Applications of characteristic classes

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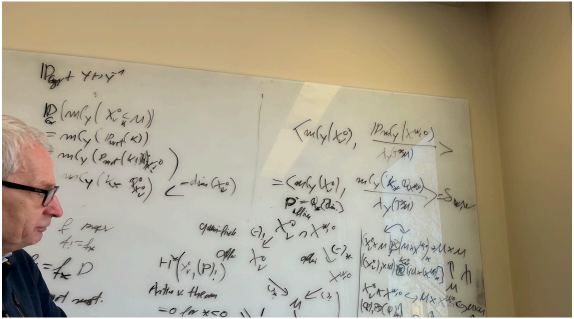


Structures on singular spaces through the lens of characteristic classes
Heidelberg, Germany
https://personal.math.vt.edu//lmihalce/Heidelberg.pdf

What we (can) do

We study characteristic classes coming from two sources:

- MacPherson's transformation:

$$c_* : \mathcal{F}(X) \to H_*(X).$$

- Motivic Chern class transformation (Brasselet-Schürmann-Yokura):

$$MC : G_0(var/X) \to K(X)[y].$$

  (The unnormalized Hirzerbruch transformation $Td_y = td_* \circ MC$, where td_* is the Todd tranformation.)

In our case X is a **flag manifold** G/P, such as the Grassmannian $\mathrm{Gr}(k;n)$ and the complete flag manifold $\mathrm{Fl}(n)$. These are homogeneous spaces, and their study leads to applications such as:

1. Formulae in combinatorics and rep. theory;
2. Constructing polynomials associated to characteristic classes;
3. Calculations of (some!) multiplicities in microlocal geometry;
4. (Cotangent) Schubert Calculus / Calculus of (Okounkov's) stable envelopes.

Number of reduced expressions (Peterson)

Let $\lambda \subset k \times (n-k)$ be a partition, e.g.

$$\lambda = \begin{array}{|c|c|c|}\hline & & \\ \hline & & \\ \hline \end{array} \rightsquigarrow \text{simple reflections} : \begin{array}{|c|c|c|}\hline s_2 & s_3 & s_4 \\ \hline s_1 & s_2 & s_3 \\ \hline \end{array} \rightsquigarrow \text{heights} : \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & 3 & 4 \\ \hline \end{array}$$

Define $w_\lambda := s_3 s_2 s_1 s_4 s_3 s_2 \in S_5$.

Theorem

(Frame-Robinson-Thrall, . . . , Proctor, Stembridge, Peterson) Let $\lambda \subset k \times (n-k)$. Then the number of reduced decompositions of w_λ is equal to

$$\mathrm{Red}(w_\lambda) = \frac{|\lambda|!}{\prod_{\square \in \lambda} \mathrm{ht}(\lambda)}$$

Example

We take $\lambda = (3,3)$ and we place simple reflections on its boxes:

s_2	s_3	s_4
s_1	s_2	s_3

giving $w_\lambda = s_3 s_2 s_1 s_4 s_3 s_2$. Then

$$\mathrm{Red}(w_{(3,3)}) = \frac{6!}{1 \times 2 \times 3 \times 2 \times 3 \times 4} = 5.$$

One may check directly that the 5 reduced decompositions of w are:

$$s_3 s_2 s_1 s_4 s_3 s_2, \quad s_3 s_2 s_4 s_1 s_3 s_2, \quad s_3 s_2 s_4 s_3 s_1 s_2, \quad s_3 s_4 s_2 s_1 s_3 s_2, \quad s_3 s_4 s_2 s_3 s_1 s_2.$$

They correspond to the (reverse) standard Young tableaux:

6	5	4
3	2	1

6	5	3
4	2	1

6	4	3
5	2	1

6	5	2
4	3	1

6	4	2
5	3	1

MacPherson's transformation

Let $\mathcal{F}(X)$ be the group of constructible functions.

