, q_r)$. For $d = d_1[C_1] + \ldots + d_r[C_r]$, set $q^d = q_1^{d_1} \cdot \ldots \cdot q_r^{d_r}$. Define the $\mathbb{Z}[[q]]$-module

$$\mathrm{QK}(X) = K(X) \otimes \mathbb{Z}[[q]].$$

Assume also that $K(X)$ has a finite basis $\mathcal{O}^0 = 1, \ldots, \mathcal{O}_n$, and denote by $\mathcal{O}^{i,\vee}$ the dual basis with respect to the intersection pairing.

Definition 2.1. *The (small)* **QK pairing** *is defined by*

$$((a,b)) = \langle a, b \rangle + \sum_{d>0} \langle a, b \rangle_d q^d.$$

Here q stands for the sequence of Novikov variables indexed by a basis of $H_2(X)$, and $q^d = q_1^{d_1} \cdot \ldots \cdot q_r^{d_r}$. The QK *pairing is a nondegenerate pairing with values in the formal power series $\mathbb{Z}[[q]]$.*

The quantum K product is the unique product $\circ$ which satisfies

$$((a \circ b, c)) = \sum_{d \geq 0} \langle a, b, c \rangle_d q^d.$$

Theorem 2.2 (Givental, Lee)**.** *The product $\circ$ equips* $\mathrm{QK}(X)$ *with a structure of a commutative, associative ring with identity $1 = [\mathcal{O}_X]$.*

From definition it follows that $\mathrm{K}(X) \simeq \mathrm{QK}(X)/\langle q \rangle$. Since $\mathrm{K}(X)$ is filtered algebra, it induces a filtration on $\mathrm{QK}(X)$, with $\deg q_i = \int_X c_1(T_X) \cap [C_i]$. The associated graded algebra is $\mathrm{Gr}\,\mathrm{QK}(X) = \mathrm{QH}^*(X)$, the **quantum cohomology** of X.

Next we unravel the definition of the QK product and we discuss two equivalent formulations of the definition.

Lemma 2.1. *Consider the product*

$$\mathcal{O}^i \circ \mathcal{O}^j = \sum N_{i,j}^{k,d} q^d \mathcal{O}^k.$$

Then we have the following equivalent formulae for the structure constants $N_{i,j}^{k,d}$:

(a)

$$N_{i,j}^{k,d} = \langle \mathcal{O}^i, \mathcal{O}^j, (\mathcal{O}^k)^\vee \rangle_d - \sum_{d'>0,s} N_{i,j}^{s,d-d'} \langle \mathcal{O}^s, (\mathcal{O}^k)^\vee \rangle_{d'}.$$

(b)

$$N_{i,j}^{k,d} = \langle \mathcal{O}^i, \mathcal{O}^j, (\mathcal{O}^k)^\vee \rangle_d \\ + \sum (-1)^s \langle \mathcal{O}^i, \mathcal{O}^j, (\mathcal{O}^{i_0})^\vee \rangle_{d_0} \cdot \langle \mathcal{O}^{i_0}, (\mathcal{O}^{i_1})^\vee \rangle_{d_1} \cdot \ldots \cdot \langle \mathcal{O}^{i_s}, (\mathcal{O}^k)^\vee \rangle_{d_s};$$

here the sum is over effective degrees $d_0, \ldots, d_s$ such that $d_0 + \ldots + d_s = d$ and $d_p > 0$ if $p > 0$.

(c) Let $\mathcal{D} \subset \overline{\mathcal{M}}_{0,3}(X,d)$ be the boundary divisors consisting of maps with reducible domain where markings $1,2$ are on the first component, and marking 3 on the last. Then

$$N_{i,j}^{k,d} = \chi(\mathcal{O}_{\overline{\mathcal{M}}_{0,3}(X,d)}(-\mathcal{D}) \cdot \mathrm{ev}_1^*(\mathcal{O}^i) \cdot \mathrm{ev}_2^*(\mathcal{O}^j) \cdot \mathrm{ev}_1^*((\mathcal{O}^k)^\vee)).$$

Note that, unlike in quantum cohomology, both 2 and 3-point invariants are needed to calculate a single structure constants. However, the proof of the associativity is essentially the same as in the cohomological case: it is obtained from equalities obtained by pulling back points in $\mathbb{P}^1 \simeq \overline{\mathcal{M}}_{0,4}$. The pull-backs are simple normal crossing boundary divisors in $\overline{\mathcal{M}}_{0,4}(X,d)$; while in cohomology the class of such a reducible divisor $D = \bigcup D_i$ is the sum if its components $[D_i]$, in K-theory this is an alternating sum

$$[\mathcal{O}_D] = \sum (-1)^{k-1} [\mathcal{O}_{D_{i_1} \cap \ldots \cap D_{i_k}}].$$

This explains the shape of the formula in part (c).

The formulae in the lemma suggest that in general the QK multiplication may not be finite. Indeed, Example 2.1 shows that the KGW invariants are in general nonzero for any degree d. It is not even clear why $1 = [\mathcal{O}_X]$ is the identity in the QK ring! In fact, the QK multiplication **is finite** for flag manifolds [BCMP13, Kat18, ACTI18].

At least for Grassmannians, we will explain this and more as an application of curve neighborhoods of Schubert varieties, and of the 'quantum=classical' statement.

