LECTURES ON QUANTUM K THEORY

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Contents

These are expanded notes of my talks at the Schubert Calculus summer school at UIUC (Urbana-Champaign, June 2023) and MSJ-SI23 (Tokyo, July 2023). For (possible) further updates of these notes, please check out my webpage
`https://personal.math.vt.edu//lmihalce/slides.html`

Date: July 26, 2023.

1. Preliminaries

Throughout the notes, $G := \mathrm{GL}_n$ denotes the complex general linear group, B, B^- a pair of opposite Borel subgroups, e.g. the upper/lower triangular matrices. The associated Weyl group is $W = S_n$, the symmetric in n letters. It is generated by simple reflections $s_i = (i, i+1)$ where $1 \le i \le n-1$. Denote by $\ell : W \to \mathbb{N}$ the length function and the longest element by w_0. A sequence $I = (1 \le i_1 < \ldots < i_k \le n-1)$ determines a collection of simple roots s_i, $i \in I$. and a parabolic subgroup $P \subset G$. The **partial flag manifold** $\mathrm{Fl}(i_1, i_2, \ldots, i_s; n)$ is the homogeneous space

$$G/P = \{F_{i_1} \subset F_{i_2} \subset \ldots \subset \mathbb{C}^n : \dim F_{i_s} = i_s\}.$$

Of particular interest will be the **Grassmannians** and the **flag manifolds**:

$$\mathrm{Gr}(k,n) = \{V \subset \mathbb{C}^n : \dim V = k\}$$

and the flag manifold by

$$\mathrm{Fl}(n) = \{F_1 \subset F_2 \subset \ldots \subset \mathbb{C}^n : \dim F_i = i\}.$$

The first example corresponds to $I = \{k\}$ and the second to $I = \{1, 2, \ldots, n-1\}$.

These are homogeneous spaces for G.[1] For a sequence I of simple roots, define W_P to be the subgroup generated by the simple reflections $s_i = (i, i+1)$ where $i \notin I$, and set $W^P := W/W_P$, the set of minimal length representatives. One can check that

$$W^P = \{w \in W : w \text{ has descents at most in positions } i_1, \ldots, i_s\}.$$

If P is a maximal parabolic, i.e. $G/P = \mathrm{Gr}(k,n)$ is a Grassmann manifold, then W^P is in bijection with **partitions** $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $0 \le \lambda_k \le \ldots \le \lambda_1 \le n-k$. The weight of such a partition is $|\lambda| = \lambda_1 + \ldots + \lambda_k$.

Fix $\{e_1, \ldots, e_n\}$ the standard basis of $\mathbb{C}^n$. The maximal torus of diagonal matrices $T \subset G$ acts on G/P and the T-fixed points are coordinate flags $\{E_w : w \in W^P\}$ where

$$E_w := \langle e_{w(1)}, \ldots, e_{w(i_1)} \rangle \subset \langle e_{w(1)}, \ldots, e_{w(i_2)} \rangle \subset \ldots \subset \langle e_{i_1}, \ldots, e_{w(i_k)} \rangle \subset \mathbb{C}^n.$$

To each $w \in W^P$ there are two Schubert varieties:

$$X_w = \overline{B.E_w}; \quad X^w = \overline{B^-.E_w},$$

where $B, B^- \subset \mathrm{GL}_n$ are opposite Borel subgroups. The orbits

$$X_w^\circ = B.E_w; \quad X^{w,\circ} = B^-.E_w$$

are called **Schubert cells**. With these conventions, $X_w^\circ \simeq \mathbb{A}^{\ell(w)}$ and $X^{w,\circ} \simeq \mathbb{A}^{\dim \mathrm{Fl}(\mathbf{i}) - \ell(w)}$. In particular,

$$\dim X_w = \mathrm{codim}\ X^w = \ell(w); \quad X_w \cap X^w = \{E_w\}; \quad X^w = w_0 X_{w_0 w w_\lambda}$$

[1] A cognizant reader may replace G by any complex semisimple Lie group, $\mathrm{Fl}(n)$ by any G/B, and Grassmannians by any cominuscule Grassmannian.

where $w_P \in W_P$ is the longest element. With these definitions, the **Bruhat order** on W^P is defined by

$$v < w \text{ in } W^P \quad \Leftrightarrow \quad X_v \subset X_w \quad \Leftrightarrow \quad X^v \supset X^w.$$

The partial flag manifolds have a stratification by Schubert cells:

$$G/P = \bigsqcup_{w \in W^P} X_w^\circ = \bigsqcup_{w \in W^P} X^{w,\circ}.$$

For further use we also recall that a partial flag variety $\mathrm{Fl}(i_1, \dots, i_s; n)$ is equipped with a tautological sequence of vector bundles:

$$0 \to \mathcal{S}_{i_1} \hookrightarrow \mathcal{S}_{i_2} \hookrightarrow \dots \hookrightarrow \mathcal{S}_{i_s} \hookrightarrow \mathbb{C}^n \twoheadrightarrow \mathcal{Q}_{n-i_1} \twoheadrightarrow \mathcal{Q}_{n-i_2} \twoheadrightarrow \dots \twoheadrightarrow \mathcal{Q}_{n-i_s} \twoheadrightarrow 0,$$

where the subscripts denote the ranks.

1.1. **Goal.** The goal of these lectures is to introduce the quantum K theory ring, some of its basic properties, and computational techniques, mainly for Grassmannians. Before we proceed, we represent schematically the relationships between various (classical and quantum) intersection rings which are available in the literature.

$$\begin{tikzcd}
& QH^*_T(X) \arrow[dl, "t\mapsto 0"'] \arrow[dd] & & QK^*_T(X) \arrow[ll, "gr"'] \arrow[dl, "e^t\mapsto 1"'] \arrow[dd, "q\mapsto 0"] \\
QH^*(X) \arrow[dd, "q\mapsto 0"] & & QK^*(X) \arrow[ll, "gr"'] \arrow[dd] & \\
& H^*_T(X) \arrow[dl, "t\mapsto 0"'] & & K_T(X) \arrow[ll, "gr"'] \arrow[dl, "e^t\mapsto 1"'] \\
H^*(X) & & K(X) \arrow[ll, "gr"'] &
\end{tikzcd}$$

We will not discuss much about the **equivariant** version of all these rings, but most techniques discussed below extend to the equivariant situation, and we will attempt to point out if any changes are needed to make statements in that generality.

A (way too early) example[2] in $\mathrm{QK}(Gr(3,6))$ (here $\deg q = 6$):

$$\begin{aligned}
\mathcal{O}^{(3,2,1)} \circ \mathcal{O}^{(2,1)} &= q\mathcal{O}^{(3)} + 2q\mathcal{O}^{(2,1)} - 2q\mathcal{O}^{(2,2)} \\
&- 2q\mathcal{O}^{(3,1)} + q\mathcal{O}^{(3,2)} + q\mathcal{O}^{(1,1,1)} - 2q\mathcal{O}^{(2,1,1)} + q\mathcal{O}^{(2,2,1)} + q\mathcal{O}^{(3,1,1)} \\
&- q\mathcal{O}^{(3,2,1)} + \mathcal{O}^{(3,3,3)}.
\end{aligned}$$

Note a positivity statement: the sign of the coefficient of $q^d\mathcal{O}^\nu$ is given by the parity of

$$|(3,2,1)| + |(2,1)| - d \cdot \deg q - |\nu| = 6 + 3 - 6d - |\nu|.$$

[2]The example was obtained utilizing A. Buch's `Equivariant Schubert Calculator` available at `https://sites.math.rutgers.edu/ asbuch/equivcalc/`

2. K theory

Good sources for the material included in this section are Brion's 'Lectures on flag manifolds' and Chriss and Ginzburg's 'Representation theory and complex algebraic geometry'.

2.1. **Generalities.** Let X be any algebraic variety. The (Grothendieck) K theory ring, denoted by $K(X)$, is defined as the ring generated by symbols $[E]$ for (algebraic) vector bundles $E \to X$, modulo the relations $[E_2] = [E_1] + [E_3]$ for any short exact sequence of vector bundles $0 \to E_1 \to E_2 \to E_3 \to 0$. The addition and multiplication are given by

$$[E_1] + [E_2] = [E_1 \oplus E_2]; \quad [E_1] \cdot [E_2] = [E_1 \otimes E_2].$$

Then $K(X)$ becomes a commutative ring with identity the (class of the) trivial, rank 1 vector bundle, which we often denote by $\mathcal{O}$. If X is complete (e.g., projective), this ring is equipped with an **intersection pairing**

$$\langle [E], [F] \rangle = \int_X [E] \cdot [F] = \chi(X, E \otimes F).$$

If X is further assumed to be a (complex, quasi-projective) **manifold**, then one can construct 'Poincaré dual classes to line bundles.

Theorem 2.1 (Resolutions of coherent sheaves)**.** *Let X be a smooth, quasi-projective variety and let $\mathcal{F}$ be a coherent sheaf on X. Then $\mathcal{F}$ has a finite resolution by locally free sheaves (aka, vector bundles) on X:*

$$0 \to E_n \to E_{n-1} \to \ldots \to E_1 \to \mathcal{F} \to 0.$$

Furthermore, one may assume that $n \leq \dim X$.

A proof can be found on [Chriss-Ginzburg, Prop. 5.1.29]. The theorem allows us to define the class of any coherent sheaf as an element in K theory:

$$[\mathcal{F}] = \sum_{i=1}^{n} (-1)^{i-1} [E_i] \in K(X).$$

Of course, the most interesting coherent sheaves are the structure sheaves of subvarieties of X. If the subvarieties in question have nice singularities, then the product of classes becomes especially nice. For the following see [Brion, Lemmas 4.1.1 and 4.1.2], who further refers to a lemma of Fulton and Pragacz.

Lemma 2.2. *Let Y, Z be equidimensional Cohen-Macaulay subvarieties of a nonsingular variety X. Assume that the intersection $Y \cap Z$ is proper, i.e., it has the expected dimension $\dim Y + \dim Z - \dim X$. Then each component of the scheme theoretic intersection $Y \cap Z$ has the expected dimension and $Y \cap Z$ is Cohen-Macaulay. Furthermore,*

$$[\mathcal{O}_Y] \cdot [\mathcal{O}_Z] = [\mathcal{O}_{Y \cap Z}] \quad \in K(X).$$

Example 2.3.

- *Any smooth variety is Cohen-Macaulay.*

- *Any Schubert variety is Cohen-Macaulay.*
- *More generally, we have a Kleiman's transversality statement: if $Y \subset X$, then for general $g_1, \dots, g_k \in G$, $Y \cap g_1 X^{w_1} \cap g_2 X^{w_2} \cap \dots \cap g_k X^{w_k}$ is either empty or purely-dimensional, of expected dimension, and Cohen-Macaulay.*
- *(To be defined later.) The moduli space of stable maps $\overline{\mathcal{M}}_{0,n}(G/P, d)$ is Cohen-Macaulay, because it is locally a smooth variety modulo a finite group.*
- *Smooth pull-backs preserve the Cohen-Macaulay property.*

2.1.1. *Functoriality.* In the literature, the Grothendieck ring of vector bundle is sometimes denoted by $K^\circ(X)$, while the Grothendieck *group* of coherent sheaves is denoted by $K_\circ(X)$. Regarding a vector bundle as a locally free sheaf, then taking tensor products gives $K_\circ(X)$ a structure of $K^\circ(X)$-module. (Note the strong similarities to cohomology/homology versions!) In particular, for any morphism $f : X \to Y$, there is a **pull-back ring homomorphism** $f^* : K^\circ(Y) \to K^\circ(X)$ given by $[E] \mapsto [f^* E]$. If f is flat and $Z \subset X$ is a subvariety, then $f^*[\mathcal{O}_Z] = [\mathcal{O}_{f^{-1}(Z)}]$. For a proper morphism $f : X \to Y$, the **push-forward** $f_* : K_\circ(X) \to K_\circ(Y)$ is defined by

$$f_*[\mathcal{F}] = \sum_{i \geq 0} (-1)^i [R^i f_* \mathcal{F}].$$

This sum is finite, as the higher direct images vanish beyond the dimension of X. The push-forward and pull-back satisfy the usual projection formula:

$$f_*(f^*[E] \otimes [\mathcal{F}]) = [E] \otimes f_*[\mathcal{F}] \quad \in K(Y).$$

2.1.2. *The topological filtration and the Chern character.* For simplicity, assume that X is smooth, so we identify $K_\circ(X) \simeq K^\circ(X)$. One big difference between K theory and (co)homology theory is that the K theory is not graded. However, one can define a **topological filtration** by defining $\mathcal{K}^i(X)$ to be the subgroup of $K_\circ(X)$ generated by sheaves $[\mathcal{F}] \in K_\circ(X)$ which have support in codimension $\geq i$. Then

$$K^\circ(X) = \mathcal{K}^0(X) \supset \mathcal{K}^1(X) \supset \dots$$

is a decreasing filtration, and $K(X)$ becomes a **filtered ring**, in the sense that $\mathcal{K}^i(X) \cdot \mathcal{K}^j(X) \subset \mathcal{K}^{i+j}(X)$.