Theorem (Deligne - Grothendieck Conjecture; MacPherson '74, M. H. Schwartz '65)

There exists a unique natural transformation $c_* : \mathcal{F}(X) \to H_*(X)$ such that:

1. *If X is projective, non-singular, $c_*(\mathbb{1}_X) = c(T_X) \cap [X]$.*
2. *c_* is functorial with respect to proper push-forwards $f : X \to Y$.*

Constructible functions $\rightsquigarrow$ characteristic classes of singular varieties:

- $\varphi = \mathbb{1}_U$ ($U \subset X$ constructible) $\rightsquigarrow$ Chern-Schwartz-MacPherson (CSM) class

$$c_{\mathrm{SM}}(U) \in H_*(X).$$

- If X-smooth, the Segre-MacPherson class is:

$$s_{\mathrm{M}}(U) = \frac{c_{\mathrm{SM}}(U)}{c(T_X)}.$$

Flag manifolds

$X = G/B$, the flag manifold, where G is complex semisimple and B is a Borel subgroup (e.g. upper triangular matrices). Let T be the maximal torus (diagonal matrices). For $G = \mathrm{SL}_n$,

$$G/B = \mathrm{Fl}(n) = \{F_\bullet : F_1 \subset F_2 \subset \ldots \subset F_n = \mathbb{C}^n\}.$$

Let W be the Weyl group. For each $w \in W$ we have Schubert cells and varieties:

$$X_w^\circ := BwB/B \simeq \mathbb{A}^{\ell(w)}; \quad X^{w,\circ} := B^- wB/B \simeq \mathbb{A}^{\binom{n}{2}-\ell(w)};$$

$$X_w := \overline{X_w^\circ}; \quad X^w := \overline{X^{w,\circ}}.$$

Let

$$[X_w], [X^w] \in H_*(G/B); \quad \mathcal{O}_w := [\mathcal{O}_{X_w}], \mathcal{O}^w := [\mathcal{O}_{X^w}] \in K(G/B),$$

the Schubert classes. Then

$$H_*(G/B) = \oplus_w \mathbb{Z}[X_w] = \oplus_w \mathbb{Z}[X^w]; \quad K(G/B) = \oplus_w \mathbb{Z}\mathcal{O}_w = \oplus_w \mathbb{Z}\mathcal{O}^w.$$

gives the (cohomological/K-theoretic) Schubert basis.

Equivariant multiplicities

It is known that

$$(G/P)^T = \bigsqcup_{w \in W^P} e_w \quad \rightsquigarrow \quad H_*^T(G/P)_{loc} = \oplus_{w \in W^P} H_T^*(pt)_{loc}[\mathbf{e}_w]_T.$$

Let $\kappa \in H_T^*(G/P)$. The **equivariant multiplicity** of κ at e_w is the coefficient m_w^κ in the expansion

$$\kappa = \sum m_w^\kappa [\mathbf{e}_w]_T.$$

Of course,

$$m_w^\kappa = \frac{\kappa|_w}{c_{top}^T(T_w(G/P))},$$

but the notion makes sense for more general (singular) spaces (Brion).

Equivariant multiplicities of Richardson varieties

Let $R_u^v := X_u \cap X^v$ be a **Richardson variety**. From Schürmann's transversality formula we obtain:

$$c_{\mathrm{SM}}(R_u^v) = s_{\mathrm{M}}(X^v) \cdot c_{\mathrm{SM}}(X_u)$$

which implies that

$$m_u^{c_{\mathrm{SM}}(R_u^v)} = \frac{s_{\mathrm{M}}(X^v)|_u \cdot c_{\mathrm{SM}}(X_u)|_u}{c^T(T_u(G/P))} = \frac{s_{\mathrm{M}}(X^v)|_u}{s_{\mathrm{M}}(X_u)|_u}.$$

IF R_u^v is smooth at e_u, then X^v is smooth at e_u and

$$\frac{s_{\mathrm{M}}(X^v)|_u}{s_{\mathrm{M}}(X_u)|_u} = \prod_{\alpha : v \le s_\alpha u < u} (1 + \frac{1}{\alpha}).$$