Informally, many calculations of KGW invariants can be traced to two facts:

- (Transversality) If $\Omega_1, \ldots, \Omega_n$ satisfy a K-theoretic transversality property, then
$$[\mathcal{O}_{\Omega_1}] \cdot \ldots \cdot [\mathcal{O}_{\Omega_n}] = [\mathcal{O}_{\Omega_1 \cap \ldots \cap \Omega_n}];$$
- (Rational connectedness + mild singularities) If X is a rational/unirational/rationally connected projective variety which has rational singularities, then $\chi(\mathcal{O}_X) = 1$.

The following result provides an important tool for proving that a variety is rationally connected.

Theorem 2.3 (Graber, Harris, Starr). *Let $f : X \to Y$ be any dominant morphism of complete irreducible complex varieties. If Y and the general fiber of f are rationally connected, then X is rationally connected.*

2.3. **Curve neighborhoods and first applications.** Throughout this section $X = G/P$ is a partial flag manifold. To perform the calculations required in formulae from Lemma 2.1, we need formulae for the two-point KGW invariants of the form $\langle \mathcal{O}^i, (\mathcal{O}^j)^\vee \rangle_d$. For flag manifolds, this was obtained in a series of papers BCMP, BM, and utilize the notion of curve neighborhoods. We present next the basic facts.

Definition 2.2. *Let $\Omega_1, \dots, \Omega_n \subset X$ be closed subvarieties and fix an effective degree $d \in H_2(X)$.*

(a) The (n-point) **Gromov-Witten variety** *is the intersection*

$$\mathrm{GW}_d(\Omega_1, \dots, \Omega_n) = \mathrm{ev}_1^{-1}(\Omega_1) \cap \dots \cap \mathrm{ev}_n^{-1}(\Omega_n) \subset \overline{\mathcal{M}}_{0,n+a}(X, d).$$

If $\Omega_2 = \dots = \Omega_n = X$ we will simply use the notation $\mathrm{GW}_d(\Omega_1) = \mathrm{GW}_d(\Omega_1, X, \dots, X)$.

(b) The (n-point) **curve neighborhood** *of $\Omega_1, \dots, \Omega_n$ is defined as the image of the corresponding Gromov-Witten variety:*

$$\Gamma_d(\Omega_1, \dots, \Omega_n) = \mathrm{ev}_{n+1}(\mathrm{GW}_d(\Omega_1, \dots, \Omega_n)).$$

As before, $\Gamma_d(\Omega) := \mathrm{ev}_{n+1}(\mathrm{GW}_d(\Omega))$.

All these may be extended to the case when one has a sequence of degrees $d_1, \dots, d_k$, by replacing the moduli space with an appropriate stratum in the boundary.

Example 2.2. *(a) If $d = 0$, then $\Gamma_0(\Omega_1, \Omega_2) = \Omega_1 \cap \Omega_2$.*

(b) Take $X = \mathbb{P}^n$ and $d > 0$. Then $\Gamma_d(pt) = \mathbb{P}^n$ and

$$\Gamma_d(pt, pt) = \begin{cases} line & d = 1 \\ \mathbb{P}^n & d \geq 2. \end{cases}$$

We also need the notion of cohomological triviality.

Definition 2.3. *Let $f : X \to Y$ be a morphism of algebraic varieties. We say that f is* **cohomologically trivial** *if $f_* \mathcal{O}_X = \mathcal{O}_Y$ and $R^i f_* \mathcal{O}_X = 0$ for i> 0.*

Most (all ?) non-trivial examples of cohomologically trivial maps arise from special cases of a theorem of Kollár:

Theorem 2.4 (Kollár)**.** *Let $f : X \to Y$ be a surjective morphism of projective varieties with rational singularities. If the general fibers of f are rationally connected, then f is cohomologically trivial.*

Initial versions of this result can be traced to work by ADDREF Chaput, Manivel and Perrin. This version can be extracted from [BCMP13].

Theorem 2.5. *Let $\Omega_1, \dots, \Omega_n$ be general translates of Schubert varieties in X. Then the following hold:*

(a) The GW *variety* $\mathrm{GW}_d(\Omega_1, \dots, \Omega_n)$ *is either empty, or locally irreducible of expected dimension, and with rational singularities. Furthermore,*

$$\langle [\mathcal{O}_{\Omega_1}], \dots, [\mathcal{O}_{\Omega_n}] \rangle_d = \chi([\mathcal{O}_{\mathrm{GW}_d(\Omega_1, \dots, \Omega_n)}]).$$

(b) The non-empty Gromov-Witten varieties $\mathrm{GW}_d(\Omega_1, \Omega_2)$ *are irreducible and rationally connected.*

(c) If Ω *is any Schubert variety, then* $\Gamma_d(\Omega)$ *is again a Schubert variety and the evaluation map* $\mathrm{ev}_i : \mathrm{GW}(\Omega) \to \Gamma_d(\Omega)$ *is cohomologically trivial.*

