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# COHOMOLOGY OF FINITE GROUPS

by

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# Preface

This volume is the outcome of lectures I gave at the Institute for Experimental Mathematics, Essen in the winter semester 1991/92. It is intended to give an introduction to the cohomology of finite groups. Unfortunately lack of time forced the omission of many important topics; for example spectral sequences and cyclic homology.

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Cohomology of Finite Groups

CONTENTS

<u>Conventions</u> Throughout, $G$ denotes a finite group, $p$ a prime, $k$ will be $\mathbb{Z}$ or a field. All modules will be finitely generated. Usually, mappings are on the right (if on the left they will be bracketed) and modules are right modules.
$\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{P} = \{1, 2, \dots\}$ .

## 1. Introduction.

1.1. <u>Definitions and Notation</u> A sequence of kG–modules

$$A : \dots\dots \longrightarrow A_2 \xrightarrow{\partial_2} A_1 \xrightarrow{\partial_1} A_0 \xrightarrow{\partial_0} 0$$

is a chain complex when $\partial_{n+1} \partial_n = 0 \ \forall n \in \mathbb{N}$ .

$$B : 0 \xrightarrow{\delta_0} B_0 \xrightarrow{\delta_1} B_1 \xrightarrow{\delta_2} B_2 \longrightarrow \dots$$

is a cochain complex when $\delta_n \delta_{n+1} = 0 \ \forall n \in \mathbb{N}$ . The $\partial_n$ are termed boundary maps, the $\delta_n$ coboundary maps. Say $A$ is projective (respectively free) if each $A_i$ is projective (respectively free).

The $n^{th}$ homology group of $A$ is $\ker \partial_n / \mathrm{im}\, \partial_{n+1}$ and is denoted by $H_n(A)$ .

The $n^{th}$ cohomology group of $B$ is $\ker \delta_{n+1} / \mathrm{im}\, \delta_n$ , denoted $H^n(B)$ .

Let $M$ be a kG–module. A resolution $(P, \epsilon)$ of $M$ (as a kG–module) is an exact sequence of kG–modules

$$(P, \epsilon) : \quad \dots \longrightarrow P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} M \longrightarrow 0 . \qquad (1)$$

Write $P$ for the chain complex

$$\dots \longrightarrow P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \longrightarrow 0 . \qquad (2)$$

(Thus $H_n(P) = 0$ for $n > 0$ , $H_0(P) = M$ . Also we still write $P$ for (2) even if (1) is only a chain complex.)

If N is a kG–module, write $\mathrm{Hom}_{kG}(P, N)$ for the cochain complex (of k–modules)

$$0 \longrightarrow \mathrm{Hom}_{kG}(P_0, N) \xrightarrow{\partial_1^*} \mathrm{Hom}_{kG}(P_1, N) \xrightarrow{\partial_2^*} \mathrm{Hom}_{kG}(P_2, N) \longrightarrow \ldots$$

where $q(\partial_n^*(f)) = (q\,\partial_n)f$, $q \in P_n$, $f \in \mathrm{Hom}_{kG}(P_{n-1}, N)$.

Lemma 1.2 Let M, N be kG–modules and $\theta_{-1} : M \longrightarrow N$ be a kG–homomorphism. Let $(P, \alpha_0): \ldots \longrightarrow P_2 \xrightarrow{\alpha_2} P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} M \longrightarrow 0$ be a chain complex of kG–modules with P projective, and $(Q, \beta_0): \ldots \longrightarrow Q_2 \xrightarrow{\beta_2} Q_1 \xrightarrow{\beta_1} Q_0 \xrightarrow{\beta_0} N \longrightarrow 0$ be a resolution of N as a kG–module.

(i) There exist kG–homomorphisms $\theta_i : P_i \longrightarrow Q_i$ such that $\alpha_i\,\theta_{i-1} = \theta_i\,\beta_i$ $\forall i \in \mathbb{N}$. (Say $\theta : P \longrightarrow Q$ is a chain map where $\theta = \bigoplus_{i\in\mathbb{N}} \theta_i$.)