Nakada's formula

Let $\mathcal{L}$ be an ample line bundle on G/P. The Chevalley formula for $s_{\mathrm{M}}(X_u) \cdot c_1^T(\mathcal{L})$ (Su) and the associativity

$$(s_{\mathrm{M}}(X_u) \cdot s_{\mathrm{M}}(X^v)) \cdot c_1^T(\mathcal{L}) = s_{\mathrm{M}}(X_u) \cdot (s_{\mathrm{M}}(X^v) \cdot c_1^T(\mathcal{L}_\lambda))$$

gives

$$\frac{s_{\mathrm{M}}(X^v)|_u}{s_{\mathrm{M}}(X_u)|_u} = \sum \frac{m_1}{\beta_1} \cdot \frac{m_2}{\beta_1 + \beta_2} \cdots \frac{m_r}{\beta_1 + \beta_2 + \ldots + \beta_r}$$

where the sum is over certain chains $v \to u$ in the Bruhat order, with weights given by roots $\beta_1, \ldots, \beta_r$.

Theorem (M.-Naruse-Su)

R_u^v is smooth at u iff a generalization of Nakada's formula holds:

$$\sum \frac{m_1}{\beta_1} \cdot \frac{m_2}{\beta_1 + \beta_2} \cdots \frac{m_r}{\beta_1 + \beta_2 + \ldots + \beta_r} = \prod_{\alpha : v \leq s_\alpha u < u} (1 + \frac{1}{\alpha}).$$

The number of reduced decompositions

In the previous formula specialize as follows:

- $G/P = \mathrm{Gr}(k;n)$;
- take terms of minimal (negative) degree;
- specialize $\beta \mapsto height(\beta)$.

The Weyl group elements correspond to partitions $u = w_\lambda, v = w_\mu$. In this case one obtains that all chains are maximal, each $m_i = 1$, each β_i is a simple root, and each chain has length exactly $|\lambda/\mu| = |\lambda| - |\mu|$. Therefore one obtains:

$$\#(chains) \times \frac{1}{(|\lambda| - |\mu|)!} = \prod_{w_\mu \le s_\alpha w_\mu < w_\lambda} \frac{1}{height(\alpha)}$$

This may be interpreted as

$$\#\mathrm{Red}(w_\mu^{-1} w_\lambda) \frac{1}{(|\lambda| - |\mu|)!} = \prod_{\square \in \lambda/\mu} \frac{1}{\mathrm{ht}(\square)}$$

Remark: This is more general than the earlier formula, which is only for $\mu = \emptyset$, i.e., when $X^v = G/P$, thus $R_u^v = X_u$.

Motivic Chern classes

Theorem (Brasselet-Schürmann-Yokura, 2010)

There exists a unique natural transformation

$$\mathrm{MC}_y : G_0(var/X) \to K(X)[y]$$

commuting with proper morphisms such that when X is smooth,

$$\mathrm{MC}[id_X : X \to X] = \lambda_y(T^*X) := \sum [\wedge^i T^*(X)] y^i$$

is the Hirzeburch λ_y class of X.
Further, if $X = pt$, then MC is a ring homomorphism.

Notation: if $Z \subset X$, denote by $\mathrm{MC}_y(Z) := \mathrm{MC}_y[Z \hookrightarrow X]$.
Initial goal: Calculate

$$\mathrm{MC}(X_w^\circ) := \mathrm{MC}[X_w^\circ \hookrightarrow G/B] \in K(G/B),$$

where X_w° is a Schubert cell in a flag manifold G/B. (Feher-Rimányi-Weber, AMSS)