Idea of proof. We may assume $\Omega_1 = X_u, \Omega_2 = X^v$. The first statement follows from applying a K-theoretic version of Kleiman-Bertini, due to Sierra. The evaluation map $\mathrm{ev}_1 : \overline{\mathcal{M}}(X, d) \to X$ is a G-equivariant locally trivial fibration in Zariski topology. Its fibre F is irreducible and unirational. By base-change, $ev_1^{-1}(X_u) \to X_u$ is also locally trivial, showing $\mathrm{GW}(X_u)$ is irreducible and rationally connected. The image $\Gamma_d(X_u) = \mathrm{ev}_2(\mathrm{GW}(X_u))$ is irreducible and B-stable, thus a B-stable Schubert variety. Then $\mathrm{GW}_d(X_u)$ has an open dense set which is a locally trivial fibration over the cell $\Gamma_d(X_u)^\circ$. The intersection $\mathrm{ev}_1^{-1}(X_u) \cap \mathrm{ev}_2^{-1}(X^v)$ is locally irreducible and it has an open dense set which is a locally trivial fibration over $\Gamma_d(X_u)^\circ \cap X^v$. If non-empty, the latter is irreducible and rational. Since all these varieties have rational singularities, and the (general) fibers of these maps are unirational, the statement follows from Theorem 2.4 and Theorem 2.3. □

An immediate consequence is:

Corollary 2.1. *Let* Ω *be any Schubert variety. Then the* 1 *and* 2*-point curve neighborhoods are irreducible.*

Proof. The curve neighborhoods in question are images of GW varieties, which are irreducible by Theorem 2.5. □

For $u \in W^P$ and d an effective degree, define the elements $u(d), u(-d) \in W^P$ by

$$X_{u(d)} = \Gamma_d(X_u); \quad X_{u(-d)} = \Gamma_d(X^u).$$

Using these elements one can immediately calculate any 2-point GW invariant.

Corollary 2.2. *Let* $u, v \in W^{\mathbf{i}}$ *be two Weyl group elements and* d *an effective degree. Then*

$$\langle \mathcal{O}^u, (\mathcal{O}^v)^\vee \rangle_d = \delta_{u(-d),v},$$

(the Kronecker delta symbol).

For $u, v \in W^P$, define $d_{min}(u, v)$ the minimum degree d for which $\mathrm{GW}_d(X_u, X^v) \neq \emptyset$. Equivalently, this is the minimal degree of a rational curve joining the fixed points $E_u, E_{w_0 v w_P}$. By results of Postnikov and Fulton-Woodward ADDREF this is the same as the minimum degree of q in the quantum cohomology product of $[X_u] \star [X^v]$. In particular, it is well defined. Using this degree, one can calculate the QK pairing between any two Schubert classes:

$$((\mathcal{O}^u, \mathcal{O}^v)) = \sum_{d \geq d_{min}(u,v)} \langle \mathcal{O}^u, \mathcal{O}^v \rangle_d \, q^d = \frac{q^{d_{min}(u,v)}}{\prod (1 - q_i)}. \tag{2.2}$$

Example 2.3. *Assume that $X = \mathbb{P}^2$. In this case $K(X)$ has a basis $1 = \mathcal{O}^0, \mathcal{O}^1, \mathcal{O}^2$, where $\mathcal{O}^i$ is the K-theoretic class representing the hyperplane of (complex) codimension i. With respect to this basis, the Poincaré metric $g_{ij} = \int_X \mathcal{O}^i \cdot \mathcal{O}^j$ is given by the matrix*

$$(g_{ij}) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

The QK *metric is obtained by adding $\frac{q}{1-q}$:*

$$((\mathcal{O}^i, \mathcal{O}^j)) = (g_{i,j}) + \frac{q}{1-q} Id.$$

Corollary 2.3 ([BCLM20])**. Assume** *that the* QK *product is finite, and consider the specialization at $q_i \mapsto 1$ for all i of the usual pairing $\chi : \mathrm{QK}(X) \to \mathrm{QK}(pt) = \mathbb{Z}[q]$. Then this is a* **ring** *homomorphism.*

Proof. Write $\mathcal{O}^u \circ \mathcal{O}^v = \sum N_{u,v}^{w,d} q^d \mathcal{O}^w$. By the Frobenius property of the QK pairing,

$$\sum N_{u,v}^{w,d} q^d \frac{1}{\prod(1-q_i)} = ((\mathcal{O}^u \circ \mathcal{O}^v, 1)) = ((\mathcal{O}^u, \mathcal{O}^v)) = \frac{q^{d_{min}(u,v)}}{\prod(1-q_i)}.$$

It follows that $\sum N_{u,v}^{w,d} = 1$. Then the statement follows from the fact that $\chi(\mathcal{O}^u) = 1$ for any u. $\square$

Note that χ is **not** a ring homomorphism for any specialization of $\mathrm{QK}(X)$ (the K-theory specialization, the quantum cohomology specialization etc).

Example 2.4. *Take $a = b = [pt]$ in $\mathbb{P}^1$. Then*

$$\chi(a \cdot b) = 0 \neq \chi(a) \cdot \chi(b) = 1 \cdot 1 = 1.$$

We will show later that in $\mathrm{QK}(\mathbb{P}^1)$, $[pt] \circ [pt] = q$ and we can already prove that $(([pt], [pt])) = \frac{q}{1-q}$.

There is a more general, and rather surprising statement, due to Kato.

Theorem 2.6 (Kato ADDREF)**.** *Let $\pi : G/P \to G/Q$ be the natural projection for $P \subset Q$. Consider the $\mathbb{Z}[[q]]$-module projection $\pi_* : \mathrm{QK}(G/P) \to \mathrm{QK}(G/Q)$ defined by extending the usual projection $\pi_* : \mathrm{K}(G/P) \to \mathrm{K}(G/Q)$ and specializing $q_i \mapsto 1$ for all i such that $s_i \in W_{\mathbf{j}} \setminus W_{\mathbf{i}}$. Then this is a ring homomorphism.*

More refined applications require more refined knowledge of the Weyl group elements giving curve neighborhoods.