(ii) If $\varphi_i : P_i \longrightarrow Q_i$ are kG–homomorphisms such that $\alpha_i\,\varphi_{i-1} = \varphi_i\,\beta_i$ $(i \in \mathbb{N})$ and $\varphi_{-1} = \theta_{-1}$ then there exist kG–homomorphisms $h_i : P_i \longrightarrow Q_{i+1}$, $h_{-1} = 0$, such that

$$\theta_i - \varphi_i = \alpha_i\,h_{i-1} + h_i\,\beta_{i+1} \quad \forall i \in \mathbb{N}.$$

(Say $\theta$ and $\varphi$ are chain homotopic.)

Lemma 1.2 can be thought of as a generalisation of Schanuel's Lemma. An important application of Lemma 1.2 occurs when $(P, \alpha_0)$ and $(Q, \beta_0)$ are projective resolutions of M. This yields

Lemma 1.3 Let M, N be kG–modules. Let

$$(P, \alpha_0): \ldots \longrightarrow P_2 \xrightarrow{\alpha_2} P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} M \longrightarrow 0$$

$$(Q, \beta_0): \ldots \longrightarrow Q_2 \xrightarrow{\beta_2} Q_1 \xrightarrow{\beta_1} Q_0 \xrightarrow{\beta_0} M \longrightarrow 0$$

be projective resolutions of M (as a kG–module). Then there exist kG–homomorphisms



$\theta_i : P_i \longrightarrow Q_i$, $\varphi_i : Q_i \longrightarrow P_i$, $i \in \mathbb{N}$, such that $\theta_i\,\varphi_i$ and $\varphi_i\,\theta_i$ induce the identity map on the $i^{th}$ cohomology group of the cochain complexes $\mathrm{Hom}_{kG}(P, N)$ and $\mathrm{Hom}_{kG}(Q, N)$ respectively.

Lemma 1.3 allows us to make the following definition.

Definition 1.4 Let M, N be kG–modules. Let

$$(P, \alpha_0): \ldots \longrightarrow P_2 \xrightarrow{\alpha_2} P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} M \longrightarrow 0$$

be a projective resolution of M. For $n \in \mathbb{N}$, $\mathrm{Ext}^n_{kG}(M, N)$ is the $n^{th}$ cohomology group of the cochain complex $\mathrm{Hom}_{kG}(P, N)$.

Remarks 1.5 (i) $\mathrm{Ext}^0_{kG}(M, N) \cong \mathrm{Hom}_{kG}(M, N)$.

(ii) By Lemma 1.3, $\mathrm{Ext}^n_{kG}(M, N)$ is well defined i.e. it is independent of the choice of projective resolution for M.

(ii) Using Lemma 1.2 we see that $\mathrm{Ext}^n_{kG}(M, -)$ is a covariant functor and $\mathrm{Ext}^n_{kG}(-, N)$ is a contravariant functor i.e. if $\theta : U \longrightarrow V$ is a kG–homomorphism, there exist natural homomorphisms

$$\theta_* : \mathrm{Ext}^n_{kG}(M, U) \longrightarrow \mathrm{Ext}^n_{kG}(M, V)$$

$$\text{and} \quad \theta^* : \mathrm{Ext}^n_{kG}(V, M) \longrightarrow \mathrm{Ext}^n_{kG}(U, M).$$

(iv) If M is a projective kG–module then $(P, \alpha_0)$ in Definition 1.4 is split exact i.e. there exist kG–homomorphisms $\beta_i : P_{i-1} \longrightarrow P_i$ $(i \in \mathbb{P})$ and $\beta_0 : M \longrightarrow P_0$ such that $\beta_i\,\alpha_i = \mathrm{id}$. It follows that the sequence

$$0 \longrightarrow \mathrm{Hom}_{kG}(M, N) \xrightarrow{\alpha_0^*} \mathrm{Hom}_{kG}(P_0, N) \xrightarrow{\alpha_1^*} \ldots$$

is also split exact. Hence if M is projective $\mathrm{Ext}^n_{kG}(M, N) = 0$ $\forall n \in \mathbb{P}$.

Exercise 1.6 Use 1.5 to show

$$\mathrm{Ext}^n_{kG}(M, U \oplus V) \cong \mathrm{Ext}^n_{kG}(M, U) \oplus \mathrm{Ext}^n_{kG}(M, V)$$

and $\mathrm{Ext}^n_{kG}(U \oplus V, M) \cong \mathrm{Ext}^n_{kG}(U, M) \oplus \mathrm{Ext}^n_{kG}(V, M)$,

U, V, M kG–modules.