Examples

1. $\mathrm{MC}(\mathbb{P}^1) = \lambda_y(T^*_{\mathbb{P}^1}) = (1+y)\mathcal{O}_{\mathbb{P}^1} - 2y\mathcal{O}_{pt}$.
2. By the motivic property:
$$\mathrm{MC}(\mathbb{A}^1 \subset \mathbb{P}^1) = \lambda_y(T^*_{\mathbb{P}^1}) - \lambda_y(T^*_{pt}) = (1+y)\mathcal{O}_{\mathbb{P}^1} - (1+2y)\mathcal{O}_{pt}.$$
3. For $y = -q$, the Hirzebruch χ_y-genus of a Schubert cell is:
$$\int_{\mathrm{Fl}(n)} \mathrm{MC}(X_w^\circ) = \mathrm{MC}[X_w^\circ \to pt] = \mathrm{MC}[\mathbb{A}^1 \to pt]^{\ell(w)} = q^{\ell(w)} = \#_{\mathbb{F}_q} X_w^\circ.$$
4. The χ_y-genus of $\mathrm{Fl}(n)$:
$$\begin{aligned}\int_{\mathrm{Fl}(n)} \mathrm{MC}_{-q}[id : \mathrm{Fl}(n) \to \mathrm{Fl}(n)] &= \sum_{w\in W} \mathrm{MC}[X_w^\circ \to pt] \\ &= \sum_{w \in W} q^{\ell(w)} \\ &= [n]_q!\end{aligned}$$

(the q-analogue of the factorial.)

How to calculate ?

1. **Interpolation/Equivariant localization methods**: Fehér, Rimányi, Tarasov, Varchenko, Weber, ..., in relation to stable envelopes (Aganagic, Maulik, Okounkov).

2. **Resolution of singularities**: AMSS (flag manifolds), Maxim - Schürmann (toric varieties). For flag manifolds, this leads to Demazure-Lusztig operators in Hecke algebras.

Intermezzo: Cotangent Schubert classes

$$\begin{array}{ccc} \mathbb{C}^* \curvearrowright & & T^*G/B \\ & & \iota \uparrow \downarrow \pi \\ \mathbb{C}^* \curvearrowright & & G/B \end{array}$$

Theorem

1. *(Fehér-Rimányi-Weber, AMSS '19). Let* $\mathrm{stab}(w) \in K_{T \times \mathbb{C}^*}(T^*G/B)$ *be the (appropriately normalized) stable envelope. Then*

$$\iota^*(\mathrm{stab}(w)) = \mathrm{MC}(X_w^\circ).$$

2. *(AMSS, in preparation) Let* $i_w : X_w^\circ \to G/B$ *be the inclusion. Then:*

$$(\star)\iota^*(gr(i_{w!}\mathbb{Q}^H_{X_w^\circ})) \otimes [\omega^\bullet_{G/B}] = \mathrm{MC}(X_w^\circ),$$

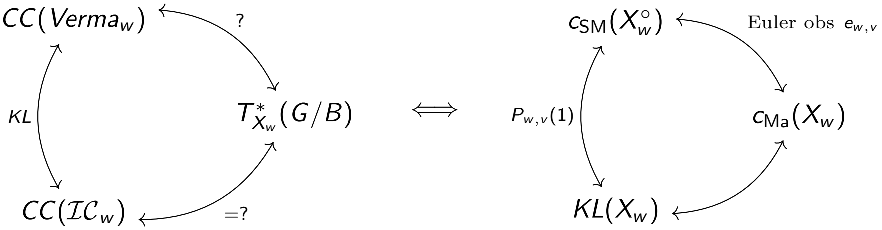
where $gr(i_{w!}\mathbb{Q}^H_{X_w^\circ})$ *is the associated graded sheaf on* $T^*(G/B)$ *determined by the 'constant' mixed Hodge module* $\mathbb{Q}^H_{X_w^\circ}$ *(cf. Tanisaki, Saito).*

The Lagrangian model for MacPherson's transformation

Theorem (Sabbah '85, Ginzburg '86, AMSS '17)

Let T be a torus and X a T-manifold, and let $T \times \mathbb{C}^*$ act on T^*X, where $\mathbb{C}^*$ acts by dilation induced by the character $\hbar^{-1}$. Consider $c_* : \mathcal{F}_T(X) \to H_*^T(X)$ to be the MacPherson transformation, extended equivariantly by Ohmoto. Then for any constructible function $\varphi \in \mathcal{F}(X)$,

$$\iota^*[CC(\varphi)]_{T\times\mathbb{C}^*} = c_*(\varphi)_\hbar.$$



Remarks.