2.4. **Calculation of curve neighborhoods.** The goal is to give an algorithm to calculate the elements $u(d)$ and $u(-d)$. To start,

$$X^{u(-d)W_P} = \Gamma_d(X^{uW_P}) = \Gamma_d(w_0 X_{w_0 uW_P}) = \Gamma_d(w_0 X_{uW_P}) = w_0 X_{u(d)W_P}.$$

This reduces the calculation of $u(-d)$ to that of $u(d)$. For 'small' degrees d, a practical method to do this calculation is based on the **moment graph** of G/P.

Example 2.5. *TODO: Add example for* $\Gamma_{(1,0)}(pt) = X_{s_1}, \Gamma_{(0,1)}(pt) = X_{s_2}, \Gamma_{(1,1)}(pt) = X_{s_1 s_2 s_1}$ *in* $\mathrm{Fl}(3)$.

For a general combinatorial procedure, we need two ingredients. The **Demazure product** $\cdot$ of two Weyl group elements is defined as follows. If $u \in W$ and $s_i \in W$ is a simple reflection,

$$u \cdot s_i = \begin{cases} us_i & \ell(us_i) > \ell(u) \\ u & \ell(us_i) < \ell(u). \end{cases}$$

If $v = s_{i_1} \ldots s_{i_k}$ is a reduced decomposition, then $u \cdot v = (((u \cdot s_{i_1}) \cdot s_{i_2}) \ldots) \cdot s_{i_k}$. This equips $(W, \cdot)$ with a structure of an associative monoid. Let also $z_d \in W$ be the unique element defined by

$$X_{u(d)} = \Gamma_d(pt) \subset \mathrm{Fl}(n).$$

The following combinatorial algorithm to calculate $u(d)$ for any flag manifold has been proved in [BM15].

Theorem 2.7. *The following hold:*

(a) In $\mathrm{Fl}(n)$, $\Gamma_d(X_u) = X_{u \cdot z_d}$.

(b) Take $\alpha > 0$ *be the largest positive root such that* $d - \alpha^\vee \geq 0$ *in* $H_2(\mathrm{Fl}(n))$. *Then*

$$z_d = z_{d-\alpha^\vee} \cdot s_\alpha = s_\alpha \cdot z_{d-\alpha^\vee}.$$

(c) Same procedure applies to any G/P*: take* $\alpha \in R^+ \setminus R_P^+$ *maximal such that* $d - \alpha^\vee \geq 0$ *in* $H_2(\mathrm{Fl}(\mathbf{i}))$. *Then*

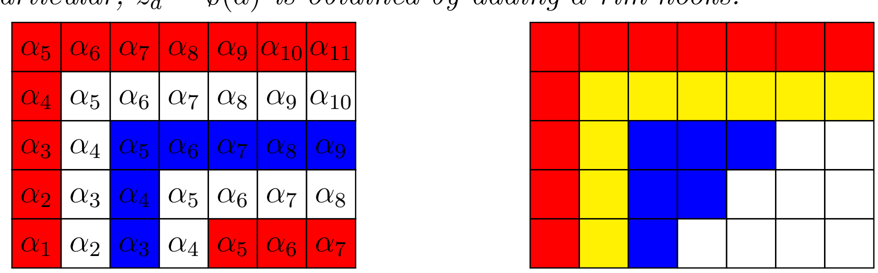
$$z_d W_P = s_\alpha \cdot z_{d-\alpha^\vee} W_P.$$

If one works in the Grassmannian $\mathrm{Gr}(k,n)$ (or more generally in **cominuscule homogeneous spaces**), the theorem above was proved earlier in [BCMP13]. In this case the Schubert classes are indexed by Young diagrams λ included in the $k \times (n-k)$ rectangle, and the curve neighborhoods have particularly nice combinatorial descriptions:

- $\lambda(d)$ is obtained from λ by adding d rim hooks of maximal length;
- $\lambda(-d)$ is obtained from λ by removing d rim hooks of maximal length.

Example 2.6. *The Schubert classes are indexed by Young diagrams* λ *included in the* $k \times (n-k)$ *rectangle. The curve neighborhoods have nice combinatorial descriptions:*

- $\lambda(d)$ *is obtained from* λ *by adding* d *rim hooks of maximal length;*
- $\lambda(-d)$ *is obtained from* λ *by removing* d *rim hooks of maximal length.*
- *In particular,* $z_d = \emptyset(d)$ *is obtained by adding* d *rim hooks.*



On the left: $z_1, z_2, \ldots$; *on the right:* $\lambda(2)$, *for* $\lambda = (3,2,1)$.

Corollary 2.4 ([BCMP13])**.** *Let X be a (cominuscule) Grassmannian. Then*

$$\Gamma_d(X_u) = \Gamma_1(\Gamma_1(\ldots(\Gamma_1(X_u)))).$$

In other words, if one point may be joined to X_u using a rational curve of degree d, then it may also be joined by a sequence of d lines.