Definition 1.7 Let M, N be kG–modules. $M^G$ is the k–module $\mathrm{Hom}_{kG}(k, M)$.

$M \otimes_k N$ is the kG–module with $(m \otimes n)g = mg \otimes ng$.

$\mathrm{Hom}_k(M, N)$ is the kG–module with $m(\theta g) = m g^{-1} \theta g$ $(\theta \in \mathrm{Hom}_k(M, N))$.

$M^*$ is the kG–module $\mathrm{Hom}_k(M, k)$.

If $\alpha \in \mathrm{Hom}_{kG}(M, N)$, define $\alpha^* \in \mathrm{Hom}_{kG}(N^*, M^*)$ by $m(\alpha^*(v)) = (m\,\alpha)v$, $v \in N^*$.

Say M is a kG–lattice when M is free as a k–module.

So $M^G = \{m \in M \mid mg = m \ \forall g \in G\}$, $\mathrm{Hom}_k(M, N)^G \cong \mathrm{Hom}_{kG}(M, N)$ and for $\theta \in M^*$, $m(\theta g) = (m g^{-1})\theta$. Also if H is a group and L is a kH–module then $M \otimes_k L$ is the $k[G \times H]$–module with $(m \otimes l)(g, h) = mg \otimes lh$.

Lemma 1.8 Let L, M, N be kG–modules. Then there exist natural kG–isomorphisms

(i) $M^* \otimes_k N \cong \mathrm{Hom}_k(M, N)$ if M is a kG–lattice.

(ii) $\mathrm{Hom}_k(L \otimes_k M, N) \cong \mathrm{Hom}_k(L, \mathrm{Hom}_k(M, N))$.

(iii) $M \cong M^{**}$ if M is a kG–lattice.

(iv) $kG \cong kG^*$.



Proof We give the isomorphisms in each case.

(i) For $f \in M^*$, $m \in M$, $n \in N$ define $\overline{f \otimes n} \in \mathrm{Hom}_k(M, N)$ by $m(\overline{f \otimes n}) = n(m f)$. Then $f \otimes n \longmapsto \overline{f \otimes n}$ induces a kG–isomorphism from $M^* \otimes_k N$ onto $\mathrm{Hom}_k(M, N)$.

(ii) For $\theta \in \mathrm{Hom}_k(L \otimes_k M, N)$, $l \in L$, $m \in M$ define $\bar\theta \in \mathrm{Hom}_k(L, \mathrm{Hom}_k(M, N))$ by $m(l\bar\theta) = (l \otimes m)\theta$.

For $\varphi \in \mathrm{Hom}_k(L, \mathrm{Hom}_k(M, N))$ define $\hat\varphi \in \mathrm{Hom}_k(L \otimes_k M, N)$ by $(l \otimes m)\hat\varphi = m(l\varphi)$.

Then $\bar{\ }$ and $\hat{\ }$ are kG–homomorphisms, inverse to each other.

(iii) For $m \in M$ define $\bar m \in M^{**}$ by $\mu\bar m = m\mu$, $\mu \in M^*$. Then $m \longmapsto \bar m$ induces a kG–isomorphism from M onto $M^{**}$.

(iv) If $\alpha = \sum_{g \in G} \alpha_g g \in kG$, $\alpha_g \in k$, define $\mathrm{tr}\,\alpha = \alpha_1$. Now define $\hat\alpha \in kG^*$ by $g\hat\alpha = \mathrm{tr}\, g^{-1}\alpha$, $g \in G$. Then $\alpha \longmapsto \hat\alpha$ induces a kG–isomorphism of kG onto $kG^*$.

Corollary 1.9 Let P be a projective kG–module.

(i) $P^*$ is a projective kG–module (not true if G is infinite).

(ii) If $0 \longrightarrow P \longrightarrow M \longrightarrow N \longrightarrow 0$ is an exact sequence of kG–lattices then it splits i.e. P is injective in the category of kG–lattices.