Let $A_*(X)$ denote the Chow group, generated by classes $[Z]$ of irreducible subvarieties $Z \subset X$ modulo rational equivalence, see [Fulton, Intersection Theory]. Let also $Gr(K(X)) = \bigoplus \mathcal{K}_i(X)/\mathcal{K}_{i+1}(X)$ be the associated graded to the topological filtration. The class of a structure sheaf passes through rational equivalence, and one obtains a ring homomorphism

$$\Psi : A_*(X) \to Gr(K(X)); \quad [Z] \mapsto [\mathcal{O}_Z].$$

In cases such as the flag manifolds (or, more generally, in the presence of a paving by affines), this is an isomorphism.

Furthermore, there is always a **Chern character** $ch : K(X) \to A_*(X)_{\mathbb{Q}}$ defined by sending the class of a line bundle $[L]$ to

$$ch[L] = e^{c_1(L)} = 1 + c_1(L) + c_1(L)^2/2! + \dots$$

For a general vector bundle $E \to X$ one uses the splitting principle to define $ch(E)$. If X is smooth, it is shown e.g. in [Ful84] that if $Z \subset X$ is closed and irreducible, then

$$ch(Z) = [Z] + l.o.t.$$

where l.o.t. are terms in homological degree strictly less than $\dim Z$. In other words $ch([\mathcal{O}_Z]) \in \oplus_{j \le i} A_j(X)$, where subscripts denote dimension. The Chern character is always a **ring isomorphism**, if one works over $\mathbb{Q}$.

2.1.3. *The Hirzebruch λ_y class.* For a rank e vector bundle $E \to X$, the **Hirzebruch λ_y class** of E is defined by

$$\lambda_y(E) = 1 + y[E] + y^2[\wedge^2 E] + \dots + y^e[\wedge^e E] \quad \in K(X)[y].$$

This class is multiplicative: if $0 \to E_1 \to E_2 \to E_3 \to 0$ is a short exact sequence then

$$\lambda_y(E_1) \cdot \lambda_y(E_3) = \lambda_y(E_2).$$

The class $\lambda_{-1}(E^*)$ is sometimes called the K-theoretic Chern class of E, denoted by $cK(E)$. This is justified by the observation that if L is a line bundle with first Chern class $c_1(L)$, then

$$ch(\lambda_{-1}(L^*)) = 1 - e^{-c_1(L)} = c_1(L) + h.o.t..$$

Furthermore, the identity

$$(1 - e^x)(1 - e^y) = (1 - e^x) + (1 - e^y) - (1 - e^{x+y})$$

implies that if L' is another line bundle, then

$$cK(L \oplus L') = cK(L) + cK(L') - cK(L \otimes L'),$$

recovering the formal group law for K theory.

Finally, note that the class $\lambda_{-1}(E)$ appears geometrically as an Euler class: if $E \to X$ is a vector bundle with a general section $s : X \to E$, then the zero locus of s has class

$$[\mathcal{O}_{Z(s)}] = \lambda_{-1}(E^*) \in K(X).$$

Virtually everything we stated above extends to the equivariant case. In that case one works with equivariant vector bundles and equivariant coherent sheaves. Informally, if X admits a G-action, an equivariant vector bundle $\pi : E \to X$ is a vector bundle such that its total space admits a G-action, π is G-equivariant, and that the restriction to fibers induces a linear map. On flag manifolds G/P, such vector bundles are induced by representations: if V is a representation of the parabolic group $P \subset G$, define

$$\mathcal{V} = G \times^P V = \{[g, v] : [g, v] \equiv [gp, p^{-1}v]\}$$

This is equipped with a projection map $\pi : \mathcal{V} \to G/P, [g, v] \mapsto gP$, giving it a structure of a G-equivariant bundle. We refer again to Chriss and Ginzburg book for an introduction in equivariant K theory and to the recent book by Anderson and Fulton for more on equivariant intersection theory.

2.2. **K theory of flag manifolds.** For now we let X to be any flag manifold. For any Schubert variety $\Omega \subset X$ with Schubert cell Ω°, define the **boundary** of Ω to be $\partial\Omega = \Omega \setminus \Omega^\circ$. This is a (Cohen-Macaulay) Weil divisor in Ω.

Remark 2.4. *Assume that $G/P = \mathrm{Gr}(k,n)$ is a Grassmann manifold and let $\iota : \mathrm{Gr}(k,n) \to \mathbb{P} = \mathbb{P}(\wedge^k \mathbb{C}^n)$ be the Plücker embedding. Then the boundary of Schubert varieties are also Cartier divisors, corresponding to the restriction of the line bundle $\mathcal{O}_{\mathbb{P}}(1)$ to X_w. More precisely, for any partition $\lambda \subset k \times (n-k)$,*

$$\mathcal{I}^\lambda = \mathcal{O}^\lambda \cdot \mathcal{O}_{\mathrm{Gr}(k,n)}(-1).$$

2.2.1. *The Schubert package.* The (Grothendieck) classes of the structure sheaves of X_w, X^w (for $w \in W^P$) are denoted by $\mathcal{O}_w, \mathcal{O}^w$ respectively. Consider the ideal sheaf of the boundary ∂X^w. This fits into an exact sequence

$$0 \to \mathcal{I}_{\partial X^w} \to \mathcal{O}_{X^w} \to \mathcal{O}_{\partial X^w} \to 0.$$

We denote the classes of $\mathcal{I}_{\partial X^w}$ and $\mathcal{I}_{\partial X_w}$ by $\mathcal{I}_w, \mathcal{I}^w$ respectively. Note that

$$\mathcal{O}_w = \mathcal{O}^{w^\vee} \text{ and } \mathcal{O}^w = \mathcal{O}_{w^\vee}$$

where $w^\vee = w_0 w w_P$ is the minimal length representative for $w_0 w$ in W^P.

Theorem 2.5. *Let $X = G/P$. Then the following hold:*

(a) The Grothendieck classes $\{\mathcal{O}^w\}_{w \in W}$ form a $\mathbb{Z}$-basis of $K(X)$, i.e.,

$$K(X) = \bigoplus_{w \in W^P} \mathbb{Z}\mathcal{O}^w = \bigoplus_{w \in W^P} \mathbb{Z}\mathcal{O}_w.$$

(b) The dual of the Schubert classes are the (opposite) boundary classes, i.e., for any $v, w \in W^P$,

$$\langle \mathcal{O}_v, \mathcal{I}^w \rangle = \langle \mathcal{O}^v, \mathcal{I}_w \rangle = \delta_{v,w}.$$

(c) Let $P \subset Q$ be two parabolic subgroups and $\pi : G/P \to G/Q$ the projection. Then for any $v \in W^P$ and $w \in W^Q$,

$$\pi_* \mathcal{O}_v = \mathcal{O}_{vW_Q}; \quad \pi^* \mathcal{O}^v = \mathcal{O}^v.$$

For proofs of parts (a), (b) we refer to [Bri05, Thm. 3.4.1]. The pull back statement in (c) follows because π is flat, and the push-forward from the Frobenius splitting properties of Schubert varieties, see e.g. [BK05, Thm. 3.3.4(a)].

To emphasize that we utilize a Poincaré dual class rather than its precise formula, we will use the notation

$$(\mathcal{O}_w)^\vee = \mathcal{I}^w; \quad (\mathcal{O}^w)^\vee = \mathcal{I}_w.$$

Remark 2.6. *This theorem implies a recursive formula to generate any Schubert class from the class of a point. Let* $\mathrm{Fl}(\hat{\imath}, n)$ *denote the partial flag manifold parametrizing* $F_1 \subset \ldots \subset \widehat{F_i} \subset \ldots \subset \mathbb{C}^n$*, and let* $p_i : \mathrm{Fl}(n) \to \mathrm{Fl}(\hat{\imath}; n)$ *be the natural projection. Then* $\partial_i = p_i^*(p_i)_*$ *is an endomorphism of* $\mathrm{K}(Fl(n))$ *called the* **Demazure operator***. From the formulae above one can show that*

$$\partial_i(\mathcal{O}^w) = \begin{cases} \mathcal{O}^{ws_i} & \textit{if } ws_i < w; \\ \mathcal{O}^w & \textit{otherwise} \end{cases}$$

We leave this as an exercise, together with the fact that the Demazure operators satisfy $\partial_i^2 = \partial_i$*, and he usual commutation and braid relations.*

Remark 2.7. *One can use part (c) in the above theorem to show that for* $w \in W^P$,

$$\pi_* \mathcal{I}_w = \begin{cases} \mathcal{I}_w & \textit{if } w \in W^Q; \\ 0 & \textit{otherwise} . \end{cases}$$

(This is another exercise.)

Using these formulae and the Möbius inversion one can write the ideal sheaf basis in terms of the Schubert classes and viceversa. Below we record two important situations (cf. [Bri05, Prop. 4.3.2], and [BCMP18, Lemma 3.5]).

Proposition 2.8. *(a) Let* $X = \mathrm{Fl}(n)$*. Then*

$$\mathcal{I}_w = \sum_{v \le w} (-1)^{\ell(w)-\ell(v)} \mathcal{O}_v; \quad \mathcal{O}_w = \sum_{v \le w} \mathcal{I}_w.$$

(b) Let $X = \mathrm{Gr}(k, n)$*. Then for any partition* $\lambda \subset k \times (n-k)$,

$$\mathcal{I}^\lambda = \sum_{\lambda \subset \mu} (-1)^{|\mu/\lambda|} \mathcal{O}^\mu; \quad \mathcal{O}^\lambda = \sum_{\lambda \subset \mu} \mathcal{I}^\mu,$$

where the sums are over partitions $\mu \supset \lambda$ *such that* μ/λ *is a* **rook strip***, i.e. the skew shape does not have two boxes in the same row or column.*

Example 2.9. *In* $\mathrm{K}(\mathrm{Gr}(2,4))$ *we have*

$$\mathcal{I}^{(1)} = \mathcal{O}^{(1)} - \mathcal{O}^{(2)} - \mathcal{O}^{(1,1)} + \mathcal{O}^{(2,1)}.$$

Part of the advertised Schubert package are the structural theorems: how to multiply $\mathcal{O}^w$ by a divisor class (the **Chevalley formula**), or by a generating set of $\mathrm{K}(G/P)$ (a **Pieri formula**). Such formulae have been found in various situations by Lenart, Sottile and Robinson, Buch, Thomas and Yong, Buch and Ravikumar .... In fact, Buch (for Grassmannians) and Thomas and Yong (for minuscule Grassmannians) found (positive) **Littlewood-Richardson rules**. A discussion on these would take us too far afar.

2.2.2. *Positivity.* Any (equivariant, quantum, K) cohomology theory of a flag manifold is expected to satisfy a positivity property. For K theory, this was discovered by Buch for Grassmannians, then proved by Brion for any flag manifold; see below. An equivariant version was proved by Anderson, Griffeth and Miller [AGM11].