This is special for (cominuscule) Grassmannians. It fails for example for $\mathrm{IG}(2,7)$ or for adjoint varieties. The corollary implies the following important simplification of the formulae from Lemma 2.1 for the QK product of Schubert classes in $\mathrm{QK}(\mathrm{Gr}(k;n))$.

Corollary 2.5. *Consider the* QK *product* $\mathcal{O}^\lambda \circ \mathcal{O}^\mu = \sum N_{\lambda,\mu}^{\nu,d} q^d \mathcal{O}^\nu$ *in* $\mathrm{QK}(\mathrm{Gr}(k;n))$. *Then*

$$N_{\lambda,\mu}^{\nu,d} = \langle \mathcal{O}^\lambda, \mathcal{O}^\mu, (\mathcal{O}^\nu)^\vee \rangle_d - \sum_\eta \langle \mathcal{O}^\lambda, \mathcal{O}^\mu, (\mathcal{O}^\eta)^\vee \rangle_{d-1} \cdot \langle \mathcal{O}^\eta, (\mathcal{O}^\nu)^\vee \rangle_1.$$

Proof. We need to show that for λ, μ fixed and fixed $d - d_0 := d_1 + \ldots + d_r \geq 2$, then

$$\sum_{d_1+\ldots+d_r = d-d_0} (-1)^r \langle \mathcal{O}^\lambda, (\mathcal{O}^{\kappa_1})^\vee \rangle_{d_1} \cdot \ldots \cdot \langle \mathcal{O}^{\kappa_r}, (\mathcal{O}^\nu)^\vee \rangle_{d_r} = 0.$$

From Corollary 2.2 it follows that this equals to

$$\begin{aligned}
\sum_{d_1+\ldots+d_r = d-d_0} (-1)^r \delta_{\lambda(-d_1),\kappa_1} \cdot \ldots \cdot \delta_{\kappa(-d_r),\nu} &= \sum (-1)^r \delta_{\lambda(-d_1-d_2-\ldots-d_r),\mu} \\
&= \sum_{r=1}^{d-d_0} (-1)^r \binom{d-d_0+r-1-r}{r-1} \\
&= 0.
\end{aligned}$$

□

This formula may be interpreted as

$$\begin{aligned}
N_{\lambda,\mu}^{\nu,d} &= \langle (\mathrm{ev}_3)_*[\mathrm{GW}_d(g_1 X^\lambda, g_2 X^\mu)] - (\mathrm{ev}_3)_*[\mathrm{GW}_{d-1,1}(g_1 X^\lambda, g_2 X^\mu)], (\mathcal{O}_\nu)^\vee \rangle \rangle \\
&= \langle \mathcal{O}_{\Gamma_d(\lambda,\mu)} - \mathcal{O}_{\Gamma_{d-1,1}(\lambda,\mu)}, (\mathcal{O}_\nu)^\vee \rangle,
\end{aligned}$$

where g_1, g_2 are general in G. In fact, the second equality is slightly incorrect, and we will see the correct form when we prove positivity theorem for $\mathrm{QK}(\mathrm{Gr}(k;n))$. This expression will be the starting point of that proof.

3. The quantum=classical statement and first applications

3.1. **The statement.** We start with Buch's notion of **kernel** and **span** of a rational curve.

Definition 3.1. *Let $f : \mathbb{P}^1 \to \mathrm{Gr}(k;n)$ be a morphism of degree d. The* **kernel** *and* **span** *of f are the linear subspaces of $\mathbb{C}^n$ defined by*

$$\ker(f) = \bigcap_{x\in\mathbb{P}^1} f(x); \quad \mathrm{span}(f) = \mathrm{span}\{f(x) : x \in \mathbb{P}^1\}$$

Proposition 3.1 (Buch, Buch-Kresch-Tamvakis)**.** *(a) If $f : \mathbb{P}^1 \to \mathrm{Gr}(k;n)$ is of degree d then $\dim\ker(f) \geq k-d$ and $\dim\mathrm{span} f \leq k+d$. Furthermore, for a general map f, equalities occur.*

(b) Let $U,V,W \subset \mathrm{Gr}(d,2d)$ be three general spaces. Then there exists a unique morphism $f : \mathbb{P}^1 \to \mathrm{Gr}(n,2n)$ of degree d such that $f(0) = U, f(1) = V, f(\infty) = W$.

Proof. Let S be the tautological bundle on $\mathrm{Gr}(k;n)$. Then $f^*(S) \subset \mathbb{C}^n$, thus $f^*S = \bigoplus_{i=1}^k \mathcal{O}_{\mathbb{P}^1}(-a_i)$ where $a_i \geq 0$ and $\sum a_i = d$. Therefore at least $k-d$ of these integers equal to 0; if one writes the coordinates of $f(x)$, all the zero contributions will be in the kernel. A similar idea works for the span.

Regarding part (b), observe that $\mathbb{C}^{2d} = U \oplus W$. Take a basis $v_1, \ldots, v_d$ of V and project to U, W: $v_i = u_i + w_i$. Define $f[s:t] = [su_1 + tw_1 : \ldots : su_d + tw_d]$. $\square$

Consider the 'kernel-span incidence':

$$\begin{tikzcd} Z_d := \mathrm{Fl}(k-d,k,k+d;n) \arrow[r, "p_d"] \arrow[d, "q_d"] & X := \mathrm{Gr}(k,n) \\ Y_d := \mathrm{Fl}(k-d,k+d;n) & \end{tikzcd}$$

Here, if $d \geq k$ then we set $Y_d := \mathrm{Fl}(k+d;n)$ and if $k+d \geq n$ then we set $Y_d := \mathrm{Gr}(k-d;n)$. In particular, if $d \geq \min\{k, n-k\}$, then Y_d is a single point.

