Motivic Chern classes of Schubert cells: applications and experiments

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Schubert Calculus

Schubert Calculus is the study of cohomology theories of flag manifolds (cohomology, K theory, equivariant versions, quantum versions, **cotangent** version etc). Denote such a theory by $\mathcal{H}(G/P)$.

Schubert basis: $\{\sigma_I\} \subset \mathcal{H}(G/P)$, giving structure constants:

$$\sigma_I \cdot \sigma_J = \sum c_{I,J}^K \sigma_K.$$

Positivity of $c_{I,J}^K$ is a central theme. Roughly, positivity is (provably/conjecturally) a consequence of:

- transversality (cohomology);
- sheaf cohomology vanishing (K - theory);
- vector bundle positivity (cotangent).

Log concavity was virtually not encountered, until recently, and it seems to appear in K-**Cotangent Schubert Calculus** . . .

K theory

X complex projective manifold. The K-theory

$$K(X) = \frac{\{[E] : E \to X \text{ vector bundle }\}}{[E] = [F] + [G]},$$

for any short exact sequence $0 \to F \to E \to G \to 0$. Addition and multiplication are given by

$$[E] + [F] := [E \oplus F]; \quad [E] \cdot [F] := [E \otimes F].$$

There is a pairing $\langle \cdot, \cdot \rangle : K(X) \times K(X) \to \mathbb{Z}$ defined by

$$\langle [E], [F] \rangle = \int_X E \otimes F = \sum (-1)^i \dim H^i(X; E \otimes F).$$

If $Y \subset X$ closed subvariety, $\mathcal{O}_Y$ has a **finite** resolution by vector bundles, thus

$$[\mathcal{O}_Y] \in K(X).$$

The Grothendieck group of varieties

Let X algebraic variety.

$$G_0(var/X) = \frac{\{[f : Y \to X] : Y - scheme\}}{[Y \to X] = [Z \to X] + [Y \setminus Z \to X]},$$

for $Z \subset Y$ a closed subvariety. For any $f : X_1 \to X_2$ have a push-forward:

$$\begin{array}{ccc} Y & \longrightarrow & X_1 \\ & \searrow & \downarrow f \\ & & X_2 \end{array}$$

$$f_! : G_0(var/X_1) \to G_0(var/X_2); \quad [g : Y \to X_1] \mapsto [f \circ g : Y \to X_2].$$

Motivic Chern classes

Theorem (Brasselet-Schürmann-Yokura, 2010)

There exists a unique natural transformation

$$\mathrm{MC}_y : G_0(var/X) \to K(X)[y]$$

commuting with proper morphisms such that when X is smooth,

$$\mathrm{MC}[id_X : X \to X] = \lambda_y(T^*X) := \sum [\wedge^i T^*(X)] y^i$$

is the Hirzeburch λ_y class of X.
Further, if $X = pt$, then MC is a ring homomorphism.

Notation: if $Z \subset X$, denote by $\mathrm{MC}_y(Z) := \mathrm{MC}_y[Z \hookrightarrow X]$.
Initial goal: Calculate

$$\mathrm{MC}(X_w^\circ) := \mathrm{MC}[X_w^\circ \hookrightarrow G/B] \in K(G/B),$$

where X_w° is a Schubert cell in a flag manifold G/B. (Feher-Rimányi-Weber, AMSS)

Flag manifolds

$X = G/B$, the flag manifold, where G is complex semisimple and B is a Borel subgroup (e.g. upper triangular matrices). Let T be the maximal torus (diagonal matrices). For $G = \mathrm{SL}_n$,

$$G/B = \mathrm{Fl}(n) = \{F_\bullet : F_1 \subset F_2 \subset \ldots \subset F_n = \mathbb{C}^n\}.$$

Let W be the Weyl group. For each $w \in W$ we have Schubert cells and varieties:

$$X_w^\circ := BwB/B \simeq \mathbb{A}^{\ell(w)}; \quad X^{w,\circ} := B^- wB/B \simeq \mathbb{A}^{\binom{n}{2}-\ell(w)};$$

$$X_w := \overline{X_w^\circ}; \quad X^w := \overline{X^{w,\circ}}.$$

Let

$$\mathcal{O}_w := [\mathcal{O}_{X_w}]; \quad \mathcal{O}^w := [\mathcal{O}_{X^w}] \in K(G/B),$$

the Schubert classes. Then

$$K(G/B) = \oplus_w \mathbb{Z}\mathcal{O}_w = \oplus_w \mathbb{Z}\mathcal{O}^w.$$

the Schubert basis.