1. If $P_{w,v}(q)$ is the Kazhdan-Lusztig polynomial, then

$$KL(X_w) = \sum P_{w,v}(1) c_{\mathrm{SM}}(X_v^\circ).$$

2. $CC(\mathcal{IC}_w) = T^*_{X_w}(G/P)$ iff $c_{\mathrm{Ma}}(X_w) = KL(X_w)$. This holds for all minuscule Grassmannians (type A: Bressler-Finkelberg-Lunts; types A,D: Boe-Fu; types E_6, E_7: M.-Singh).
3. $CC(\mathcal{IC}_w)$ is reducible in general (Kashiwara-Saito, Tanisaki, Boe-Fu, Braden, Williamson, ...)
4. Equivalently, let $c_{\mathrm{Ma}}(X_w) = \sum e_{w,v} c_{\mathrm{SM}}(X_v^\circ)$. Then

$$CC(\mathcal{IC}_w) \textit{ irreducible} \quad \Longleftrightarrow \quad e_{w,v} = P_{w,v}(1).$$

Bold conjecture : $e_{w,v} \geq 0$. (True for all cominuscule G/P's: see above for types A,D,E; Levan-Raicu in type C.)

Demazure-Lusztig operators

Fix $1 \leq i \leq n-1$, and let $s_i = (i, i+1) \in W$ (simple transposition). Consider the projection: $p_i : \mathrm{Fl}(n) \to \mathrm{Fl}(\hat{i}; n)$. The Demazure operator is

$$\partial_i := (p_i)^*(p_i)_* : K(\mathrm{Fl}(n)) \to K(\mathrm{Fl}(n)).$$

$$\partial_i \mathcal{O}_w = \begin{cases} \mathcal{O}_{ws_i} & ws_i > w \\ \mathcal{O}_w & ws_i < w. \end{cases}$$

The Demazure-Lusztig operators are:

$$\mathcal{T}_i = \lambda_y(T^*_{p_i})\partial_i - id; \quad \mathcal{T}_i^\vee = \partial_i \lambda_y(T^*_{p_i}) - id.$$

Lemma (Lusztig)

The operators $\mathcal{T}_i$ satisfy the following properties:

1. *(commutativity)* $\mathcal{T}_i\mathcal{T}_j = \mathcal{T}_j\mathcal{T}_i$ *if* $|i-j| \geq 2$;
2. *(braid relations)* $\mathcal{T}_i\mathcal{T}_{i+1}\mathcal{T}_i = \mathcal{T}_{i+1}\mathcal{T}_i\mathcal{T}_{i+1}$;
3. *(quadratic relations):* $(\mathcal{T}_i + y)(\mathcal{T}_i + id) = 0$.

Same properties are satisfied by $\mathcal{T}_i^\vee$ *and* $\langle \mathcal{T}_i(a), b\rangle = \langle a, \mathcal{T}_i^\vee(b)\rangle$.

Theorem (MC classes and their family: mostly AMSS '19, M-Su '19)

1. *(Hecke recursions) Let $w \in W$ and assume that $ws_i > w$. Then*

$$\mathrm{MC}(X(ws_i)^\circ) = \mathcal{T}_i(\mathrm{MC}(X(w)^\circ)).$$

In particular, $\mathrm{MC}(X(w)^\circ) = \mathcal{T}_{w^{-1}}(\mathcal{O}_{id})$.