Theorem 3.1 (Quantum = classical [BM11])**.** *Let $a,b,c \in K_T(\mathrm{Gr}(k;n))$ and $d \geq 0$ a degree. If $d \geq k$ then we set $d-k := 0$ and if $k+d \geq n$ then we set $k+d := n$. Then the following equality holds in $K_T(pt)$:*

$$\langle a,b,c\rangle_d = \int_{Y_d} (q_d)_*(p_d^*a) \cdot (q_d)_*(p_d^*b) \cdot (q_d)_*(p_d^*c).$$

The cohomological version of this theorem was obtained by Buch, Kresch and Tamvakis [BKT03].

Idea of proof. The proof of this is based on the 'quantum = classical' diagram which we explain below. Let $M_d := \overline{\mathcal{M}}_{0,3}(X,d)$,

$$\mathrm{Bl}_d = \{((K,S),f) \in Y_d \times M_d, K \subset \ker(f), \mathrm{span}(f) \subset S\}$$

$$Z_d^{(3)} = \{K \subset V_1, V_2, V_3 \subset S : (K, V_i, S) \in Z_d\}$$

There is the following commutative diagram from [BM11]:

$$
\begin{tikzcd}
\mathrm{Bl}_d \arrow[rr, "\pi"] \arrow[d, "\phi"] & & M_d \arrow[d, "\mathrm{ev}_i"] \\
Z_d^{(3)} \arrow[r, "e_i"] & Z_d \arrow[r, "p_d"] \arrow[d, "q_d"] & X \\
& Y_d &
\end{tikzcd}
\tag{3.1}
$$

The map $\pi : \mathrm{Bl}_d \to M_d$ is birational, and if $d \leq \min\{k, n-k\}$ then $\phi : \mathrm{Bl}_d \to Z_d^{(3)}$ is also birational. A diagram chase proves the theorem in this case. The key point for general d is that the general fibre of ϕ is rationally connected, thus ϕ is cohomologically trivial. This is proved in type A in [BM11] by putting local coordinates, and in other cominuscule types in ADDREF [Chaput-Perrin]. □

There is a version of the 'quantum=classical' which goes from a Grassmannian to another Grassmannian. Form the following incidence diagram:

$$
\begin{tikzcd}
Z_d := \mathrm{Fl}(k-d, k, k+d; n) \arrow[r, "p_d'"] \arrow[d, "q_d"] & \mathrm{Fl}(k-d, k; n) \arrow[r, "p_d''"] \arrow[d, "q_d'"] & X := \mathrm{Gr}(k; n) \\
Y_d := \mathrm{Fl}(k-d, k+d; n) \arrow[r, "pr"] & \mathrm{Gr}(k-d; n) &
\end{tikzcd}
\tag{3.2}
$$

Here all maps are the natural projections. As before, denote by $p_d : \mathrm{Fl}(k-d, k, k+d; n) \to \mathrm{Gr}(k; n)$ the composition $p_d := p_d'' \circ p_d'$.

Corollary 3.1. *Let $a, b, c \in K_T(\mathrm{Gr}(k; n))$ and $d \geq 0$ a degree. Assume that $(q_d)_*(p_d^*(a)) = pr^*(a')$ for some $a' \in K_T(\mathrm{Gr}(k-d; n)$. Then*

$$\langle a, b, c \rangle_d = \int_{\mathrm{Gr}(k-d;n)} a' \cdot (q_d')_*(p_d''^*(b)) \cdot (q_d')_*(p_d''^*(c)).$$

3.2. **Pieri rule.** One can prove that $q_d' p_d''(\mathcal{O}^\lambda) = \mathcal{O}^{\overline{\lambda}_d}$, where $\overline{\lambda}_d$ is the result if removing the top d rows of λ. Similarly, if one uses $\mathrm{Gr}(k+d; n)$ instead of $\mathrm{Gr}(k-d; n)$, one needs to remove the leftmost d columns. Therefore one has explicit explicit calculations of the coefficients in the products

$$\mathcal{O}^i \circ \mathcal{O}^\lambda = \sum N_{i,\lambda}^{\mu,d} q^d \mathcal{O}^\mu$$

in terms of the classical coefficients for $\mathcal{O}^i \circ \mathcal{O}^\lambda$, found by Lenart ADDREF.

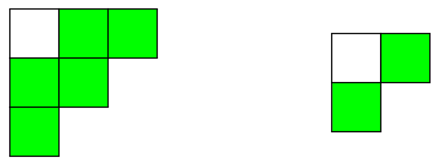
Recall that the **outer rim** of a partition λ consists of the set of boxes which do not have any box strictly SE. One obtains the following formula:

Theorem 3.2 (Pieri rule)**.** *The constants $N_{i,\lambda}^{\mu,d} = 0$ for $d \geq 2$. Furthermore, $N_{i,\lambda}^{\mu,1}$ is nonzero only if $\ell(\lambda) = k$, and μ can be obtained from λ by removing a subset of the* *boxes in the outer rim of λ, with at least one box removed from each row. When these conditions hold, we have*

$$N_{i,\lambda}^{\mu,1} = (-1)^e \binom{r}{e}$$

where $e = |\mu| + n - i - |\lambda|$ and r is the number of rows of μ that contain at least one box from the outer rim of λ, excluding the bottom row of this rim.