Theorem 2.10 (Positivity theorem; Buch [Buc02], Brion [Bri02])**.** *Consider the Schubert expansion in $K(G/P)$:*

$$\mathcal{O}^u \cdot \mathcal{O}^v = \sum c_{u,v}^w \mathcal{O}^w.$$

Then $(-1)^{\ell(u)+\ell(v)-\ell(w)} c_{u,v}^w \geq 0$.

The proof relies on a more general result proved by Brion, stated next, which relies on the Kawamata-Viehweg vanishing theorem.

Definition 2.11. *A variety X has* **rational singularities** *if has a proper resolution of singularities $\pi : X' \to X$ such that (as sheaves) $\pi_* \mathcal{O}_{X'} = \mathcal{O}_X$ and $R^i \pi_* \mathcal{O}_{X'} = 0$ for $i > 0$.*

A variety with rational singularities must be normal and Cohen-Macaulay. Schubert varieties have rational singularities, and so have general intersections of them.

Brion proved the following general positivity statement.

Theorem 2.12. *Let $X = G/P$ and $Y \subset X$ be a subvariety with rational singularities. Consider the expansion*

$$[\mathcal{O}_Y] = \sum a_w \mathcal{O}_w.$$

Then $(-1)^{\ell(w)-\dim Y} a_w \geq 0$.

Proof. We give the proof in the case Y is smooth and $X = \mathbb{P}^n$. Then

$$a_w = \chi(Y \cdot (\mathcal{O}^w)^\vee)) = \chi([\mathcal{O}_Y] \cdot \mathcal{O}_{\mathbb{P}^i} \cdot \mathcal{O}(-1)).$$

If nonempty, the general intersection is a (possibly disconnected) union of smooth varieties. The Kodaira vanishing applied to each component of this intersection implies that

$$\chi([\mathcal{O}_Y] \cdot \mathcal{O}_{\mathbb{P}^i} \cdot \mathcal{O}(-1)) = (-1)^{\dim Y - n - i} H^{\dim Y - n - i}(Y \cap \mathbb{P}^i; \mathcal{O}(-1))$$

proving the claim. □

2.2.3. *Presentations.* We give (Whitney) presentations of the K theory rings in two extremal cases: $X = \mathrm{Gr}(k;n)$ and $X = \mathrm{Fl}(n)$. The proofs are left as exercises. (See also [Las90] and for Grassmannians [GMSZ22].)

Proposition 2.13. *Let $X = \mathrm{Gr}(k;n)$ equipped with the tautological sequence $0 \to \mathcal{S} \to \mathbb{C}^n \to \mathcal{Q} \to 0$. Then*

$$\lambda_y(\mathcal{S}) \cdot \lambda_y(\mathcal{Q}) = \lambda_y(\mathbb{C}^n)$$

and a formal version of this leads to the full ideal of relations in $\mathrm{K}_T(\mathrm{Gr}(k;n))$.

Proposition 2.14. *Let* $X = \mathrm{Fl}(n)$ *equipped with the tautological sequence* $0 \subset \mathcal{S}_1 \subset \mathcal{S}_2 \subset \ldots \subset \mathcal{S}_{n-1} \subset \mathbb{C}^n$. *Then*

$$\lambda_y(\mathcal{S}_1) \cdot \lambda_y(\mathcal{S}_2/\mathcal{S}_1) \cdot \ldots \cdot \lambda_y(\mathbb{C}^n/\mathcal{S}_{n-1}) = \lambda_y(\mathbb{C}^n).$$

A formal version of these equations leads to the full ideal of relations in $\mathrm{K}_T(\mathrm{Fl}(n))$.

3. Definition of quantum K theory and first properties

Plan:

(1) Definitions of QK theory, of curve neighborhoods, and applications;
(2) 'Quantum = classical' , structure theorems, presentations.

3.1. **The moduli space.** Let X be a projective manifold - very soon $X = G/P$. For an effective degree $d \in H_2(X;\mathbb{Z})$, denote by $\overline{\mathcal{M}}_{0,n}(X,d)$ the Kontsevich moduli space of (genus 0, n pointed) stable maps of degree d. This is a projective scheme, with points stable maps:

$$f : (C, p_1, \dots p_n) \to X; \quad f_*[C] = d.$$

Here C is a tree of $\mathbb{P}^1$'s, and f satisfies a **stability condition**. f C' is a component such that $f(C') = cst$, then C' must have at least three mariked points. A marked point is either a node or a marking p_i. There is a natural equivalence relation on this data ensuring that there are finitely many automorphisms. The moduli space comes equipped with evaluation maps $\mathrm{ev}_i : \overline{\mathcal{M}}_{0,n}(X,d) \to X$, sending $f \mapsto f(p_i)$. If $n \geq 3$ and $d = 0$, then $\overline{\mathcal{M}}_{0.n}(X,0) = X \times \overline{\mathcal{M}}_{0,n}$, the product of X with the Mumford moduli space of stable curves. The evaluation maps are all equal to the projection to X.

More generally, for a sequence of effective degrees $d_1, \dots, d_r \in H_2(X)$, we can consider the fibre product

$$\overline{\mathcal{M}}_{0,n_1+\ldots+n_r}(X,(d_1,\dots,d_r)) := \overline{\mathcal{M}}_{0,n_1+1}(X,d_1) \times_X \ldots \times_X \overline{\mathcal{M}}_{0,n_r+1}(X,d_r)$$

This may be identified with a boundary component inside $\overline{\mathcal{M}}_{0,n_1+\ldots n_r)}(X,(d_1+\ldots+d_r))$. We list some important properties of the Kontsevich moduli space.

Theorem 3.1. *Let $X = G/P$ be a flag manifold. Then the following hold:*

- $\overline{\mathcal{M}}_{0,n}(X,d)$ *has finite quotient singularities, hence rational singularities - this follows from construction, see e.g.* [FP97]
- $\overline{\mathcal{M}}_{0,n}(G/P,d)$ *is a connected, thus irreducible variety (Thomsen* [Tho98]*);*
- $\overline{\mathcal{M}}_{0,n}(X,d)$ *is a rational variety (Kim and Pandharipande).*

3.2. **Definition of quantum K theory (after Givental and Lee** [Giv00, Lee04]**).** From now on we will take $X = G/P$ to be any partial flag manifold, or any homogeneous space. This results in fewer technicalities, such as the replacement of the 'virtual fundamental sheaves' of Kontsevich moduli spaces by structure sheaves. For the general construction, consult [Lee04].

We define next the K**-theoretic Gromov-Witten invariants** (KGW). Let $a_1, \dots, a_n \in \mathrm{K}(X)$ and $d \in H_2(X)$. The KGW invariant is

$$\langle a_1, \dots, a_n \rangle_d = \int_{\overline{\mathcal{M}}_{0,n}(X,d)} \mathrm{ev}_1^*(a_1) \cdot \ldots \cdot \mathrm{ev}_n^*(a_n). \tag{3.1}$$

In general the moduli space is not smooth, but since X is, one may write each of the classes a_i as a finite alternating sum of classes of vector bundles. Then (3.1) may be written as a finite alternating sum of sheaf Euler characteristics of vector bundles. In the latter case, the product $\cdot$ is the tensor product $\otimes$.

Example 3.2. *Consider $X = G/P$ a partial flag manifold. Then*

$$\langle 1, \ldots, 1\rangle_d = 1$$

for any degree d. Indeed, from Theorem 3.1 we deduce that $H^i(\mathcal{O}_{\overline{\mathcal{M}}_{0,n}}(X,d)) = 0$ for $i > 0$, hence $\chi(\mathcal{O}_{\overline{\mathcal{M}}_{0,n}}(X,d)) = 1$. More generally, as explained in [Giv00, Cor. 1]*, if $\pi : \overline{\mathcal{M}}_{0,n+1}(X,d) \to \overline{\mathcal{M}}_{0,n}(X,d)$, then $\pi_*[\mathcal{O}_{\overline{\mathcal{M}}_{0,n+1}(X,d)}] = [\mathcal{O}_{\overline{\mathcal{M}}_{0,n}(X,d)}]$ since all the fibers are rational curves. This implies that*

$$\langle a_1, \ldots, a_n, 1\rangle_d = \langle a_1, \ldots, a_n\rangle_d, \tag{3.2}$$

which is the simplest case of the string equation*; see also Lee's paper* [Lee04, §4.4].

Recall that $H_2(X)$ has a basis of effective curve classes, say $[C_1], \ldots, [C_r]$. Consider the sequence of Novikov variables $q = (q_1, \ldots, q_r)$. For $d = d_1[C_1] + \ldots + d_r[C_r]$, set $q^d = q_1^{d_1} \cdot \ldots \cdot q_r^{d_r}$. Define the $\mathbb{Z}[[q]]$-module

$$\mathrm{QK}(X) = \mathrm{K}(X) \otimes \mathbb{Z}[[q]].$$

Assume also that $\mathrm{K}(X)$ has a finite basis $\mathcal{O}^0 = 1, \ldots, \mathcal{O}^n$, and denote by $\mathcal{O}^{i,\vee}$ the dual basis with respect to the intersection pairing.

(For $X = G/P$ a flag variety, one may take Schubert classes $\{\mathcal{O}^w\}_{w\in W^P}$, with the dual basis given by the boundary classes $\mathcal{I}_w$.)

Definition 3.3. *The (small)* **QK pairing** *is defined by*

$$((a,b)) = \langle a, b\rangle + \sum_{d>0} \langle a, b\rangle_d q^d.$$

Here q stands for the sequence of Novikov variables indexed by a basis of $H_2(X)$, and $q^d = q_1^{d_1} \cdot \ldots \cdot q_r^{d_r}$. The QK *pairing is a nondegenerate pairing with values in the formal power series $\mathbb{Z}[[q]]$.*

The quantum K product is the unique product $\circ$ which satisfies

$$((a \circ b, c)) = \sum_{d\geq 0} \langle a, b, c\rangle_d q^d.$$

Example 3.4. *It follows from Example 3.2 that if $X = \mathrm{Gr}(k,n)$ then*

$$((1,1)) = 1 + q + q^2 + \ldots = \frac{1}{1-q}.$$

More generally, if $X = G/P$, then

$$((1,1)) = \frac{1}{\prod_{i=1}^{rank\ H_2(G/P)}(1-q_i)},$$

As a fun exercise, one can use the string equation (3.2) *to check that $a \circ 1 = a$.*

Theorem 3.5 (Givental, Lee)**.** *The product $\circ$ equips $\mathrm{QK}(X)$ with a structure of a commutative, associative ring with identity $1 = [\mathcal{O}_X]$.*

From definition it follows that $\mathrm{K}(X) \simeq \mathrm{QK}(X)/\langle q \rangle$. Since $\mathrm{K}(X)$ is filtered algebra, it induces a filtration on $\mathrm{QK}(X)$, with $\deg q_i = \int_X c_1(T_X) \cap [C_i]$. The associated graded algebra is

$$\mathrm{Gr}\,\mathrm{QK}(X) = \mathrm{QH}^*(X),$$

the **quantum cohomology** of X.

Next we unravel the definition of the QK product and we discuss two equivalent formulations of the definition.

Lemma 3.6. *Consider the product*

$$\mathcal{O}^u \circ \mathcal{O}^v = \sum N_{u,v}^{w,d} q^d \mathcal{O}^w.$$

Then we have the following equivalent formulae for the structure constants $N_{u,v}^{w,d}$:

(a)

$$N_{u,v}^{w,d} = \langle \mathcal{O}^u, \mathcal{O}^v, (\mathcal{O}^w)^\vee \rangle_d - \sum_{d'>0,\kappa} N_{u,v}^{\kappa,d-d'} \langle \mathcal{O}^\kappa, (\mathcal{O}^w)^\vee \rangle_{d'}.$$

(b)

$$\begin{aligned} N_{u,v}^{w,d} =& \langle \mathcal{O}^u, \mathcal{O}^v, (\mathcal{O}^w)^\vee \rangle_d \\ &+ \sum (-1)^s \langle \mathcal{O}^u, \mathcal{O}^v, (\mathcal{O}^{\kappa_0})^\vee \rangle_{d_0} \cdot \langle \mathcal{O}^{\kappa_0}, (\mathcal{O}^{\kappa_1})^\vee \rangle_{d_1} \cdot \ldots \cdot \langle \mathcal{O}^{\kappa_s}, (\mathcal{O}^k)^\vee \rangle_{d_s}; \end{aligned}$$

here the sum is over effective degrees $d_0, \ldots, d_s$ such that $d_0 + \ldots + d_s = d$ and $d_p > 0$ if $p > 0$.