Examples

1. $\mathrm{MC}(\mathbb{P}^1) = \lambda_y(T^*_{\mathbb{P}^1}) = (1+y)\mathcal{O}_{\mathbb{P}^1} - 2y\mathcal{O}_{pt}$.
2. By the motivic property:

$$\mathrm{MC}(\mathbb{A}^1 \subset \mathbb{P}^1) = \lambda_y(T^*_{\mathbb{P}^1}) - \lambda_y(T^*_{pt}) = (1+y)\mathcal{O}_{\mathbb{P}^1} - (1+2y)\mathcal{O}_{pt}.$$

3. For $y = -q$, the Hirzebruch χ_y-genus of a Schubert cell is:

$$\int_{\mathrm{Fl}(n)} \mathrm{MC}(X_w^\circ) = \mathrm{MC}[X_w^\circ \to pt] = \mathrm{MC}[\mathbb{A}^1 \to pt]^{\ell(w)} = q^{\ell(w)} = \#_{\mathbb{F}_q} X_w^\circ.$$

4. The χ_y-genus of $\mathrm{Fl}(n)$:

$$\begin{aligned}\int_{\mathrm{Fl}(n)} \mathrm{MC}_{-q}[id : \mathrm{Fl}(n) \to \mathrm{Fl}(n)] &= \sum_{w \in W} \mathrm{MC}[X_w^\circ \to pt] \\ &= \sum_{w \in W} q^{\ell(w)} \\ &= [n]_q!\end{aligned}$$

(the q-analogue of the factorial.)

How to calculate ?

1. **Interpolation/Equivariant localization methods**: Fehér, Rimányi, Tarasov, Varchenko, Weber, ..., in relation to stable envelopes (Aganagic, Maulik, Okounkov).

2. **Resolution of singularities**: AMSS (flag manifolds), Maxim - Schürmann (toric varieties). For flag manifolds, this leads to Demazure-Lusztig operators in Hecke algebras.

Cotangent Schubert classes

$$
\begin{array}{ccc}
\mathbb{C}^* \curvearrowright & & T^*G/B \\
& & \iota \big\uparrow \big\downarrow \pi \\
\mathbb{C}^* \curvearrowright & & G/B
\end{array}
$$

Theorem

1. *(Fehér-Rimányi-Weber). Let* $\mathrm{stab}(w) \in K_{T\times\mathbb{C}^*}(T^*G/B)$ *be the (appropriately normalized) stable envelope. Then*

$$\iota^*(\mathrm{stab}(w)) = \mathrm{MC}(X_w^\circ).$$

2. *(AMSS, in preparation) Let* $i_w : Y(w)^\circ \to G/B$ *be the inclusion. Then:*

$$(\star)\iota^*(gr(i_{w!}\mathbb{Q}^H_{X_w^\circ})) \otimes [\omega^\bullet_{G/B}] = \mathrm{MC}(X_w^\circ),$$

where $gr(i_{w!}\mathbb{Q}^H_{X_w^\circ})$ *is the associated graded sheaf on* $T^*(G/B)$ *determined by the 'constant' mixed Hodge module* $\mathbb{Q}^H_{X_w^\circ}$ *(cf. Tanisaki, Saito).*