Example 3.1. *On $X = \mathrm{Gr}(3,6)$ we have $N_{2,(3,2,1)}^{(2,1),1} = -2$, with $e = 1$ and $r = 2$.*



3.3. **A presentation of** $\mathrm{QK}(\mathrm{Gr}(k;n))$**.** Let $0 \to \mathcal{S} \to \mathbb{C}^n \to \mathcal{Q} \to 0$ be the tautological sequence, where $\mathrm{rk}(\mathcal{S}) = k$. An influential result by Witten [Wit95] proves that $(\mathrm{QH}^*(\mathrm{Gr}(k;n)), \star)$, the quantum cohomology ring of the Grassmannian, is determined by the 'quantum Whitney relations':

$$c(\mathcal{S}) \star c(\mathcal{Q}) = c(\mathbb{C}^n) + (-1)^k q, \tag{3.3}$$

where $c(E) = 1 + c_1(E) + \ldots + c_e(E)$ is the total Chern class of the rank e bundle E. This equation leads to a presentation of $\mathrm{QH}^*(\mathrm{Gr}(k;n))$ by generators and relations:

$$\mathrm{QH}^*(\mathrm{Gr}(k;n)) = \frac{\mathbb{Z}[q][e_1(x), \ldots, e_k(x); e_1(\tilde{x}), \ldots, e_{n-k}(\tilde{x})]}{\left\langle \left(\sum_{i=0}^k e_i(x)\right) \left(\sum_{j=0}^{n-k} e_j(\tilde{x})\right) = 1 + (-1)^k q \right\rangle}. \tag{3.4}$$

The idea if proof is explained in [FP97] (and it is originally due to Ruan-Tian) and it goes as follows.

Proposition 3.2. *Consider a ring $R := \mathbb{Z}[q][e_1, \ldots, e_k, e_1(\tilde{x}), \ldots, e_{n-k}(\tilde{x})]/\langle P_1, \ldots, P_n \rangle$ where P_i's are polynomials in e_i's, $\tilde{e}_j$'s, and q. Assume that:*

- *The specializatons $P_i|_{q=0}$ generate the ideal of relations for $H^*(X)$;*
- *Each $P_i = 0$ in $\mathrm{QH}^*(X)$.*

Then $R \simeq \mathrm{QH}^*(X)$.

The idea is to extend this to QK theory. For that we start by writing down the relations in $\mathrm{QK}(\mathrm{Gr}(k;n))$. One can show that $\lambda_y(\mathcal{S}) \cdot \lambda_y(\mathcal{Q}) = \lambda_y(\mathbb{C}^n)$ in the (equivariant) K-theory ring of $\mathrm{Gr}(k;n)$. They utilize the Hirzebruch λ_y-class $\lambda_y(E) = 1 + yE + \ldots + y^e \wedge^e E$ of a vector bundle E. Our first theorem is an analogue of the quantum Whitney relations (3.3).

Theorem 3.3 (Gu-M-Sharpe-Zou)**.** *The following equality holds in $\mathrm{QK}_T(X)$:*

$$\lambda_y(\mathcal{S}) \star \lambda_y(\mathcal{Q}) = \lambda_y(\mathbb{C}^n) - \frac{q}{1-q} y^{n-k} (\lambda_y(\mathcal{S}) - 1) \star \det \mathcal{Q}. \tag{3.5}$$

Corollary 3.2. *Let $X = (X_1, \dots, X_k)$ and $\tilde{X} = (\tilde{X}_1, \dots, \tilde{X}_{n-k})$. The quantum* K *theory ring* $\mathrm{QK}(\mathrm{Gr}(k;n))$ *has a presentation with generators and relations*

$$\frac{\mathbb{Z}[[q]][e_1(X), \dots, e_k(X), e_1(\tilde{X}), \dots, e_{n-k}(\tilde{X})]}{\langle \prod_{i=1}^k (1+yX_i) \prod_{j=1}^{n-k} = (1+y)^n - \frac{q}{1-q} y^{n-k} \tilde{X}_1 \cdot \ldots \cdot \tilde{X}_{n-k} (\prod_{i=1}^k (1+yX_i) - 1) \rangle}$$

While in cohomology Chern classes of a vector bundle and its dual differ by a sign, the relation is more subtle in K-theory. For example

$$\wedge^i(\mathcal{S}) \cdot \det(S^*) = \wedge^{k-i}(\mathcal{S}^*).$$

(Take Chern character.) The quantum analogue of this is the following.

Theorem 3.4 (Gu-M-Sharpe-Zou)**.** *The following holds in* $\mathrm{QK}_T(\mathrm{Gr}(k;n))$*:*

$$(\lambda_y(\mathcal{S}) - 1) \star \det(\mathcal{Q}) = (1-q)((\lambda_y(\mathcal{S}) - 1) \cdot \det(Q)).$$

Equivalently, for any $i > 0$,

$$\wedge^i(\mathcal{S}) \star \det(\mathcal{Q}) = (1-q) \wedge^{k-i} (\mathcal{S}^*) \cdot \det(\mathbb{C}^n).$$

(Here we included $\det \mathbb{C}^n$, because that's how this statement generalizes to the equivariant setting.)