(iii) If $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ is an exact sequence of kG–lattices then $0 \longrightarrow \mathrm{Hom}_{kG}(N, P) \longrightarrow \mathrm{Hom}_{kG}(M, P) \longrightarrow \mathrm{Hom}_{kG}(L, P) \longrightarrow 0$ is exact i.e. $\mathrm{Hom}_{kG}(-, P)$ is exact on the category of kG–lattices.

Proof (i) Follows from Lemma 1.8 (iv).

(ii) The exact sequence yields an exact sequence of kG–lattices

$0 \longrightarrow N^* \longrightarrow M^* \longrightarrow P^* \longrightarrow 0$.

This splits by (i). Hence

$0 \longrightarrow P^{**} \longrightarrow M^{**} \longrightarrow N^{**} \longrightarrow 0$ splits. Now apply Lemma 1.8 (iii).

(iii) Follows from (ii).

<u>Corollary 1.10</u> Let P be a projective kG–module and L be a kG–lattice. Then $\mathrm{Ext}^n_{kG}(L, P) = 0 \ \forall n \in \mathbb{P}$.

<u>Proof</u> Let $Q: \dots \longrightarrow Q_2 \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow L \longrightarrow 0$ be a projective resolution of L. By Corollary 1.9 (iii) $0 \longrightarrow \mathrm{Hom}_{kG}(L, P) \longrightarrow \mathrm{Hom}_{kG}(Q_0, P) \longrightarrow \mathrm{Hom}_{kG}(Q_1, P) \longrightarrow \dots$ is exact.

<u>Lemma 1.11</u> Let L be a kG–lattice and P a projective kG–module. Then $L \otimes_k P$ is projective.

<u>Proof</u> Let M be a kG–module. By Lemma 1.8 (ii) and the remarks after Definition 1.7 there is a natural isomorphism

$$\mathrm{Hom}_{kG}(P \otimes_k L, M) \cong \mathrm{Hom}_{kG}(P, \mathrm{Hom}_k(L, M)).$$

Since L is a lattice, $\mathrm{Hom}_k(L, -)$ is exact. Since P is a projective kG–module $\mathrm{Hom}_{kG}(P, -)$ is exact. It follows that $\mathrm{Hom}_{kG}(P \otimes_k L, -)$ is exact i.e. $P \otimes_k L$ is a projective kG–module.



<u>Lemma 1.12</u> (Mayer–Vietoris sequence) Let $0 \longrightarrow A \xrightarrow{\theta} B \xrightarrow{\varphi} C \longrightarrow 0$ be an exact sequence of chain complexes i.e. a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_1 & \xrightarrow{\theta_1} & B_1 & \xrightarrow{\varphi_1} & C_1 & \longrightarrow & 0 \\
 & & \alpha_1\downarrow & & \beta_1\downarrow & & \gamma_1\downarrow & & \\
0 & \longrightarrow & A_0 & \xrightarrow{\theta_0} & B_0 & \xrightarrow{\varphi_0} & C_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Then there exists a long exact sequence (natural)

$$\dots \longrightarrow H_{n+1}(C) \xrightarrow{\partial_{n+1}} H_n(A) \xrightarrow{(\theta_n)_*} H_n(B) \xrightarrow{(\varphi_n)_*} H_n(C) \xrightarrow{\partial_n} H_{n-1}(A) \longrightarrow$$
$$\dots \longrightarrow H_0(C) \longrightarrow 0 .$$

<u>Sketch Proof</u> $(\theta_n)_*$ and $(\varphi_n)_*$ are induced by $\theta_n$ and $\varphi_n$ respectively. To define $\partial_n$, suppose $a \in H_n(C) = \ker \gamma_n / \mathrm{im}\ \gamma_{n+1}$. Choose $b \in \ker \gamma_n \subseteq C_n$ representing a, so $b\gamma_n = 0$. Choose $c \in B_n$ such that $c\varphi_n = b$ ($\varphi_n$ is onto). Then $c\beta_n\varphi_{n-1} = c\varphi_n\gamma_n = b\gamma_n = 0$. Therefore $c\beta_n = d\theta_{n-1}$ for some $d \in A_{n-1}$. Now check that $a \longmapsto d$ induces a well–defined homomorphism $\partial_n : H_n(C) \longrightarrow H_{n-1}(A)$ and that the resulting sequence is exact.