(c) Let $\mathcal{D} \subset \overline{\mathcal{M}}_{0,3}(X,d)$ be the boundary divisors consisting of maps with reducible domain where markings $1,2$ are on the first component, and marking 3 on the last. Then

$$N_{u,v}^{w,d} = \chi(\mathcal{O}_{\overline{\mathcal{M}}_{0,3}(X,d)}(-\mathcal{D}) \cdot \mathrm{ev}_1^*(\mathcal{O}^u) \cdot \mathrm{ev}_2^*(\mathcal{O}^v) \cdot \mathrm{ev}_1^*((\mathcal{O}^w)^\vee)).$$

Note that, unlike in quantum cohomology, both 2 and 3-point invariants are needed to calculate a single structure constants. However, the proof of the associativity is essentially the same as in the cohomological case: it is obtained from equalities obtained by pulling back points in $\mathbb{P}^1 \simeq \overline{\mathcal{M}}_{0,4}$. The pull-backs are simple normal crossing boundary divisors in $\overline{\mathcal{M}}_{0,4}(X,d)$; while in cohomology the class of such a reducible divisor $D = \bigcup D_i$ is the sum if its components $[D_i]$, in K-theory this is an alternating sum

$$[\mathcal{O}_D] = \sum (-1)^{k-1} [\mathcal{O}_{D_{i_1} \cap \ldots \cap D_{i_k}}].$$

This explains the shape of the formula in part (c).

The formulae in the lemma suggest that in general the QK multiplication may not be finite. Indeed, Example 3.2 shows that the KGW invariants are in general nonzero for any degree d. It is not even clear why $1 = [\mathcal{O}_X]$ is the identity in the QK ring! In fact, the QK multiplication **is finite** for flag manifolds [BCMP13, Kat18, ACTI18].

At least for Grassmannians, we will explain this and more as an application of curve neighborhoods of Schubert varieties, and of the 'quantum=classical' statement.

Informally, many calculations of KGW invariants can be traced to two geometric facts:

- (Transversality) If $\Omega_1, \dots, \Omega_n$ satisfy a K-theoretic transversality property, then
$$[\mathcal{O}_{\Omega_1}] \cdot \ldots \cdot [\mathcal{O}_{\Omega_n}] = [\mathcal{O}_{\Omega_1 \cap \ldots \cap \Omega_n}];$$
- (Rational connectedness + mild singularities) If X is a rational/unirational/rationally connected projective variety which has rational singularities, then $\chi(\mathcal{O}_X) = 1$.

The following result provides an important tool for proving that a variety is rationally connected.

Theorem 3.7 (Graber, Harris, Starr)**.** *Let $f : X \to Y$ be any dominant morphism of complete irreducible complex varieties. If Y and the general fiber of f are rationally connected, then X is rationally connected.*

4. Some theorems

In this section we give informal statements of theorems we will talk about in these lectures, and directly related to Schubert Calculus. Quantum K theory draws from many areas, and obviously this list only scratches the surface of what has been done.

Theorem 4.1 (2-point KGW invariants)**.** *(a) Let $X = G/P$ and let $u, v \in W^P$. Then for each d there is an explicitly defined element $u(d) \in W^P$ and the 2-point K theoretic GW invariants are given by*
$$\langle \mathcal{O}_u, \mathcal{I}^v \rangle_d = \delta_{u(d),v}.$$
The Schubert variety $X_{u(d)}$ is the **curve neighborhood** *of X_u.*

(b) The QK metric may be calculated by
$$((\mathcal{O}^u, \mathcal{O}^v)) = \frac{q^{d_{min}(u,v)}}{\prod(1-q_i)}$$
where $q^{d_{min}(u,v)}$ is the minimum power of q in the **quantum cohomology** *product $[X^u] \star [X^v]$.*

Theorem 4.2 (Finiteness)**.** *Let $X = G/P$. The quantum K product is finite, i.e., for any $u, v \in W^P$, $\mathcal{O}^u \circ \mathcal{O}^v \in \mathrm{K}(X) \otimes \mathbb{Z}[q]$.*

Theorem 4.3 ('Quantum = classical')**.** *Assume $X = \mathrm{Gr}(k, n)$ is a Grassmannian. Consider the incidence diagram*

$$\begin{tikzcd} Z_d := \mathrm{Fl}(k-d, k, k+d; n) \arrow[r, "p_d"] \arrow[d, "q_d"] & X := \mathrm{Gr}(k, n) \\ Y_d := \mathrm{Fl}(k-d, k+d; n) & \end{tikzcd}$$

Here, if $d \geq k$ then we set $Y_d := \mathrm{Fl}(k{+}d; n)$ and if $k{+}d \geq n$ then we set $Y_d := \mathrm{Gr}(k{-}d; n)$. In particular, if $d \geq \min\{k, n-k\}$, then Y_d is a single point. Then for any $a, b, c \in \mathrm{K}(X)$ and any effective degree d

$$\langle a, b, c\rangle_d = \int_{Y_d} (q_d)_* p_d^*(a) \cdot (q_d)_* p_d^*(b) \cdot (q_d)_* p_d^*(c).$$

The 'quantum = classical' theorem has many applications, including:

- explicit combinatorial **Pieri/Chevalley formulae** for any (co)minuscule Grassmannians X;
- **Presentations** of $\mathrm{QK}(\mathrm{Gr}(k, n))$ by generators and relations which quantize the Whitney presentation.
- An extension of **Seidel representation** and combinatorics of quantum shapes, generalizing Postnikov's cylinder.

The 'quantum = classical' also made it possible to prove the following:

Theorem 4.4 (Positivity)**.** *Let $X = \mathrm{Gr}(k, n)$ and consider*

$$\mathcal{O}^u \circ \mathcal{O}^v = \sum N_{u,v}^{w,d} q^d \mathcal{O}^w.$$

Then $(-1)^{\ell(w)+nd-\ell(u)-\ell(v)} N_{u,v}^{w,d} \geq 0$.

As the reader will observe, we are not saying much about the quantum K ring of (partial) flag manifolds, beyond Grassmannians. For this, recent results by Syu Kato establish a ring isomorphism between a localization of $\mathrm{QK}(\mathrm{Fl}(n))$ (and more generally $\mathrm{QK}(G/B)$) and the K-theory of 'semi-infinite flag manifolds'. Under this isomorphism, multiplications by (antidominant) line bundles in the QK ring correspond to certain line bundle multiplications on the semi-infinite side. In papers by Lenart, Maeno, Naito, Sagaki, it is built a combinatorial model to multiply by line bundles in $\mathrm{QK}(\mathrm{Fl}(n))$. In addition, this leads to presentations of $\mathrm{QK}(\mathrm{Fl}(n))$, and to proofs that the (double) quantum Grothedieck polynomials represent Schubert classes in the quantum K ring.

Another direction we do not cover is the relation to integrable systems, either via the Bethe Ansatz (as in Gorbounov-Korff), or generalizations of Toda lattice (Koroteev *et al*). Related to this is an area with a high level of activity, that of quantum K theory of cotangent bundles of flag manifolds, or of Nakajima quiver varieties.

5. Curve neighborhoods and first applications

Throughout this section $X = G/P$ is a partial flag manifold. To perform the calculations required in formulae from Lemma 3.6, we need formulae for the two-point KGW invariants of the form $\langle \mathcal{O}^i, (\mathcal{O}^j)^\vee\rangle_d$. These rely on the notion of **curve neighborhoods**. For flag manifolds, this was obtained in a series of papers [BCMP13, BM15], and earlier version also appeared in papers by Chaput, Manivel, and Perrin. We present next the basic facts.

Definition 5.1. *Let $\Omega_1, \dots, \Omega_n \subset X$ be closed subvarieties and fix an effective degree $d \in H_2(X)$.*

(a) The (n-point) **Gromov-Witten variety** *is the intersection*

$$\mathrm{GW}_d(\Omega_1, \dots, \Omega_n) = \mathrm{ev}_1^{-1}(\Omega_1) \cap \dots \cap \mathrm{ev}_n^{-1}(\Omega_n) \subset \overline{\mathcal{M}}_{0,n+a}(X,d).$$

If $\Omega_2 = \ldots = \Omega_n = X$ we will simply use the notation $\mathrm{GW}_d(\Omega_1) = \mathrm{GW}_d(\Omega_1, X, \dots, X)$.

(b) The (n-point) **curve neighborhood** *of $\Omega_1, \dots, \Omega_n$ is defined as the image of the corresponding Gromov-Witten variety:*

$$\Gamma_d(\Omega_1, \dots, \Omega_n) = \mathrm{ev}_{n+1}(\mathrm{GW}_d(\Omega_1, \dots, \Omega_n)).$$

As before, $\Gamma_d(\Omega) := \mathrm{ev}_{n+1}(\mathrm{GW}_d(\Omega))$.

All these may be extended to the case when one has a sequence of degrees $d_1, \dots, d_k$, by replacing the moduli space with an appropriate stratum in the boundary.

Example 5.2. *(a) If $d = 0$, then $\Gamma_0(\Omega_1, \Omega_2) = \Omega_1 \cap \Omega_2$.*

(b) Take $X = \mathbb{P}^n$ and $d > 0$. Then $\Gamma_d(pt) = \mathbb{P}^n$ and

$$\Gamma_d(pt, pt) = \begin{cases} line & d = 1 \\ \mathbb{P}^n & d \geq 2. \end{cases}$$

We also need the notion of cohomological triviality.

Definition 5.3. *Let $f : X \to Y$ be a morphism of algebraic varieties. We say that f is* **cohomologically trivial** *if $f_* \mathcal{O}_X = \mathcal{O}_Y$ and $R^i f_* \mathcal{O}_X = 0$ for $i > 0$.*

Most non-trivial examples of cohomologically trivial maps arise from special cases of a theorem of Kollár:

Theorem 5.4 (Kollár)**.** *Let $f : X \to Y$ be a surjective morphism of projective varieties with rational singularities. If the general fibers of f are rationally connected, then f is cohomologically trivial.*

Initial versions of the next result can be traced to work byw Chaput, Manivel and Perrin. This version can be extracted from [BCMP13].

Theorem 5.5. *Let $\Omega_1, \dots, \Omega_n$ be general translates of Schubert varieties in X. Then the following hold:*

(a) The GW variety $\mathrm{GW}_d(\Omega_1, \dots, \Omega_n)$ is either empty, or locally irreducible of expected dimension, and with rational singularities. Furthermore,

$$\langle [\mathcal{O}_{\Omega_1}], \dots, [\mathcal{O}_{\Omega_n}] \rangle_d = \chi([\mathcal{O}_{\mathrm{GW}_d(\Omega_1, \dots, \Omega_n)}]).$$

(b) The non-empty Gromov-Witten varieties $\mathrm{GW}_d(\Omega_1, \Omega_2)$ are irreducible and rationally connected. In particular, the 2-point curve neighborhood $\Gamma_d(\Omega_1, \Omega_2)$ is also irreducible and rationally connected.

(c) If Ω is any Schubert variety, then $\Gamma_d(\Omega)$ is again a Schubert variety and the evaluation map $\mathrm{ev}_i : \mathrm{GW}(\Omega) \to \Gamma_d(\Omega)$ is cohomologically trivial.

