Zero divisors and group von Neumann algebras

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www.math.vt.edu/people/plinnell/ (homepage)
www.ams.org/mathscinet/search/author.html?mrauthid=114455 (MathSciNet)
http://arxiv.org/a/linnell_p_1 (arXiv)

Group algebras

Definition

Let G be a group and let k be a field. Then the group algebra kG is the k-vector space with basis G, so $kG = \{\sum_{g\in G} a_g g \mid a_g \in k,\ a_g = 0$ for all but finitely many $g\}$, and multiplication

$$\sum_g a_g g \sum_h b_h h = \sum_{g,h} a_g b_h gh = \sum_{g\in G} \Big(\sum_{x\in G} a_x b_{x^{-1}g}\Big) g.$$

Example

Let $G = \mathbb{Z}^n$. Then $kG \cong k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the Laurent polynomial ring in n variables.

Zero divisor conjecture

Definition

Let G be a group. Then G is torsion free if all nonidentity elements have infinite order.

Example

- $\mathbb{Z}^n$ is torsion free.
- Let p be an odd prime, let d be a positive integer, and let $\mathbb{Z}_p$ denote the p-adic integers. Let $C_p = \{A \in \mathrm{GL}_d(\mathbb{Z}_p) \mid A \equiv I \mod p\}$, a congruence subgroup. Then C_p is torsion free.

Conjecture (Zero divisor conjecture)

Let k be a field and let G be a torsion-free group. Then kG is a domain (i.e. has no nonzero zerodivisors).

Zero divisor examples

Proposition

The zero divisor conjecture is true for $G = \mathbb{Z}^n$.

Proof.

$k[\mathbb{Z}^n] \cong k[x_1^{\pm 1}, \dots, x_n^{\pm n}]$ and Laurent polynomial rings are domains. □

Proposition

The zero divisor conjecture is true for

- *Solvable groups.*
- *Congruence subgroups C_p if k has characteristic 0 or p.*

Group von Neumann algebra

Let $\ell^2(G)$ denote the Hilbert space with Hilbert basis G:

$$\ell^2(G) = \{\sum_{g \in G} a_g g \mid \sum_{g \in G} |a_g|^2 < \infty\}$$

Multiplication (convolution)
$\ell^2(G) \times \ell^2(G) \to \ell^\infty(G) = \{\sum_{g \in G} a_g g \mid \sup_{g \in G} |a_g| < \infty\}$:

$$\sum_{g \in G} a_g g \sum_{g \in G} b_g g = \sum_{h,g \in G} a_h b_g gh = \sum_{g \in G} (\sum_{x \in G} a_{gx^{-1}} b_x) g$$

Then the group von Neumann algebra $\mathcal{N}(G)$ is
$\{\alpha \in \ell^2(G) \mid \alpha\beta \in \ell^2(G)\ \forall \beta \in \ell^2(G)\}$. So $\mathcal{N}(G)$ is a subspace of $\ell^2(G)$ which is also an algebra.

Group von Neumann algebra examples

Example

- If G is finite, then $\mathcal{N}(G) \cong \mathbb{C}G$.
- If $G = \mathbb{Z}$, then $\mathcal{N}(G) \cong \mathcal{M}(\mathbb{T})$.

Here $\mathbb{T}$ is the torus $\{z \in \mathbb{C} \mid |z| = 1\}$ and $\mathcal{M}(\mathbb{T})$ denotes the bounded measurable functions on $\mathbb{T}$ with the operations of pointwise addition and multiplication.

Atiyah conjecture

Conjecture (Special case of Atiyah conjecture)

Let G be a torsion-free group. If $0 \neq \alpha \in \mathbb{C}G$ and $0 \neq \beta \in \mathcal{N}(G)$, then $\alpha\beta \neq 0$.

Proposition

The Atiyah conjecture is true for $G = \mathbb{Z}$.

Proof.

$\mathcal{N}(\mathbb{Z}) \cong \mathcal{M}(\mathbb{T})$ and $\mathbb{C}\mathbb{Z}$ corresponds to the polynomial functions on $\mathbb{T}$. A nonzero polynomial has only finitely many zeros, so can be zero only on a set of measure 0. □

Results on the Atiyah conjecture

Theorem

The Atiyah conjecture is true when G is

- *solvable*
- *a congruence subgroup C_p*
- *G is left orderable.*

Left orderable groups

Definition

A group G is left orderable means G has a total order $\leq$ such that $x \leq y$ implies $gx \leq gy$ for all $g, x, y \in G$.

- Left orderable groups are torsion free.
- Not all torsion-free groups are left orderable.
- $\mathbb{Z}$, $\mathbb{R}$ with the usual order.
- $\mathbb{Z}^n$

Proposition

A countable group G is left orderable if and only if it is isomorphic to a subgroup of $\mathrm{Homeo}^+(\mathbb{R})$*, the orientation preserving homeomorphisms of* $\mathbb{R}$*.*

The space of left orders

Definition

The left orders $\mathrm{LO}(G)$ can be given a topology. For $g \in G$, let $O_g = \{< \in \mathrm{LO}(G) \mid 1 < g\}$. Then a subbase of open sets for this topology is $\{O_g \mid g \in G \setminus 1\}$.

Proposition

- $\mathrm{LO}(G)$ *is a compact Hausdorff space*
- *If G is finitely generated, it is metrizable*
- *G acts on $\mathrm{LO}(G)$ by homeomorphisms*

Can apply theorems from ergodic theory on this space, such as the Poincaré recurrence theorem.