<u>Mayer–Vietoris sequence for cochain complexes.</u>

Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be an exact sequence of cochain complexes i.e. a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & A_1 & \xrightarrow{\theta_1} & B_1 & \xrightarrow{\varphi_1} & C_1 & \longrightarrow & 0 \\
 & & \alpha_1\uparrow & & \beta_1\uparrow & & \gamma_1\uparrow & & \\
0 & \longrightarrow & A_0 & \xrightarrow{\theta_0} & B_0 & \xrightarrow{\varphi_0} & C_0 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Then there exists a long exact sequence (natural)

$$0 \longrightarrow H^0(A) \xrightarrow{(\theta_0)_*} H^0(B) \xrightarrow{(\varphi_0)_*} H^0(C) \xrightarrow{\partial_1} H^1(A) \xrightarrow{(\theta_1)_*} H^1(B) \longrightarrow \dots$$

<u>Corollary 1.13</u> Let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence of kG–modules. Let $U$ be a kG–module. Then there exist (natural) exact sequences

(i) $0 \longrightarrow Ext^0_{kG}(U, L) \longrightarrow Ext^0_{kG}(U, M) \longrightarrow Ext^0_{kG}(U, N) \longrightarrow Ext^1_{kG}(U, L) \longrightarrow \dots$

(ii) $0 \longrightarrow Ext^0_{kG}(N, U) \longrightarrow Ext^0_{kG}(M, U) \longrightarrow Ext^0_{kG}(L, U) \longrightarrow Ext^1_{kG}(N, U) \longrightarrow \dots$

$\dots \longrightarrow Ext^n_{kG}(N, U) \longrightarrow Ext^n_{kG}(M, U) \longrightarrow Ext^n_{kG}(L, U) \longrightarrow \dots$

<u>Proof</u> (i) Let $(P, \epsilon)$ be a projective resolution of $U$. Then we have an exact sequence of cochain complexes $0 \longrightarrow Hom_{kG}(P, L) \longrightarrow Hom_{kG}(P, M) \longrightarrow Hom_{kG}(P, N) \longrightarrow 0$.

Now apply Lemma 1.12.

(ii) Let $(P, \epsilon)$, $(R, v)$ be projective resolutions for $L$, $N$ respectively. By the Horseshoe Lemma there is a projective resolution $(Q, \mu)$ of $M$ and a commutative diagram with exact rows and columns



$$\begin{array}{ccccccccccccccc}
 & & 0 & & 0 & & & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & P_{n+1} & \longrightarrow & P_n & \longrightarrow & \dots & \longrightarrow & P_1 & \longrightarrow & P_0 & \xrightarrow{\lambda} & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & Q_{n+1} & \longrightarrow & Q_n & \longrightarrow & \dots & \longrightarrow & Q_1 & \longrightarrow & Q_0 & \xrightarrow{\mu} & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & R_{n+1} & \longrightarrow & R_n & \longrightarrow & \dots & \longrightarrow & R_1 & \longrightarrow & R_0 & \xrightarrow{v} & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & & & 0 & & 0 & & 0 & &
\end{array}$$

Since $0 \longrightarrow P_n \longrightarrow Q_n \longrightarrow R_n \longrightarrow 0$ is split exact for all $n \in \mathbb{N}$ it follows that $0 \longrightarrow Hom_{kG}(R, U) \longrightarrow Hom_{kG}(Q, U) \longrightarrow Hom_{kG}(P, U) \longrightarrow 0$ is an exact sequence of cochain complexes. Now apply Lemma 1.12.

<u>Inflation and Restriction maps</u>

Let $\alpha : H \longrightarrow G$ be a homomorphism of groups and let $M$, $N$ be kG–modules. Then $M$, $N$ are also kH–modules by defining $mh = m(h\alpha)$ for $m \in M$ or $N$ and $h \in H$. Let $(P, \epsilon)$ be a projective resolution of $M$ with kH–modules. Let $(Q, v)$ be a projective resolution of $M$ with kG–modules. Viewing $(Q, v)$ as a resolution with kH–modules (now not necessarily projective), Lemma 1.2 (i) shows that there exists a kH–chain map $\theta : P \longrightarrow Q$ extending the identity map on $M$. This gives a chain map

$$\theta^* : Hom_{kG}(Q, N) \longrightarrow Hom_{kH}(P, N)$$

defined by $\theta^*(f) = \theta f$ for $f \in Hom_{kG}(Q, N)$.