Idea of proof. Part (a) follows from a K-theoretic Kleiman-Bertini type statement, due to Sierra. For (b) we may assume $\Omega_1 = X_u, \Omega_2 = X^v$. The evaluation map $\mathrm{ev}_1 : \overline{\mathcal{M}}(X,d) \to X$ is a G-equivariant locally trivial fibration in Zariski topology. Its fibre F is irreducible and unirational. By base-change, $ev_1^{-1}(X_u) \to X_u$ is also locally trivial, showing $\mathrm{GW}(X_u)$ is irreducible and rationally connected. The image $\Gamma_d(X_u) = \mathrm{ev}_2(\mathrm{GW}(X_u))$ is irreducible and B-stable, thus a B-stable Schubert variety. Then $\mathrm{GW}_d(X_u)$ has an open dense set which is a locally trivial fibration over the cell $\Gamma_d(X_u)^\circ$. The intersection $\mathrm{ev}_1^{-1}(X_u) \cap \mathrm{ev}_2^{-1}(X^v)$ is locally irreducible and it has an open dense set which is a locally trivial fibration over $\Gamma_d(X_u)^\circ \cap X^v$. If non-empty, the latter is irreducible and rational. Since all these varieties have rational singularities, and the (general) fibers of these maps are unirational, the statement follows from Theorem 5.4 and Theorem 3.7. $\square$

Part (c) of the theorem implies that for any $u \in W^P$ and d an effective degree one may define the elements $u(d), u(-d) \in W^P$ by

$$X_{u(d)} = \Gamma_d(X_u); \quad X^{u(-d)} = \Gamma_d(X^u).$$

Using these elements one can immediately calculate any 2-point GW invariant.

Corollary 5.6. *Let $X = G/P$ and let $u, v \in W^P$ be two Weyl group elements and d an effective degree. Then*

$$\langle \mathcal{O}^u, (\mathcal{O}^v)^\vee \rangle_d = \delta_{u(-d),v},$$

(the Kronecker delta symbol).

Proof. From definition,

$$\begin{aligned}
\langle \mathcal{O}^u, (\mathcal{O}^v)^\vee \rangle_d &= \chi(\overline{\mathcal{M}}_{0,3}(X,d); \mathrm{ev}_1^*(\mathcal{O}^u) \cdot \mathrm{ev}_2^*((\mathcal{O}^v)^\vee)) \\
&= \chi(G/P; (\mathrm{ev}_2)_*(\mathrm{ev}_1^*(\mathcal{O}^u) \cdot \mathrm{ev}_2^*((\mathcal{O}^v)^\vee))) \\
&= \chi(G/P; [\mathcal{O}_{\Gamma_d(X^u)} \cdot (\mathcal{O}^v)^\vee) \\
&= \chi(G/P); \mathcal{O}^{u(-d)} \cdot (\mathcal{O}^v)^\vee) \\
&= \delta_{u(-d),v}.
\end{aligned}$$

Here the third equality follows from the projection formula, and the last from the duality. $\square$

Definition 5.7. *For $u, v \in W^P$, define $d_{min}(u,v)$ the minimum degree d for which q^d appears in the quantum cohomology product $[X^u] \star [X^v]$.*

This minimum degree is obviously well defined if $Pic(G/P) \simeq \mathbb{Z}$ (i.e., when P is a maximal parabolic), and more generally it is well defined by results of Postnikov and Fulton-Woodward [FW04]. One can prove that this is the same as the minimum degree d of q such that $GW_d(X^u, w_0 X^v) \neq \emptyset$.

Using this degree, one can calculate the QK pairing between any two Schubert classes:

$$((\mathcal{O}^u, \mathcal{O}^v)) = \sum_{d \geq d_{min}(u,v)} \langle \mathcal{O}^u, \mathcal{O}^v \rangle_d \; q^d = \frac{q^{d_{min}(u,v)}}{\prod (1-q_i)}. \tag{5.1}$$

(This generalizes equivariantly, but one needs to use opposite classes $((\mathcal{O}^u, \mathcal{O}_v))$. Recall that non-equivariantly, $\mathcal{O}_v = \mathcal{O}^{v^\vee}$.)

Example 5.8. *Assume that $X = \mathbb{P}^2$. In this case $K(X)$ has a basis $1 = \mathcal{O}^0, \mathcal{O}^1, \mathcal{O}^2$, where $\mathcal{O}^i$ is the K-theoretic class representing the hyperplane of (complex) codimension i. With respect to this basis, the Poincaré metric $g_{ij} = \int_X \mathcal{O}^i \cdot \mathcal{O}^j$ is given by the matrix*

$$(g_{ij}) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

The QK *metric is obtained by adding $\frac{q}{1-q}$:*

$$((\mathcal{O}^i, \mathcal{O}^j)) = (g_{i,j}) + \frac{q}{1-q} Id.$$

Corollary 5.9 ([BCLM20])**. Assume** *that the* QK *product is finite, and consider the specialization at $q_i \mapsto 1$ for all i of the usual pairing $\chi : \mathrm{QK}(X) \to \mathrm{QK}(pt) = \mathbb{Z}[q]$. Then this is a* **ring** *homomorphism.*

Proof. Write $\mathcal{O}^u \circ \mathcal{O}^v = \sum N_{u,v}^{w,d} q^d \mathcal{O}^w$. By the Frobenius property of the QK pairing,

$$\sum N_{u,v}^{w,d} q^d \frac{1}{\prod(1-q_i)} = ((\mathcal{O}^u \circ \mathcal{O}^v, 1)) = ((\mathcal{O}^u, \mathcal{O}^v)) = \frac{q^{d_{min}(u,v)}}{\prod(1-q_i)}.$$

It follows that $\sum N_{u,v}^{w,d} = 1$. Then the statement follows from the fact that $\chi(\mathcal{O}^u) = 1$ for any u. $\square$

Note that χ is **not** a ring homomorphism for any specialization of $\mathrm{QK}(X)$ (the K-theory specialization, the quantum cohomology specialization etc).

Example 5.10. *Take $a = b = [pt]$ in $\mathbb{P}^1$. Then*

$$\chi(a \cdot b) = 0 \neq \chi(a) \cdot \chi(b) = 1 \cdot 1 = 1.$$

We will show later that in $\mathrm{QK}(\mathbb{P}^1)$, $[pt] \circ [pt] = q$ and we can already prove that $(([pt], [pt])) = \frac{q}{1-q}$.

There is a more general, and rather surprising statement, due to Kato.

Theorem 5.11 (Kato)**.** *Let $\pi : G/P \to G/Q$ be the natural projection for $P \subset Q$. Consider the $\mathbb{Z}[[q]]$-module projection $\pi_* : \mathrm{QK}(G/P) \to \mathrm{QK}(G/Q)$ defined by extending the usual projection $\pi_* : \mathrm{K}(G/P) \to \mathrm{K}(G/Q)$ and specializing $q_i \mapsto 1$ for all i such that $s_i \in W_Q \setminus W_P$. Then this is a ring homomorphism.*

More refined applications require more refined knowledge of the Weyl group elements giving curve neighborhoods.

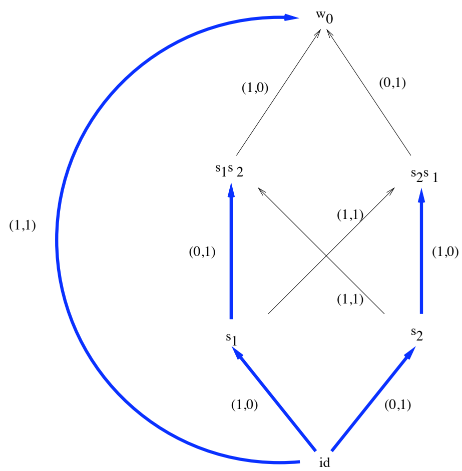
5.1. **Calculation of curve neighborhoods.** The goal is to give an algorithm to calculate the elements $u(d)$ and $u(-d)$. To start,

$$X^{u(-d)W_P} = \Gamma_d(X^{uW_P}) = \Gamma_d(w_0 X_{w_0 u W_P}) = \Gamma_d(w_0 X_{uW_P}) = w_0 X_{u(d)W_P}.$$

This reduces the calculation of $u(-d)$ to that of $u(d)$. For 'small' degrees d, a practical method to do this calculation is based on the **moment graph** of G/P.

The moment graph of G/P has vertices corresponding to $u \in W^P$ and edges $u \xrightarrow{d(i,j)} v$ if $\ell(v) > \ell(u)$ and $u \cdot (i,j) = v$ for $i < j$. The edge has (multi)degree $\varepsilon_i - \varepsilon_j$ modulo Δ_P (the simple roots which are already in P). Then $\Gamma_d(X_u)$ is the (unique!) maximal element in the Bruhat order obtained from tracing a path from u of degree $\leq d$.

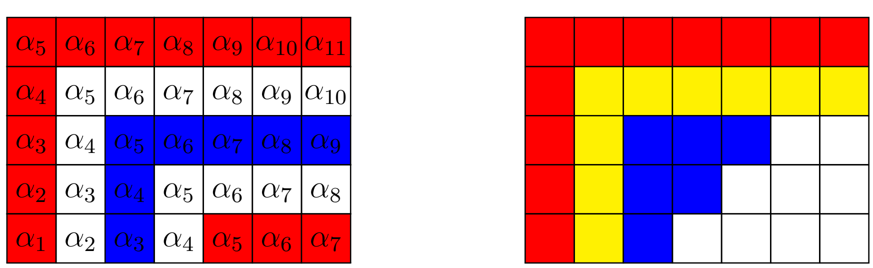
Example 5.12. *Below is the moment graph for* Fl(3). *With blue we drew the paths giving* $\Gamma_{(1,0)}(pt) = X_{s_1}, \Gamma_{(0,1)}(pt) = X_{s_2}, \Gamma_{(1,1)}(pt) = X_{s_1 s_2 s_1}$.



5.1.1. *Curve neighborhoods of Grassmannians.* We now turn to the calculation of curve neighborhoods for Grassmannians. In this case, (or more generally in **cominuscule Grassmannians**) a formula follows from results in [BCMP13], and a procedure is explicitly reviewed in [BCMP18]. Recall that in this case the Schubert classes are indexed by Young diagrams λ included in the $k \times (n-k)$ rectangle, and the curve neighborhoods have particularly nice combinatorial descriptions:

- $\lambda(d)$ is obtained from λ by adding d rim hooks of maximal length;
- $\lambda(-d)$ is obtained from λ by removing d rim hooks of maximal length.

Example 5.13.



On the left: $\emptyset(1), \emptyset(2), \ldots$*; on the right:* $\lambda(2)$, *for* $\lambda = (3,2,1)$.

The key geometric fact which explains this formula for Grassmannians is the following:

Corollary 5.14 ([BCMP13]). *Let X be a (cominuscule) Grassmannian. Then*

$$\Gamma_d(X_u) = \Gamma_1(\Gamma_1(\ldots(\Gamma_1(X_u)))).$$

In other words, if one point may be joined to X_u using a rational curve of degree d, then it may also be joined by a sequence of d lines.

This is special for (cominuscule) Grassmannians. It fails for example for submaximal isotropic Grassmanian $\mathrm{IG}(2,7)$ or for adjoint varieties. The corollary implies the following important simplification of the formulae from Lemma 3.6 for the QK product of Schubert classes in $\mathrm{QK}(\mathrm{Gr}(k;n))$.