2. *(Poincaré duality) Let $\mathcal{D} : K(\mathrm{Fl}(n))[y] \to K(\mathrm{Fl}(n))[y]$ be defined by $\mathcal{D}[E] = E^\vee \otimes \omega_{G/B}[\dim G/B]$ and $\mathcal{D}[y] = y^{-1}$. Then*

$$\langle \mathrm{MC}(X_w^\circ), \frac{\mathcal{D}\mathrm{MC}(X^{v,\circ})}{\lambda_y(T^*(G/B))} \rangle = (\star)\delta_{v,w}.$$

3. *(Hecke / Serre duality) If $\overline{\mathcal{T}_w} := \mathcal{T}_{w^{-1}}^{-1}$, $\overline{y} := y^{-1}$ is the Hecke involution, then*

$$\mathcal{D}\mathrm{MC}(X_w^\circ) = \overline{\mathcal{T}_{w^{-1}}}(\mathcal{O}_{id}).$$

4. *(Segre classes) We have*

$$\mathrm{SMC}(X^{w,\circ}) := \frac{\mathrm{MC}(X^{w,\circ})}{\lambda_y(T^*(G/B))} = (\star)\mathcal{T}^\vee_{w^{-1}w_0}(\mathcal{O}_{pt}).$$

Big cell in $\mathrm{Fl}(3)$

The motivic class for the open cell in $\mathrm{Fl}(3)$ is:

$$\begin{aligned}\mathrm{MC}(X(s_1s_2s_1)^\circ) =&(1+y)^3\mathcal{O}_{s_1s_2s_1} - (1+y)^2(1+2y)(\mathcal{O}_{s_1s_2}+\mathcal{O}_{s_2s_1})+\\ &(1+y)(5y^2+4y+1)(\mathcal{O}_{s_1}+\mathcal{O}_{s_2})\\ &-(8y^3+11y^2+5y+1)\mathcal{O}_{id}\end{aligned}$$

Observe (AMSS '23):

1. Divisibility by $(1+y)^{\ell(v)}$;
2. Specialize: $y=-1 \rightsquigarrow \mathcal{O}_{id}$;
3. Specialize: $y=0 \rightsquigarrow \mathcal{O}_{X_{s_1s_2s_1}}(-\partial X_{s_1s_2s_1})$ (ideal sheaf of the boundary);
4. Coefficient of $y^{top}=y^3$: $K_{X_{s_1s_2s_1}}$ (the dualizing sheaf).

Remark. The specialization $y=1$ in $MC(\mathcal{IC}(X_w))$ is (expected to be) related to the $\mathbb{L}$-class (Banagl-Schürmann-Wrazidlo '23).

Positivity I: transition matrices

$$\begin{aligned}\mathrm{MC}(X(s_1 s_2 s_1)^\circ) =& (1+y)^3 \mathcal{O}_{s_1 s_2 s_1} - (1+y)^2(1+2y)(\mathcal{O}_{s_1 s_2} + \mathcal{O}_{s_2 s_1}) + \\ & (1+y)(5y^2+4y+1)(\mathcal{O}_{s_1} + \mathcal{O}_{s_2}) \\ & -(8y^3+11y^2+5y+1)\mathcal{O}_{id}\end{aligned}$$

Consider the Schubert expansion:

$$\mathrm{MC}(X_w^\circ) = \sum_v c_{v,w}(y)\mathcal{O}_v.$$

Conjecture

(AMSS (all Lie types), Fehér-Rimanyi-Weber (type A))

1. *(Positivity):*
$$(-1)^{\ell(w)-\ell(v)} c(v;w)(y) \in \mathbb{Z}_{\geq 0}[y].$$
2. *(Log concavity) The polynomial* $(-1)^{\ell(w)-\ell(v)} c_{v;w}(y)$ *has no internal zeros and it is log concave.*