This induces a natural homomorphism of k–modules

$$\alpha^*_n : Ext^n_{kG}(M, N) \longrightarrow Ext^n_{kH}(M, N) \quad \forall n \in \mathbb{N}.$$

By Lemma 1.2 (ii), $\alpha^*_n$ does not depend on $\theta$.

Special Cases (i) $\alpha$ is inclusion i.e. $H \le G$. Then $\alpha^*$ is the restriction map from $G$ to $H$, denoted $\mathrm{res}_{G,H}$.

(ii) $\alpha$ is an epimorphism. Then $\alpha^*$ is the inflation map from $G$ to $H$, denoted $\mathrm{inf}_{G,H}$.

Transfer map Let $H \le G$, let $M, N$ be kG–modules and let $(P, \epsilon)$ be a projective resolution of $M$. Let $\{x_1, \ldots, x_n\}$ be a right transversal for $H$ in $G$, so $G = Hx_1 \uplus Hx_2 \uplus \ldots \uplus Hx_n$. We have a natural map of cochain complexes $\mathrm{Hom}_{kH}(P, N) \longrightarrow \mathrm{Hom}_{kG}(P, N)$ defined by $\theta \longmapsto \sum_{i=1}^{n} x_i^{-1} \theta x_i$, which is independent of the choice of transversal. This induces a natural homomorphism of k–modules

$$\mathrm{Ext}^n_{kH}(M, N) \longrightarrow \mathrm{Ext}^n_{kG}(M, N)$$

denoted $\mathrm{tr}_{HG}$, the transfer map from $H$ to $G$.

Lemma 1.14 Let $H \le G$, let $\ell = [G : H]$, let $M, N$ be kG–modules, let $n \in \mathbb{N}$ and let $\alpha \in \mathrm{Ext}^n_{kG}(M, N)$. Then $\mathrm{tr}_{H,G}(\mathrm{res}_{H,G}\, \alpha) = \ell\alpha$.

Proof Let $(P, \epsilon)$ be a projective resolution of $M$ and let $\{x_1, \ldots, x_\ell\}$ be a right transversal for $H$ in $G$. If $\alpha$ is represented by $\theta \in \mathrm{Hom}_{kG}(P_n, N)$ then $\mathrm{tr}_{H,G}\mathrm{res}_{G,H}$ is induced by

$$\begin{array}{ccccc} \mathrm{Hom}_{kG}(P_n, N) & \longrightarrow & \mathrm{Hom}_{kH}(P_n, N) & \longrightarrow & \mathrm{Hom}_{kG}(P_n, N) \\ \theta & \longmapsto & \theta & \longmapsto & \sum_{i=1}^{\ell} x_i^{-1} \theta x_i = \ell\theta \end{array}$$

since $\theta$ commutes with the $x_i$'s.



Corollary 1.15 Suppose $k$ is $\mathbb{Z}$ or a finite field, $M$ is a kG–lattice and $N$ is a kG–module. Then $\mathrm{Ext}^n_{kG}(M, N)$ is a finite group with exponent dividing $|G|$ for all $n \in \mathbb{P}$.

Proof Since $\mathrm{Ext}^n_{k1}(M, N) = 0$ for all $n \in \mathbb{P}$, Lemma 1.14 shows that $|G|\, \mathrm{Ext}^n_{kG}(M, N) = 0$. Also it is clear from the definition of Ext in terms of resolutions that $\mathrm{Ext}^n_{kG}(M, N)$ is finitely generated as a k–module. The result follows.

Exercise Suppose $k$ is $\mathbb{Z}$ or a finite field and $M, N$ are kG–modules. Show that $\mathrm{Ext}^n_{kG}(M, N)$ is a finite group for all $n \in \mathbb{P}$ and that its exponent divides $|G|$ for all $n \ge 2$.