Corollary 5.15. *Consider the* QK *product* $\mathcal{O}^\lambda \circ \mathcal{O}^\mu = \sum N_{\lambda,\mu}^{\nu,d} q^d \mathcal{O}^\nu$ *in* $\mathrm{QK}(\mathrm{Gr}(k;n))$. *Then*

$$N_{\lambda,\mu}^{\nu,d} = \langle \mathcal{O}^\lambda, \mathcal{O}^\mu, (\mathcal{O}^\nu)^\vee \rangle_d - \sum_\eta \langle \mathcal{O}^\lambda, \mathcal{O}^\mu, (\mathcal{O}^\eta)^\vee \rangle_{d-1} \cdot \langle \mathcal{O}^\eta, (\mathcal{O}^\nu)^\vee \rangle_1.$$

Proof. We need to show that for λ, μ fixed and fixed $d - d_0 := d_1 + \ldots + d_r \geq 2$, then

$$\sum_{d_1+\ldots+d_r = d-d_0} (-1)^r \langle \mathcal{O}^\lambda, (\mathcal{O}^{\kappa_1})^\vee \rangle_{d_1} \cdot \ldots \cdot \langle \mathcal{O}^{\kappa_r}, (\mathcal{O}^\nu)^\vee \rangle_{d_r} = 0.$$

From Corollary 5.6 it follows that this equals to

$$\begin{aligned}\sum_{d_1+\ldots+d_r = d-d_0} (-1)^r \delta_{\lambda(-d_1),\kappa_1} \cdot \ldots \cdot \delta_{\kappa(-d_r),\mu} &= \sum (-1)^r \delta_{\lambda(-d_1-d_2-\ldots-d_r),\mu} \\ &= \sum_{r=1}^{d-d_0} (-1)^r \binom{d-d_0+r-1-r}{r-1} \\ &= (1-1)^{d-d_0-1} = 0.\end{aligned}$$

□

This formula may be interpreted as

$$\begin{aligned} N_{\lambda,\mu}^{\nu,d} &= \langle (\mathrm{ev}_3)_*[\mathrm{GW}_d(g_1 X^\lambda, g_2 X^\mu)] - (\mathrm{ev}_3)_*[\mathrm{GW}_{d-1,1}(g_1 X^\lambda, g_2 X^\mu)], (\mathcal{O}_\nu)^\vee \rangle \rangle \\ &= \langle [\mathcal{O}_{\Gamma_d(\lambda,\mu)}] - [\mathcal{O}_{\Gamma_{d-1,1}(\lambda,\mu)}], (\mathcal{O}_\nu)^\vee \rangle, \end{aligned}$$

where g_1, g_2 are general in G. In fact, the second equality is slightly incorrect: while we can prove that $(\mathrm{ev}_3)_*[\mathrm{GW}_d(g_1 X^\lambda, g_2 X^\mu)] = [\mathcal{O}_{\Gamma_d(\lambda,\mu)}]$, we do *not know* whether $(\mathrm{ev}_3)_*[\mathrm{GW}_{d-1,1}(g_1 X^\lambda, g_2 X^\mu)] = [\mathcal{O}_{\Gamma_{d-1,1}(\lambda,\mu)}]$. But this is true in many cases, and analyzing this carefully lies at the heart of the proof of positivity for $\mathrm{QK}(\mathrm{Gr}(k;n))$ from [BCMP].

5.1.2. *Curve neighborhoods for arbitrary flag manifolds.* For a general combinatorial procedure, we need two ingredients. The **Demazure product** $\cdot$ of two Weyl group elements is defined as follows. If $u \in W$ and $s_i \in W$ is a simple reflection,

$$u \cdot s_i = \begin{cases} us_i & \ell(us_i) > \ell(u) \\ u & \ell(us_i) < \ell(u). \end{cases}$$

If $v = s_{i_1} \dots s_{i_k}$ is a reduced decomposition, then $u \cdot v = (((u \cdot s_{i_1}) \cdot s_{i_2}) \dots) \cdot s_{i_k}$. This equips $(W, \cdot)$ with a structure of an associative monoid. Let also $z_d \in W$ be the unique element defined by

$$X_{u(d)} = \Gamma_d(pt) \subset \mathrm{Fl}(n).$$

The following combinatorial algorithm to calculate $u(d)$ for any flag manifold has been proved in [BM15].

Theorem 5.16. *The following hold:*

(a) In $\mathrm{Fl}(n)$, $\Gamma_d(X_u) = X_{u \cdot z_d}$.

(b) Take $\alpha > 0$ *be the largest positive root such that* $d - \alpha^\vee \geq 0$ *in* $H_2(\mathrm{Fl}(n))$. *Then*

$$z_d = z_{d-\alpha^\vee} \cdot s_\alpha = s_\alpha \cdot z_{d-\alpha^\vee}.$$

(c) Same procedure applies to any G/P*: take* $\alpha \in R^+ \setminus R_P^+$ *maximal such that* $d - \alpha^\vee \geq 0$ *in* $H_2(G/P)$. *Then*

$$z_d W_P = s_\alpha \cdot z_{d-\alpha^\vee} W_P.$$

In an exercise you are asked about recovering the formulae for z_d using the recursion, in the case of $\mathrm{Fl}(3)$.

The following is conjectural expression for the 'Chevalley' KGW invariants of any partial flag manifold. It can be thought as a replacement for the 'divisor axiom' in quantum K theory.

Conjecture 1. *[Buch-M., 2011] Let* $u, v \in W^P$ *and let* $s_i \in W^P$ *be a simple reflection. Then*

$$\langle \mathcal{O}^{s_i}, \mathcal{O}^u, \mathcal{O}^v \rangle_d = \begin{cases} \langle \mathcal{O}^u, \mathcal{O}^v \rangle_d & \text{if } d_i > 0; \\ \langle \mathcal{O}^{s_i}, \mathcal{O}^{u(-d)}, \mathcal{O}^v \rangle_0 & \text{if } d_i = 0. \end{cases}$$

The conjecture was proved for (cominuscule) Grassmannians [BM11] and recently for incidence flag manifolds $\mathrm{Fl}(1, n-1; n)$ by Weihong Xu [Xu21].

6. The 'quantum=classical' statement and applications

6.1. **The statement.** We start with Buch's notion of **kernel** and **span** of a rational curve.

Definition 6.1. *Let $f : \mathbb{P}^1 \to \mathrm{Gr}(k;n)$ be a morphism of degree d. The* **kernel** *and* **span** *of f are the linear subspaces of $\mathbb{C}^n$ defined by*

$$\ker(f) = \bigcap_{x \in \mathbb{P}^1} f(x); \quad \mathrm{span}(f) = \mathrm{span}\{f(x) : x \in \mathbb{P}^1\}$$

Proposition 6.2 (Buch, Buch-Kresch-Tamvakis)**.** *(a) If $f : \mathbb{P}^1 \to \mathrm{Gr}(k;n)$ is of degree d then* $\dim\ker(f) \geq k-d$ *and* $\dim \mathrm{span} f \leq k+d$*. Furthermore, for a general map f, equalities occur.*

(b) Let $U,V,W \subset \mathrm{Gr}(d,2d)$ be three general spaces. Then there exists a unique morphism $f : \mathbb{P}^1 \to \mathrm{Gr}(n,2n)$ of degree d such that $f(0) = U, f(1) = V, f(\infty) = W$.

Proof. Let S be the tautological bundle on $\mathrm{Gr}(k;n)$. Then $f^*(S) \subset \mathbb{C}^n$, thus $f^*S = \bigoplus_{i=1}^k \mathcal{O}_{\mathbb{P}^1}(-a_i)$ where $a_i \geq 0$ and $\sum a_i = d$. A map $f : \mathbb{P}^1 \to \mathrm{Gr}(k;n)$ is then given by

$$\sum_{j=0}^{a_i} \alpha_j u^{-j} v^{j-a_i} \mapsto \sum_{j=0}^{a_i} \alpha_j \otimes v_j^{(i)}.$$

We have

$$\sum_{i=1}^{k} (1+a_i) = k+d$$

$v_j^{(i)}$'s, showing that the span is at most of dimension $k+d$. But at least $k-d$ of a_i's equal to 0, giving that (for these a_i's) $v_0^{(i)}$ are in the kernel; there are at least $k-d$ of these.

Regarding part (b), observe that $\mathbb{C}^{2d} = U \oplus W$. Take a basis $v_1, \dots, v_d$ of V and project to U,W: $v_i = u_i + w_i$. Define $f[s:t] = [su_1 + tw_1 : \ldots : su_d + tw_d]$. ☐

Consider the 'kernel-span incidence':

$$\begin{tikzcd} Z_d := \mathrm{Fl}(k-d,k,k+d;n) \arrow[r, "p_d"] \arrow[d, "q_d"] & X := \mathrm{Gr}(k,n) \\ Y_d := \mathrm{Fl}(k-d,k+d;n) & \end{tikzcd}$$

Here, if $d \geq k$ then we set $Y_d := \mathrm{Fl}(k+d;n)$ and if $k+d \geq n$ then we set $Y_d := \mathrm{Gr}(k-d;n)$. In particular, if $d \geq \min\{k, n-k\}$, then Y_d is a single point.

Theorem 6.3 (Quantum = classical [BM11])**.** *Let $a,b,c \in K_T(\mathrm{Gr}(k;n))$ and $d \geq 0$ a degree. If $d \geq k$ then we set $d-k := 0$ and if $k+d \geq n$ then we set $k+d := n$. Then* *the following equality holds in $K_T(pt)$:*

$$\langle a, b, c\rangle_d = \int_{Y_d} (q_d)_*(p_d^* a) \cdot (q_d)_*(p_d^* b) \cdot (q_d)_*(p_d^* c).$$

The cohomological version of this theorem was obtained by Buch, Kresch and Tamvakis [BKT03].

Idea of proof. The proof of this is based on the 'quantum = classical' diagram which we explain below. Let $M_d := \overline{\mathcal{M}}_{0,3}(X, d)$,

$$\mathrm{Bl}_d = \{((K, S), f) \in Y_d \times M_d, K \subset \ker(f), \mathrm{span}(f) \subset S\}$$
$$Z_d^{(3)} = \{K \subset V_1, V_2, V_3 \subset S : (K, V_i, S) \in Z_d\}$$

There is the following commutative diagram from [BM11]:

$$\begin{tikzcd} \mathrm{Bl}_d \arrow[rr, "\pi"] \arrow[d, "\phi"] & & M_d \arrow[d, "\mathrm{ev}_i"] \\ Z_d^{(3)} \arrow[r, "e_i"] & Z_d \arrow[r, "p_d"] \arrow[d, "q_d"] & X \\ & Y_d & \end{tikzcd} \tag{6.1}$$

The map $\pi : \mathrm{Bl}_d \to M_d$ is birational, and if $d \leq \min\{k, n - k\}$ then $\phi : \mathrm{Bl}_d \to Z_d^{(3)}$ is also birational. A diagram chase proves the theorem in this case. The key point for general d is that the general fibre of ϕ is rationally connected, thus ϕ is cohomologically trivial. This is proved in type A in [BM11] by putting local coordinates, and in other cominuscule types in [CP11]. $\square$

There is a version of the 'quantum=classical' which goes from a Grassmannian to another Grassmannian. Form the following incidence diagram:

$$\begin{tikzcd} Z_d := \mathrm{Fl}(k-d, k, k+d; n) \arrow[r, "p'_d"] \arrow[d, "q_d"'] & \mathrm{Fl}(k-d, k; n) \arrow[r, "p''_d"] \arrow[d, "q'_d"'] & X := \mathrm{Gr}(k; n) \\ Y_d := \mathrm{Fl}(k-d, k+d; n) \arrow[r, "pr"] & \mathrm{Gr}(k-d; n) & \end{tikzcd} \tag{6.2}$$

Here all maps are the natural projections. As before, denote by $p_d : \mathrm{Fl}(k-d, k, k+d; n) \to \mathrm{Gr}(k; n)$ the composition $p_d := p''_d \circ p'_d$.

Corollary 6.4. *Let $a, b, c \in K_T(\mathrm{Gr}(k; n))$ and $d \geq 0$ a degree. Assume that $(q_d)_*(p_d^*(a)) = pr^*(a')$ for some $a' \in K_T(\mathrm{Gr}(k - d; n)$. Then*

$$\langle a, b, c\rangle_d = \int_{\mathrm{Gr}(k-d;n)} a' \cdot (q'_d)_*(p''^*_d(b)) \cdot (q'_d)_*(p''^*_d(c)).$$

A similar statement holds, relating to the QK(Gr$(k + d; n)$).