Demazure-Lusztig operators

Fix $1 \leq i \leq n-1$, and let $s_i = (i, i+1) \in W$ (simple transposition). Consider the projection: $p_i : G/B \to G/P_i$, with P_i-minimal parabolic. The Demazure operator is

$$\partial_i := (p_i)^*(p_i)_* : K(G/B) \to K(G/B).$$

$$\partial_i \mathcal{O}_w = \begin{cases} \mathcal{O}_{ws_i} & ws_i > w \\ \mathcal{O}_w & ws_i < w. \end{cases}$$

The Demazure-Lusztig operators are:

$$\mathcal{T}_i = \lambda_y(T^*_{p_i})\partial_i - id; \quad \mathcal{T}_i^\vee = \partial_i \lambda_y(T^*_{p_i}) - id.$$

Lemma (Lusztig)

The operators $\mathcal{T}_i$ satisfy the following properties:

1. *(commutativity)* $\mathcal{T}_i\mathcal{T}_j = \mathcal{T}_j\mathcal{T}_i$ *if* $|i-j| \geq 2$;
2. *(braid relations)* $\mathcal{T}_i\mathcal{T}_{i+1}\mathcal{T}_i = \mathcal{T}_{i+1}\mathcal{T}_i\mathcal{T}_{i+1}$;
3. *(quadratic relations):* $(\mathcal{T}_i + y)(\mathcal{T}_i + id) = 0$.

Same properties are satisfied by $\mathcal{T}_i^\vee$ *and* $\langle \mathcal{T}_i(a), b\rangle = \langle a, \mathcal{T}_i^\vee(b)\rangle$.

Theorem (MC classes and their family: mostly AMSS '19, M-Su '19)

1. *(Hecke recursions) Let $w \in W$ and assume that $ws_i > w$. Then*

$$\mathrm{MC}(X(ws_i)^\circ) = \mathcal{T}_i(\mathrm{MC}(X(w)^\circ)).$$

   In particular, $\mathrm{MC}(X(w)^\circ) = \mathcal{T}_{w^{-1}}(\mathcal{O}_{id})$.

2. *(Poincaré duality) Let $\mathcal{D} : K(\mathrm{Fl}(n))[y] \to K(\mathrm{Fl}(n))[y]$ be defined by $\mathcal{D}[E] = E^\vee \otimes \omega_{G/B}[\dim G/B]$ and $\mathcal{D}[y] = y^{-1}$. Then*

$$\langle \mathrm{MC}(X_w^\circ), \frac{\mathcal{D}\mathrm{MC}(X^{v,\circ})}{\lambda_y(T^*(G/B))} \rangle = (\star)\delta_{v,w}.$$

3. *(Hecke / Serre duality) If $\overline{\mathcal{T}_w} := \mathcal{T}_{w^{-1}}^{-1}$, $\overline{y} := y^{-1}$ is the Hecke involution, then*

$$\mathcal{D}\mathrm{MC}(X_w^\circ) = \overline{\mathcal{T}_{w^{-1}}}(\mathcal{O}_{id}).$$

4. *(Segre classes) We have*

$$\mathrm{SMC}(X^{w,\circ}) := \frac{\mathrm{MC}(X^{w,\circ})}{\lambda_y(T^*(G/B))} = (\star)\mathcal{T}^\vee_{w^{-1}w_0}(\mathcal{O}_{pt}).$$

Big cell in $\mathrm{Fl}(3)$

The motivic class for the open cell in $\mathrm{Fl}(3)$ is:

$$\begin{aligned}\mathrm{MC}(X(s_1s_2s_1)^\circ) =& (1+y)^3\mathcal{O}_{s_1s_2s_1} - (1+y)^2(1+2y)(\mathcal{O}_{s_1s_2}+\mathcal{O}_{s_2s_1}) + \\ & (1+y)(5y^2+4y+1)(\mathcal{O}_{s_1}+\mathcal{O}_{s_2}) \\ & -(8y^3+11y^2+5y+1)\mathcal{O}_{id}\end{aligned}$$

Observe (AMSS, in progress):

1. Divisibility by $(1+y)^{\ell(v)}$;
2. Specialize: $y=-1 \rightsquigarrow \mathcal{O}_{id}$;
3. Specialize: $y=0 \rightsquigarrow \mathcal{O}_{X_{s_1s_2s_1}}(-\partial X_{s_1s_2s_1})$ (ideal sheaf of the boundary);
4. Coefficient of $y^{top}=y^3$: $K_{X_{s_1s_2s_1}}$ (the dualizing sheaf).