An example and one non-example

Example

Take flag manifold of Lie type G_2 (dimension 6). Then:

$$c_{id;w_0}(y) = 64y^6 + 141y^5 + 125y^4 + 69y^3 + 29y^2 + 8y + 1.$$

Example

The λ_y class of $\mathrm{Fl}(3)$ is:

$$\sum_w \mathrm{MC}(X_w^\circ) = \lambda_y(\mathrm{Fl}(3)) = (1+y)^3\mathcal{O}_{\mathrm{Fl}(3)} - 2y(1+y)^2(\mathcal{O}_{s_1s_2} + \mathcal{O}_{s_2s_1}) + y(1+y)(5y-1)(\mathcal{O}_{s_1} + \mathcal{O}_{s_2}) - y(8y^2+y-1)\mathcal{O}_{pt}.$$

Not positive!

CSM specialization

The cohomological analogues of the motivic Chern classes are called the **Chern-Schwartz-MacPherson** classes. Consider the expansion

$$\mathrm{csm}(X(w)^\circ) = \sum c'(v;w)[X(v)].$$

Then

$$c'(v;w) = \frac{c(v;w)}{(1+y)^{\ell(v)}}|_{y=-1}$$

Example

$$\mathrm{csm}(X(s_1s_2s_1)^\circ) = [X(s_1s_2s_1)] + [X(s_1s_2)] + [X(s_2s_1)] + 2([X(s_1)] + [X(s_2)]) + [pt].$$

Theorem (J. Huh (Grassmannians); AMSS '17 (all G/P))

The coefficients $c'(v;w) \geq 0$.

Intermezzo 2: A transversality formula and point counting

Theorem (Schürmann)

Let $X_1, X_2 \subset X$ intersecting appropriately transversal. Then

$$\mathrm{MC}(X_1 \cap X_2) = \mathrm{SMC}(X_1) \cdot \mathrm{MC}(X_2).$$

Example

Take $\mathrm{Fl}(2) = \mathbb{P}^1$. Then

$$\int_{\mathbb{P}^1} \mathrm{MC}_{-q}(\mathbb{A}^1 \cap g_1\mathbb{A}^1 \cap g_2\mathbb{A}^1) = q - 2 = \#_{\mathbb{F}_q}(\mathbb{A}^1 \cap g_1\mathbb{A}^1 \cap g_2\mathbb{A}^1).$$

Example

For $u \leq v$, define $R_v^u := X_v^\circ \cap X^{u,\circ}$, the Richardson 'cell'. Then

$$\int_{\mathrm{Fl}(n)} \mathrm{MC}_{-q}(R_v^u) = R_{v,u}(q) = \#_{\mathbb{F}_q} R_v^u,$$

where $R_{v,u}$ is the Kazhdan-Lusztig R-polynomial.

Positivity II: Structure constants

Let $\mathrm{DMC}(X^{w,\circ})$ denote the Poincaré dual of $\mathrm{MC}(X^\circ_w)$. Consider the multiplication

$$\mathrm{DMC}(X^{u,\circ}) \cdot \mathrm{DMC}(X^{v,\circ}) = \sum_w c^w_{u,v}(y)\mathrm{DMC}(X^{w,\circ}).$$

Observe:

$$c^w_{u,v}(y) = \int_{\mathrm{Fl}(n)} \mathrm{DMC}(X^{u,\circ}) \cdot \mathrm{DMC}(X^{v,\circ}) \cdot \mathrm{MC}(X^\circ_w)$$
$$\neq \chi_y(X^{u,\circ} \cap g_1 X^{v,\circ} \cap g_2 X^\circ_w).$$

Conjecture (Knutson-M.-Zinn-Justin)

The polynomials

$$(-1)^{\ell(u)+\ell(v)-\ell(w)} c^w_{u,v}(-y)$$

have non-negative coefficients (and are log concave).

The positivity was recently proved for partial flag manifolds with ≤ 4 steps, by Knutson and Zinn-Justin using integrable systems.