6.2. **A Pieri/Chevalley rule.** One can prove that $(q_d')_*(p_d'')^*(\mathcal{O}^\lambda) = \mathcal{O}^{\overline{\lambda}_d}$, where $\overline{\lambda}_d$ is the result of removing the top d rows of λ. Similarly, if one uses $\mathrm{Gr}(k+d;n)$ instead of $\mathrm{Gr}(k-d;n)$, one needs to remove the leftmost d columns. Therefore one has explicit explicit calculations of the coefficients in the products

$$\mathcal{O}^i \circ \mathcal{O}^\lambda = \sum N_{i,\lambda}^{\mu,d} q^d \mathcal{O}^\mu$$

in terms of the classical coefficients for $\mathcal{O}^i \cdot \mathcal{O}^\lambda$, found by Lenart [Len00]. We illustrate the calculation for the QK **Chevalley formula** of $\mathrm{Gr}(k,n)$, following mainly [BM11], see also [BCMP18]. Recall that if $\lambda \subset \mu$ are two partitions, the skew shape μ/λ is a **rook strip** if the skew shape μ/λ has no two boxes on the same row, and on the same column.

Theorem 6.5 (The QK Chevalley formula)**.** *The following holds in* $\mathrm{QK}(\mathrm{Gr}(k,n))$*:*

$$\mathcal{O}^1 \circ \mathcal{O}^\lambda = \sum_\mu (-1)^{|\mu/\lambda|}\mathcal{O}^\mu + \sum_\nu (-1)^{\nu/\lambda(-1)}\mathcal{O}^\nu,$$

where the first sum is over those μ such that μ/λ is a non-empty rook strip; the second sum is empty unless $\lambda_1 = n-k$, $\ell(\lambda) = k$, in which case the sum is over ν such that $\nu = \mu(-1)$ and μ/λ is a rook strip.

Example 6.6. *In* $\mathrm{QK}(\mathrm{Gr}(3,7))$ *we consider the multiplication* $\mathcal{O}^1 \circ \mathcal{O}^{(4,3,1)}$*. Note that*

$$(4,3,1)(-1) = \ydiagram[*(white)]{4,4,1}*[*(green)]{2+2,3,1}\,(-1) = \ydiagram{2}$$

$$\mathcal{O}^1 \circ \mathcal{O}^{(4,3,1)} = q\mathcal{O}^2 - q\mathcal{O}^3 - q\mathcal{O}^{2,1} + q\mathcal{O}^{3,1} + \mathcal{O}^{4,3,2} + \mathcal{O}^{4,4,1} - \mathcal{O}^{4,4,2},$$

or, in terms of shapes,

$$\mathcal{O}^{\ydiagram{1}} \circ \mathcal{O}^{\ydiagram{4,3,1}} = q\mathcal{O}^{\ydiagram{2}} - q\mathcal{O}^{\ydiagram{3}} - q\mathcal{O}^{\ydiagram{2,1}} + q\mathcal{O}^{\ydiagram{3,1}} + \mathcal{O}^{\ydiagram{4,3,2}} + \mathcal{O}^{\ydiagram{4,4,1}} - \mathcal{O}^{\ydiagram{4,4,2}}$$

Idea of proof for Theorem 6.5. The classical part follows from Lenart's Pieri rule. For the quantum part, note that by Corollary 5.15 we have

$$N_{\lambda,(1)}^{\mu,d} = \langle \mathcal{O}^\lambda, \mathcal{O}^{(1)}, (\mathcal{O}^\mu)^\vee\rangle_d - \sum_\eta \langle \mathcal{O}^\lambda, \mathcal{O}^{(1)}, (\mathcal{O}^\eta)^\vee\rangle_{d-1} \cdot \langle \mathcal{O}^\eta, (\mathcal{O}^\mu)^\vee\rangle_1.$$

By an exercise in the homework,

$$\langle \mathcal{O}^\lambda, \mathcal{O}^{(1)}, (\mathcal{O}^\mu)^\vee\rangle_d = \langle \mathcal{O}^\lambda, (\mathcal{O}^\mu)^\vee\rangle_d$$

whenever $d > 1$. In particular, if $d \geq 2$, the right hand side contains the same terms occurring in $1 \circ \mathcal{O}^\lambda$, therefore it must vanish in this case. Thus only q^1 may appear. In this case, the right hand side is equal to

$$\delta_{\lambda(-1),\nu} - \sum_\eta \langle \mathcal{O}^\lambda, \mathcal{O}^{(1)}, (\mathcal{O}^\eta)^\vee\rangle_0 \cdot \langle \mathcal{O}^\eta, (\mathcal{O}^\mu)^\vee\rangle_1 = \delta_{\lambda(-1),\nu} - \sum_\eta N_{\lambda,(1)}^{\eta,0} \cdot \delta_{\eta(-1),\nu}$$

A combinatorial exercise shows that the latter expression is the one claimed. $\square$

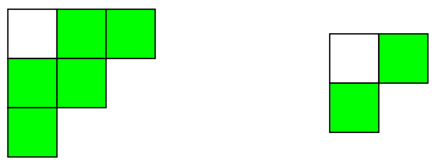
We now give the general **Pieri formula**. Recall that the **outer rim** of a partition λ consists of the set of boxes which do not have any box strictly SE. One obtains the following formula:

Theorem 6.7 (Pieri rule). *The constants $N_{i,\lambda}^{\mu,d} = 0$ for $d \geq 2$. Furthermore, $N_{i,\lambda}^{\mu,1}$ is nonzero only if $\ell(\lambda) = k$, and μ can be obtained from λ by removing a subset of the boxes in the outer rim of λ, with at least one box removed from each row. When these conditions hold, we have*

$$N_{i,\lambda}^{\mu,1} = (-1)^e \binom{r}{e}$$

where $e = |\mu| + n - i - |\lambda|$ and r is the number of rows of μ that contain at least one box from the outer rim of λ, excluding the bottom row of this rim.

Example 6.8. *On $X = \mathrm{Gr}(3,6)$ we have $N_{2,(3,2,1)}^{(2,1),1} = -2$, with $e = 1$ and $r = 2$.*



7. Presentations of the quantum K ring of flag manifolds

Presentations of the (equivariant) quantum K rings of the complete flag manifold have been recently obtained by Lenart, Naito and Sagaki (non-equivariantly) and extended to the equivariant setting by Maeno, Naito and Sagaki. The source of these presentation is given by the theory of quantum Grothendieck polynomials pioneered by Lenart and Maeno [LM06]. In what follows we give (sometimes conjectural) presentations, which aim to generalize the usual (K-theoretic) Whitney relations. These presentations are also related to the Wilson line operator approach ti quantum K theory, arising in physics.

7.1. A presentation for the QK ring of Grassmannians. Let $0 \to \mathcal{S} \to \mathbb{C}^n \to \mathcal{Q} \to 0$ be the tautological sequence, where $\mathrm{rk}(\mathcal{S}) = k$. An influential result by Witten [Wit95] proves that $(\mathrm{QH}^*(\mathrm{Gr}(k;n)), \star)$, the quantum cohomology ring of the Grassmannian, is determined by the 'quantum Whitney relations':

$$c(\mathcal{S}) \star c(\mathcal{Q}) = c(\mathbb{C}^n) + (-1)^k q, \tag{7.1}$$

where $c(E) = 1 + c_1(E) + \ldots + c_e(E)$ is the total Chern class of the rank e bundle E. This equation leads to a presentation of $\mathrm{QH}^*(\mathrm{Gr}(k;n))$ by generators and relations:

$$\mathrm{QH}^*(\mathrm{Gr}(k;n)) = \frac{\mathbb{Z}[q][e_1(x), \ldots, e_k(x); e_1(\tilde{x}), \ldots, e_{n-k}(\tilde{x})]}{\left\langle \left(\sum_{i=0}^k e_i(x)\right) \left(\sum_{j=0}^{n-k} e_j(\tilde{x})\right) = 1 + (-1)^k q \right\rangle}. \tag{7.2}$$

The idea of proof is explained in [FP97] (and it is originally due to Ruan-Tian) and it goes as follows.

Proposition 7.1. *Consider a graded ring $R := \mathbb{Z}[q][e_1, \ldots, e_k, e_1(\tilde{x}), \ldots, e_{n-k}(\tilde{x})]/\langle P_1, \ldots, P_n \rangle$ where P_i's are polynomials in e_i's, $\tilde{e}_j$'s, and q. Assume that:*

- *The specializatons $P_i|_{q=0}$ generate the ideal of relations for $H^*(X)$;*
- *Each $P_i = 0$ in $\mathrm{QH}^*(X)$.*

Then $R \simeq \mathrm{QH}^*(X)$.

The idea is to extend this to QK theory. For that we start by writing down the relations in $\mathrm{QK}(\mathrm{Gr}(k;n))$. One can show that $\lambda_y(\mathcal{S}) \cdot \lambda_y(\mathcal{Q}) = \lambda_y(\mathbb{C}^n)$ in the (equivariant) K-theory ring of $\mathrm{Gr}(k;n)$. They utilize the Hirzebruch λ_y-class $\lambda_y(E) = 1 + yE + \ldots + y^e \wedge^e E$ of a vector bundle E. Our first theorem is an analogue of the quantum Whitney relations (7.1).

Theorem 7.2 (Gu-M-Sharpe-Zou [GMSZ22]). *The following equality holds in $\mathrm{QK}_T(X)$:*

$$\lambda_y(\mathcal{S}) \star \lambda_y(\mathcal{Q}) = \lambda_y(\mathbb{C}^n) - \frac{q}{1-q} y^{n-k} (\lambda_y(\mathcal{S}) - 1) \star \det \mathcal{Q}. \tag{7.3}$$

Corollary 7.3. *Let $X = (X_1, \ldots, X_k)$ and $\tilde{X} = (\tilde{X}_1, \ldots, \tilde{X}_{n-k})$. The quantum K theory ring $\mathrm{QK}(\mathrm{Gr}(k;n))$ has a presentation with generators and relations*

$$\frac{\mathbb{Z}[[q]][e_1(X), \ldots, e_k(X), e_1(\tilde{X}), \ldots, e_{n-k}(\tilde{X})]}{\langle \prod_{i=1}^k (1 + yX_i) \prod_{j=1}^{n-k} = (1+y)^n - \frac{q}{1-q} y^{n-k} \tilde{X}_1 \cdot \ldots \cdot \tilde{X}_{n-k} (\prod_{i=1}^k (1 + yX_i) - 1) \rangle}$$

While in cohomology Chern classes of a vector bundle and its dual differ by a sign, the relation is more subtle in K-theory. For example

$$\wedge^i(\mathcal{S}) \cdot \det(S^*) = \wedge^{k-i}(\mathcal{S}^*).$$

(Take Chern character.) The quantum analogue of this is the following.

Theorem 7.4 (Gu-M-Sharpe-Zou)**.** *The following holds in* $\mathrm{QK}_T(\mathrm{Gr}(k;n))$*:*

$$(\lambda_y(\mathcal{S}) - 1) \star \det(\mathcal{Q}) = (1-q)((\lambda_y(\mathcal{S}) - 1) \cdot \det(Q)).$$

Equivalently, for any $i > 0$*,*

$$\wedge^i(\mathcal{S}) \star \det(\mathcal{Q}) = (1-q) \wedge^{k-i} (\mathcal{S}^*) \cdot \det(\mathbb{C}^n).$$

(Here we included $\det \mathbb{C}^n$, because that's how this statement generalizes to the equivariant setting.)

To prove such statements again one uses the 'quantum=classical'. We illustrate with the following corollary.

Corollary 7.5. *Fix arbitrary* $b, c \in K_T(\mathrm{Gr}(k;n))$ *and any degree* $d \geq 0$*. Then the equivariant KGW invariant* $\langle \lambda_y(\mathcal{S}), b, c \rangle_d$ *satisfies:*

$$\langle \lambda_y(\mathcal{S}), b, c \rangle_d = \int_{\mathrm{Gr}(k-d;n)} \lambda_y(\mathcal{S}_{k-d}) \cdot q_* p^*(b) \cdot q_* p^*(c).$$

In particular, the 2*-point KGW invariant* $\langle b, c \rangle_d$ *satisfies:*

$$\langle b, c \rangle_d = \int_{\mathrm{Gr}(k-d;n)} q_* p^*(b) \cdot q_* p^*(c).$$

Once the relations are proved, to show that they form the *full set of relations*, one needs a K theoretic generalization of the Proposition 7.1 above. This is based on the following two results from commutative algebra (see the Appendix of [GMSZ22]).