Let $\mathrm{DMC}(X^{w,\circ})$ denote the Poincaré dual of $\mathrm{MC}(X_w^\circ)$.

Positivity and log concavity I: transition matrices

$$\mathrm{MC}(X(s_1s_2s_1)^\circ) = (1+y)^3\mathcal{O}_{s_1s_2s_1} - (1+y)^2(1+2y)(\mathcal{O}_{s_1s_2} + \mathcal{O}_{s_2s_1}) + (1+y)(5y^2+4y+1)(\mathcal{O}_{s_1} + \mathcal{O}_{s_2}) - (8y^3+11y^2+5y+1)\mathcal{O}_{id}$$

Consider the Schubert expansion:

$$\mathrm{MC}(X_w^\circ) = \sum_v c_{v,w}(y)\mathcal{O}_v.$$

Conjecture

(AMSS (all Lie types), Fehér-Rimanyi-Weber (type A))

1. *(Positivity):*
$$(-1)^{\ell(w)-\ell(v)}c(v;w)(q) \in \mathbb{Z}_{\geq 0}[y].$$
2. *(Log concavity) The polynomial* $(-1)^{\ell(w)-\ell(v)}c_{v;w}(q)$ *has no internal zeros and it is log concave.*

Examples and non-examples

Example

Take flag manifold of Lie type G_2 (dimension 6). Then:

$$c_{id;w_0}(y) = 64y^6 + 141y^5 + 125y^4 + 69y^3 + 29y^2 + 8y + 1 = (\star) \int_{G_2/B} \mathrm{DMC}(X^{id,\circ}).$$

Example

The λ_y class of $\mathrm{Fl}(3)$ is:

$$\sum_w \mathrm{MC}(X_w^\circ) = \lambda_y(\mathrm{Fl}(3)) = (1+y)^3 \mathcal{O}_{\mathrm{Fl}(3)} - 2y(1+y)^2(\mathcal{O}_{s_1 s_2} + \mathcal{O}_{s_2 s_1}) + y(1+y)(5y-1)(\mathcal{O}_{s_1} + \mathcal{O}_{s_2}) - y(8y^2 + y - 1)\mathcal{O}_{pt}.$$

Not positive!

CSM specialization

The cohomological analogues of the motivic Chern classes are called the **Chern-Schwartz-MacPherson** classes. Consider the expansion

$$\mathrm{csm}(X(w)^\circ) = \sum c'(v;w)[X(v)].$$

Then

$$c'(v;w) = \frac{c(v;w)}{(1+y)^{\ell(v)}}|_{y=-1}$$

Example

$$\mathrm{csm}(X(s_1 s_2 s_1^\circ) = [X(s_1 s_2 s_1)] + [X(s_1 s_2)] + [X(s_2 s_1)] + 2([X(s_1)] + [X(s_2)]) + [pt].$$

Theorem (J. Huh (Grassmannians); AMSS (all G/P))

The coefficients $c'(v;w) \geq 0$.

Intermezzo: A transversality formula and point counting

Theorem (Schürmann)

Let $X_1, X_2 \subset X$ intersecting appropriately transversal. Then

$$\mathrm{MC}(X_1 \cap X_2) = \mathrm{SMC}(X_1) \cdot \mathrm{MC}(X_2).$$

Example

Take $\mathrm{Fl}(2) = \mathbb{P}^1$. Then

$$\int_{\mathbb{P}^1} \mathrm{MC}_{-q}(\mathbb{A}^1 \cap g_1\mathbb{A}^1 \cap g_2\mathbb{A}^1) = q - 2 = \#_{\mathbb{F}_q}(\mathbb{A}^1 \cap g_1\mathbb{A}^1 \cap g_2\mathbb{A}^1).$$

Example

For $u \leq v$, define $R_v^u := X_v^\circ \cap X^{u,\circ}$, the Richardson 'cell'. Then

$$\int_{\mathrm{Fl}(n)} \mathrm{MC}_{-q}(R_v^u) = R_{v,u}(q) = \#_{\mathbb{F}_q} R_v^u,$$

where $R_{v,u}$ is the Kazhdan-Lusztig R-polynomial.