To prove such statements again one uses the 'quantum=classical'. We illustrate with the following corollary.

Corollary 3.3. *Fix arbitrary $b, c \in K_T(\mathrm{Gr}(k;n))$ and any degree $d \geq 0$. Then the equivariant KGW invariant $\langle \lambda_y(\mathcal{S}), b, c\rangle_d$ satisfies:*

$$\langle \lambda_y(\mathcal{S}), b, c\rangle_d = \int_{\mathrm{Gr}(k-d;n)} \lambda_y(\mathcal{S}_{k-d}) \cdot q_* p^*(b) \cdot q_* p^*(c).$$

In particular, the 2-point KGW invariant $\langle b, c\rangle_d$ satisfies:

$$\langle b, c\rangle_d = \int_{\mathrm{Gr}(k-d;n)} q_* p^*(b) \cdot q_* p^*(c).$$

Based on ideas from physics, one considers the 'twisted superpotential' (see Morrison-Plesser, Closset-Kim and others)

$$\begin{aligned} W \;=\; & \frac{k}{2} \sum_{a=1}^k (\ln X_a)^2 \;-\; \frac{1}{2} \left(\sum_{a=1}^k \ln X_a \right)^2 \\ & +\; \ln\left((-1)^{k-1} q\right) \sum_{a=1}^k \ln X_a \;+\; n \sum_{a=1}^k \mathrm{Li}_2\left(X_a\right). \end{aligned} \tag{3.6}$$

Here Li_2 is the dilogarithm, and the only thing we need is that it satisfies

$$y \frac{\partial}{\partial y} \mathrm{Li}_2(y) \;=\; -\ln(1-y), \tag{3.7}$$

The variables X_i are interpreted as the exponentials of the Chern roots $X_i = e^{x_i}$. In this context, the exterior powers $\wedge^i \mathcal{S}, \wedge^j \mathcal{Q}$ arise as certain **Wilson line operators** considered in the physics literature (Jockers, Mayr et al, Ueda et al). The Coulomb branch (or vacuum) equations for W are

$$\exp\left(\frac{\partial W}{\partial \ln X_i}\right) = 1, \quad 1 \le i \le k. \tag{3.8}$$

These equations are not $S_k \times S_{n-k}$ symmetric, so one needs to symmetrize them. For that, it is convenient to work with the 'shifted Wilson line operators', or, equivalently, with variables

$$z_i = 1 - X_i, \quad (1 \le i \le k).$$

The Coulomb branch equations show that z_i are the roots of a 'characteristic polynomial':

$$f(\xi, z, q) \;=\; \xi^n \;+\; \sum_{i=0}^{n-1} (-1)^{n-i} \xi^i g_{n-i}(z, \lambda, q), \tag{3.9}$$

where $g_j(z, \lambda, q)$ is symmetric in z_i's. (See example below.) This means that $f(\xi, z_i, q) = 0$ for $1 \le i \le k$.

Theorem 3.5 (Gorbounov-Korff, Gu-Sharpe-M.-Zou)**.** *The Vieta relations applied to the characteristic polynomial $f(\xi, z_i, q)$ generate an ideal I such that*

$$\mathbb{C}[[q]][z_1, \dots, z_k; \hat{z}_1, \dots, \hat{z}_{n-k}]/I$$

is isomorphic to $\mathrm{QK}(\mathrm{Gr}(k; n))$.

Example 3.2. *The Coulomb branch relations for* $\mathrm{Gr}(2; 5)$ *are*

$$\sum_{i+j=\ell} e_i(z) e_j(\hat{z}) = g_\ell(z, q) \quad ,$$

for $1 \le \ell \le 5$, where the polynomials $g_\ell(z, \lambda, q)$ are given by

$$g_1 = z_1 z_2; g_2 = g_3 = 0; g_4 = g_5 = -q.$$

In fact, One may solve for $e_i(\hat{z})$ in terms of $e_i(z)$ to obtain:

$$\begin{aligned} e_1(\hat{z}) &= -G_1(z); \\ e_2(\hat{z}) &= G_2(z); \\ e_3(\hat{z}) &= -G_3(z). \end{aligned}$$

Here $G_i(z)$ are the Grothendieck polynomials, given by

$$\begin{aligned} G_1(z) &= z_1 + z_2 - z_1 z_2; \\ G_2(z) &= z_1^2 + z_1 z_2 + z_2^2 - z_1^2 z_2 - z_1 z_2^2; \\ G_3(z) &= z_1^3 + z_1^2 z_2 + z_1 z_2^2 + z_2^3 - z_1^3 z_2 - z_1^2 z_2^2 - z_1 z_2^3. \end{aligned}$$

4. Positivity

Conjecture 1. *(Lenart-Maeno, Buch-M., Buch-Chaput-M.-Perrin) Consider the* QK *product* $\mathcal{O}^\lambda \circ \mathcal{O}^\mu = \sum N_{\lambda,\mu}^{\nu,d} q^d \mathcal{O}^\nu$ *in* $\mathrm{Gr}(k;n)$. *Then*

$$(-1)^{|\lambda|+|\mu|-|\nu|-nd} N_{\lambda,\mu}^{\nu,d} \geq 0.$$

This conjecture was recently proved in [BCMP], in the general case of minuscule Grassmannians.

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