Proposition 7.6. *Let* R *be a Noetherian integral domain, and let* $I \subset R$ *be an ideal. Assume that* R *is complete in the* I*-adic topology. Let* M, N *be finitely generated* R*-modules.*

Assume that the R*-module* N*, and the* R/I*-module* N/IN*, are both free modules of the same rank* $p < \infty$*, and that we are given an* R*-module homomorphism* $f : M \to N$ *such that the induced* R/I*-module map* $\overline{f} : M/IM \to N/IN$ *is an isomorphism of* R/I*-modules.*

Then f *is an isomorphism.*

Proposition 7.7. *Let* M *be an* R*-module complete with respect to an ideal* I*. Assume that* M *is equipped with decreasing filtration* (M_n) *such that* $I.M_n \subset M_{n+1}$*, it is separated (i.e.,* $\bigcap_n M_n = 0$*) and it is good (i.e., the associated graded* $\mathrm{gr}M = \bigoplus_i M_i/M_{i+1}$ *is a finitely generated* $\mathrm{gr}R$*-module). Then* M *is a finitely generated* R*-module.*

In our case, $R = \mathrm{K}_T(pt)[[q]]$, $I = \langle q \rangle$, M is the conjectured presentation and $N = \mathrm{QK}_T(X)$. The fact that the relations hold implies that there is a well defined ring homomorphism $f : M \to N$, and the naturality of the construction implies that f is compatible with the corresponding filtrations. Note that R is I-complete and that $\overline{f} : M/IM \to \mathrm{QH}^*(X)$ gives a known presentation of the non-quantum ring $\mathrm{K}_T(X)$.

7.2. **The Coulomb branch presentation.** The study of presentations generated by bundles (rather than Schubert classes) was actually inspired by results in physics [JM20, JMNT20, JM19, UY20], see also the recent [DN, §5]. Informally, the (Schur) bundles correspond to 'Wilson line operators' and the QK ring arises as

$$\mathrm{QK}(X) = \text{ algebra of Wilson operators / relations }.$$

In the physics literature (cf. e.g. [MP95, CK16]), one considers the 'twisted superpotential'

$$\begin{aligned} W \;&=\; \frac{k}{2}\sum_{a=1}^{k} (\ln X_a)^2 \;-\; \frac{1}{2}\left(\sum_{a=1}^{k} \ln X_a\right)^2 \\ &\quad + \ln\left((-1)^{k-1}q\right)\sum_{a=1}^{k} \ln X_a \;+\; n\sum_{a=1}^{k} \mathrm{Li}_2\left(X_a\right). \end{aligned} \tag{7.4}$$

Here

$$\mathrm{Li}_2(z) = \int_1^{1-z} \frac{\ln(t)}{1-t}dt$$

is the **dilogarithm**, and the only thing we need is that it satisfies

$$y\frac{\partial}{\partial y}\mathrm{Li}_2(y) \;=\; -\ln(1-y), \tag{7.5}$$

The variables X_i are interpreted as the exponentials of the Chern roots $X_i = e^{x_i}$. In this context, the exterior powers $\wedge^i \mathcal{S}, \wedge^j \mathcal{Q}$ are the aforementioned Wilson line operators considered in the physics literature. The Coulomb branch (or vacuum) equations for W are

$$\exp\left(\frac{\partial W}{\partial \ln X_i}\right) = 1, \quad 1 \le i \le k. \tag{7.6}$$

This implies that

$$(-1)^{k-1} q \left(X_a\right)^k \;=\; \left(\prod_{b=1}^{k} X_b\right)\left(1 - X_a\right)^n. \tag{7.7}$$

These equations turn out to be equal to the **Bethe Ansatz** equations in an integrable system studied by Gorbounov and Korff [GK17]. There is also an equivariant version of these identities. Let $T_i \in \mathrm{K}_T(pt)$ denote equivariant parameters. These appear in the pertinent physical theories as exponentials of "twisted masses." Concretely, in cases with twisted masses, the superpotential (7.4) for $\mathrm{Gr}(k;n)$ generalizes to [UY20]

$$\begin{aligned} W &= \frac{k}{2}\sum_{a=1}^{k}(\ln X_a)^2 - \frac{1}{2}\left(\sum_{a=1}^{k}\ln X_a\right)^2 \\ &\quad + \ln\left((-1)^{k-1}q\right)\sum_{a=1}^{k}\ln X_a + \sum_{i=1}^{n}\sum_{a=1}^{k}\mathrm{Li}_2\left(X_a T_i^{-1}\right). \end{aligned}$$

Simplifying

$$\exp\left(\frac{\partial W}{\partial \ln X_a}\right) = 1 \tag{7.8}$$

for each $1 \le a \le k$, we find

$$(-1)^{k-1}q(X_a)^k\prod_{j=1}^{n}T_j = \left(\prod_{b=1}^{k}X_b\right)\cdot\prod_{i=1}^{n}(T_i - X_a). \tag{7.9}$$

The equations are not $S_k \times S_{n-k}$ symmetric, so one needs to symmetrize them. For that, it is convenient to work with the 'shifted Wilson line operators', or, equivalently, with variables

$$z_i = 1 - X_i, \quad (1 \le i \le k).$$

The Coulomb branch equations show that z_i are the roots of a 'characteristic polynomial':

$$f(\xi, z, q) = \xi^n + \sum_{i=0}^{n-1}(-1)^{n-i}\xi^i g_{n-i}(z, \lambda, q), \tag{7.10}$$

where $g_j(z,\lambda,q)$ is symmetric in z_i's. (See example below.) This means that $f(\xi, z_i, q) = 0$ for $1 \le i \le k$.

Theorem 7.8 (Gorbounov-Korff, Gu-Sharpe-M.-Zou)**.** *The Vieta relations applied to the characteristic polynomial $f(\xi, z_i, q)$ generate an ideal I such that*

$$\mathbb{C}[[q]][z_1,\dots,z_k;\hat{z}_1,\dots,\hat{z}_{n-k}]/I$$

is isomorphic to $\mathrm{QK}(\mathrm{Gr}(k;n))$.

Example 7.9. *The Coulomb branch relations for* $\mathrm{Gr}(2;5)$ *are*

$$\sum_{i+j=\ell} e_i(z)e_j(\hat{z}) = g_\ell(z,q) \quad ,$$

for $1 \le \ell \le 5$, where the polynomials $g_\ell(z,\lambda,q)$ are given by

$$g_1 = z_1 z_2; g_2 = g_3 = 0; g_4 = g_5 = -q.$$

In fact, One may solve for $e_i(\hat{z})$ *in terms of* $e_i(z)$ *to obtain:*

$$\begin{aligned} e_1(\hat{z}) &= -G_1(z); \\ e_2(\hat{z}) &= G_2(z); \\ e_3(\hat{z}) &= -G_3(z). \end{aligned}$$

Here $G_i(z)$ *are the Grothendieck polynomials, given by*

$$\begin{aligned} G_1(z) &= z_1 + z_2 - z_1 z_2; \\ G_2(z) &= z_1^2 + z_1 z_2 + z_2^2 - z_1^2 z_2 - z_1 z_2^2; \\ G_3(z) &= z_1^3 + z_1^2 z_2 + z_1 z_2^2 + z_2^3 - z_1^3 z_2 - z_1^2 z_2^2 - z_1 z_2^3. \end{aligned}$$

7.3. **A Whitney presentation for QK of partial flag manifolds.** Consider now the partial flag manifold $X = \mathrm{Fl}(i_1, \dots, i_k; n)$ equipped with the tautological sequence $0 = \mathcal{S}_0 \subset \mathcal{S}_1 \subset \cdots \subset \mathcal{S}_k \subset \mathcal{S}_{k+1} = \mathbb{C}^n$ where $\mathcal{S}_j$ has rank r_j.

Conjecture 2 (Gu-M-Sharpe-Xu-Zhang-Zou)**.** *For* $j = 1, \dots, k$*, the following relations hold in* $\mathrm{QK}^T(X)$*:*

$$\lambda_y(\mathcal{S}_j) \star \lambda_y(\mathcal{S}_{j+1}/\mathcal{S}_j) = \lambda_y(\mathcal{S}_{j+1}) - y^{r_{j+1}-r_j} \frac{q_j}{1-q_j} \det(\mathcal{S}_{j+1}/\mathcal{S}_j) \star (\lambda_y(\mathcal{S}_j) - \lambda_y(\mathcal{S}_{j-1})).$$

Note that this generalizes the presentation from Theorem 7.2 above. This presentation is currently being proved in the case $\mathrm{Fl}(1, n-1; n)$ and for $\mathrm{Fl}(n)$, the latter being conditional on the validity of Conjecture 1. There is also an analogue of the twisted superpotential from (7.4), see [GMS+23].

8. Open problems

8.0.1. *Structural theorems, polynomial representatives.* In short, there have been rules to multiply by generators in quantum K theory, obtained either as a 'quantum=classical' statement (for cominuscule Grassmannians), or, more recently in $\mathrm{QK}(G/B)$ from connections to the K theory of semi-infinite flag manifolds (Kato, Lenart, Maeno, Naito, Sagaki, ...). Besides extending the scope of these rules to as many flag manifolds as possible, we would also like to obtain explicit combinatorial rules to multiply by λ_y classes of tautological bundles. For example, we would like to have rules in $\mathrm{QK}(\mathrm{Gr}(k,n)$ to multiply by $\wedge^i \mathcal{S}, \wedge^j \mathcal{Q}$ for any i, j.

8.0.2. *Divisor axiom.* Prove Conjecture 1. This conjecture is known for (cominuscule) Grassmannians (Buch-Chaput-M.-Perrin) and for incidence varieties (W. Xu).

8.0.3. *Positivity.* Prove:

Conjecture 3. *(Lenart-Maeno, Buch-M., Buch-Chaput-M.-Perrin) Consider the* QK *product* $\mathcal{O}^u \circ \mathcal{O}^v = \sum N_{u,v}^{w,d} q^d \mathcal{O}^w$ *in* $\mathrm{QK}(G/P)$*. Then*

$$(-1)^{\ell(w)+\deg q^d - \ell(u) - \ell(v)} N_{u,v}^{w,d} \geq 0.$$

This conjecture was recently proved in [BCMP], in the general case of (minuscule) Grassmannians, but the general case is wide open.

8.0.4. *Is there a maximum quantum degree ?* It is known that the quantum degrees form integer intervals in the multiplication in $\mathrm{QK}(\mathrm{Gr}(k,n))$ and more generally for the quantum K ring of cominuscule Grassmannians. Examples show that the quantum (multi) degrees appearing in $\mathrm{QK}(\mathrm{Fl}(n))$ do not form convex sets. However, it is known that a unique minimum quantum degree exists (Postnikov, Buch-Chung-Li-M.), and examples suggest that for any $u, v \in W$, $\mathcal{O}^u \circ \mathcal{O}^v$ has a **unique maximum degree**.

8.0.5. *Relation to integrable systems and other QK theory theories (cotangent bundles, quasimap QK theory, quantum K theory with level).* There are many questions here:

- It is expected that 'Givental QK theory' and 'quasimap QK theory' agree for flag manifofds. Can one make precise the connection ?
- Integrable systems such as the Bethe Ansatz (or nested version thereof) appear in the study of quantum cohomology and quantum K theory. Various objects in the integrable system (spin basis, on/off-shell Bethe vectors, transfer matrices) are known, or expected to have geometric meaning in terms of (Schubert classes, fixed points, multiplication operators). Building a dictionary (integrable system) $\leftrightarrow$ geometry should benefit both areas, and may shed light on deeper phenomena.

  For example, the 'row-to-row' transfer matrix from the study of the $\mathrm{QH}_T^*(\mathrm{Gr}(k;n))$ Gorbounov and Korff [GK17] are quantum multiplication operators by $c^T(\mathcal{Q})$, the equivariant total Chern class of the tautological quotient bundle.

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