Example

Consider $\mathrm{Fl}(5)$ (dimension 10), and w_0 the longest element in S_5. Then

$$c_{id,id}^{w_0}(y) = y^{10} - 22y^9 + 92y^8 - 130y^7 + 76y^6 - 18y^5 + 2y^4.$$

Observe:

$$c_{id,id}^{w_0}(-1) = 341 = \chi_{-1}(X^{id,\circ} \cap g_1 X^{id,\circ} \cap g_2 X^{id,\circ}).$$

(The Euler characteristic of the intersection of 3 translates of open cells in $\mathrm{Fl}(5)$.)

Theorem (Simpson - Schürmann - Wang '23)

Let $u, v, w \in W$. Then

$$(-1)^{\ell(u)+\ell(v)-\ell(w)} \chi(X^{u,\circ} \cap X^{v,\circ} \cap X_w^{\circ}) \geq 0.$$

These are precisely the structure constants obtained by multiplying Poincaré duals of CSM classes of Schubert cells, i.e., of Segre-MacPherson classes (cf. AMSS'17).

Conclusion

1. Localization properties of characteristic classes, recovers, and sometimes improves, existing formulae in combinatorics.
2. There are several longstanding problems, such as calculating Mather classes of Schubert varieties.
3. The study of CSM, Mather, and motivic Chern classes leads to (conjecturally) positive, and log concave polynomials, coming from two sources:
   - Transition matrices (MC to Schubert classes);
   - Structure constants for multiplication.

   Question. What (Hodge) geometry determines these properties ?

Alles Gute zum Geburtstag, Jörg!

Dimension polynomials (AMSS, M.-Singh '22)

Assume that $X = G/P$ (e.g., any partial flag manifold). Take $\kappa \in H_*(X)$ and expand

$$\kappa = \sum_w a_w [X(w)].$$

The **dimension polynomial** of κ is defined by

$$D(\kappa) = \sum a_w x^{\ell(w)}.$$

Example

Recall that in $H_*(\mathrm{Fl}(3))$,

$$\kappa = \mathrm{csm}(X(s_1 s_2 s_1)^\circ) = [X(s_1 s_2 s_1)] + [X(s_1 s_2)] + [X(s_2 s_1)] + 2([X(s_1)] + [X(s_2)]) + [pt] \in$$

Then $D(\kappa) = x^3 + 2x^2 + 2x + 1$.

We know that if $\kappa = \mathrm{csm}(X(w)^\circ) \in H_*(G/P)$ or $\kappa = c_{Ma}(\Omega_\lambda) \in H_*(\mathrm{Gr}(k;n))$ (the **Mather class** of the Schubert **variety**), then $D(\kappa) \in \mathbb{Z}_{\geq 0}[x]$.

Obligatory conjecture

Conjecture

1. *The dimension polynomial for the CSM class of any Schubert cell is unimodal. If $G = \mathrm{GL}_n$ (i.e., X is a partial flag manifold) then $D(\mathrm{csm}(X(w)^\circ))$ is log concave.*
2. *Assume that $X = \mathrm{Gr}(k; n)$. Then the dimension polynomial of the Mather class of any Schubert variety is log concave.*

Example

Consider

$$c_{Ma}\left(\ydiagram{3,2,1}\right) = \ydiagram{3,2,1} + 4\,\ydiagram{3,2} + 4\,\ydiagram{3,1,1} + 4\,\ydiagram{2,2,1} + 15\,\ydiagram{3,1} + 15\,\ydiagram{2,2} + 15\,\ydiagram{2,1,1}$$
$$+ 17\,\ydiagram{3} + 52\,\ydiagram{2,1} + 17\,\ydiagram{1,1,1} + 54\,\ydiagram{2} + 54\,\ydiagram{1,1} + 60\,\ydiagram{1} + 24\ \emptyset.$$

Then

$$D\left(c_{Ma}\left(\ydiagram{3,2,1}\right)\right) = x^6 + 12x^5 + 45x^4 + 86x^3 + 108x^2 + 60x + 24.$$