Positivity and log concavity II: integrals of dual MC classes

Conjecture (Knutson-M.-Zinn-Justin)

The following polynomials have non-negative coefficients and are log concave:

$$(1+y)^{\dim} \int_{\mathrm{Fl}(n)} \mathrm{DMC}(X^{w,\circ}); \quad (1+y)^{\dim} \int_{\mathrm{Fl}(n)} \mathrm{DMC}(X^{u,\circ}) \cdot \mathrm{DMC}(X^{v,\circ}).$$

Example

$$\int_{\mathrm{Fl}(5)} \mathrm{DMC}(X^{id,\circ}) \cdot \mathrm{DMC}(X^{id,\circ}) = 59049y^{10} + 120138y^9 + 112132y^8 + 66978y^7 + 30120y^6 + 11112y^5 + 3427y^4 + 850y^3 + 155y^2 + 18y + 1.$$

Positivity and log concavity III: Structure constants

Consider the multiplication

$$\mathrm{DMC}(X^{u,\circ}) \cdot \mathrm{DMC}(X^{v,\circ}) = \sum_{w} c_{u,v}^{w}(y)\mathrm{DMC}(X^{w,\circ}).$$

Observe:

$$c_{u,v}^{w}(y) = \int_{\mathrm{Fl}(n)} \mathrm{DMC}(X^{u,\circ}) \cdot \mathrm{DMC}(X^{v,\circ}) \cdot \mathrm{MC}(X_{w}^{\circ})$$
$$\neq \chi_y(X^{u,\circ} \cap g_1 X^{v,\circ} \cap g_2 X_{w}^{\circ}).$$

Conjecture (Knutson-M.-Zinn-Justin)

The polynomials

$$(-1)^{\ell(u)+\ell(v)-\ell(w)} c_{u,v}^{w}(-y)$$

have non-negative coefficients and are log concave.

The positivity was recently proved for partial flag manifolds with ≤ 4 steps, by Knutson and Zinn-Justin using integrable systems.

Example

Consider $\mathrm{Fl}(5)$ (dimension 10), and w_0 the longest element in S_5. Then

$$c_{id,id}^{w_0}(y) = y^{10} - 22y^9 + 92y^8 - 130y^7 + 76y^6 - 18y^5 + 2y^4.$$

Observe:

$$c_{id,id}^{w_0}(-1) = 341 = \chi_{-1}(X^{id,\circ} \cap g_1 X^{id,\circ} \cap g_2 X^{id,\circ}).$$

(The Euler characteristic of the intersection of 3 translates of open cells in $\mathrm{Fl}(5)$.)

Conjecture

Let $u, v, w \in W$. Then

$$(-1)^{\ell(u)+\ell(v)-\ell(w)} \chi(X^{u,\circ} \cap X^{v,\circ} \cap X_w^\circ) \geq 0.$$

This conjecture appears in the context of Segre-MacPherson classes, which are the cohomological analogues of $\mathrm{DMC}/\mathrm{SMC}$ classes.

Conclusion

The study of motivic Chern and related classes leads to (conjecturally) positive, and log concave polynomials, coming from two sources:

1. Transition matrices (*MC* to Schubert classes; *DMC* to *MC* classes);
2. Structure constants for multiplication.

In some instances, this theory seems better behaved than the classical KL theory: e.g. the KL-R polynomials are not log concave, but similar 'cotangent' polynomials are expected to be.

Question. Does Hodge geometry determine these properties ?

MERCI À TOUS!