Lemma 1.16 Let $H \le G$, let $M$ be a kH–module and let $N$ be a kG–module. Then there exist natural isomorphisms

(i) $\mathrm{Hom}_{kH}(N, M) \cong \mathrm{Hom}_{kG}(N, M \otimes_{kH} kG)$

(ii) $\mathrm{Hom}_{kH}(M, N) \cong \mathrm{Hom}_{kG}(M \otimes_{kH} kG, N)$.

Lemma 1.17 Let $H \le G$, let $M$ be a kH–module, let $N$ be a kG–module and let $n \in \mathbb{N}$. Then there exist natural isomorphisms

(i) $\mathrm{Ext}^n_{kH}(N, M) \cong \mathrm{Ext}^n_{kG}(N, M \otimes_{kH} kG)$

(ii) $\mathrm{Ext}^n_{kH}(M, N) \cong \mathrm{Ext}^n_{kG}(M \otimes_{kH} kG, N)$.

Proof (i) Let $(P, \epsilon)$ be a projective resolution of $N$ as a kG–module. Then

$$\begin{aligned} \mathrm{Ext}^n_{kG}(N, M \otimes_{kH} kG) &= H^n(\mathrm{Hom}_{kG}(P, M \otimes_{kH} kG)) \\ &\cong H^n(\mathrm{Hom}_{kH}(P, M)) \quad \text{by Lemma 1.16 (i)} \\ &= \mathrm{Ext}^n_{kH}(N, M). \end{aligned}$$

(ii) Exercise (similar to (i)).

<u>Lemma 1.18</u> Let $K$ be a field containing $k$, let $M, N$ be $kG$–modules and let $n \in \mathbb{N}$. Then

$$\mathrm{Ext}^n_{kG}(M, N) \otimes_k K \cong \mathrm{Ext}^n_{kG}(M \otimes_k K, N \otimes_k K).$$

<u>Proof</u> Let $(P, \epsilon)$ be a projective resolution of $M$. The result follows from the natural isomorphism

$$\mathrm{Hom}_{kG}(P, N) \otimes_k K \longrightarrow \mathrm{Hom}_{kG}(P \otimes_k K, N \otimes_k K)$$

defined by sending $\theta \otimes u$ $(\theta \in \mathrm{Hom}_{kG}(P, N), u \in K)$ to the map $q \otimes v \xrightarrow{\theta \otimes u} q\theta \otimes vu$ $(q \in P, v \in K)$.

<u>Definition</u> Let $M$ be a $\mathbb{Z}G$–module and $n \in \mathbb{N}$. Then

$$H^n(G, M) := \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, M).$$

<u>Remarks 1.19</u> (i) $H^0(G, M) = M^G$.

(ii) $H^n(G, M)$ is a finite group with exponent dividing the order of $G$ $\forall n \in \mathbb{P}$ (use Corollary 1.15).

(iii) If $K$ is a field containing $k$ then $H^n(G, M \otimes_k K) \cong H^n(G, M) \otimes_k K$ $\forall n \in \mathbb{N}$ (use Lemma 1.18).

(iv) If $M$ is a $kG$–module then $M$ is also a $\mathbb{Z}G$–module (at least if we drop the requirement that all modules are finitely generated) and we have $H^n(G, M) \cong \mathrm{Ext}^n_{kG}(k, M)$ $\forall n \in \mathbb{N}$ (exercise).

(v) Let $M$ be a $kG$–lattice, let $N$ be a $kG$–module and let $n \in \mathbb{N}$. Then there is a natural isomorphism

$$\mathrm{Ext}^n_{kG}(M, N) \cong H^n(G, M^* \otimes_k N).$$



To prove this, use $\mathrm{Hom}_{kG}(P \otimes_k M, N) \cong \mathrm{Hom}_{kG}(P, M^* \otimes_k N)$ which follows from Lemma 1.8 (ii), and Lemma 1.11 which tells us that $P$ projective implies $P \otimes_k M$ is projective.

<u>Proposition 1.20</u> Let $M$ be a $\mathbb{Z}G$–module with trivial $G$–action i.e. $M = M^G$. Then $H^1(G, M) \cong \mathrm{Hom}(G, M)$ naturally.

<u>Remarks</u> $\mathrm{Hom}(G, M) = \mathrm{Hom}(G/G', M)$. Thus

$H^1(G, \mathbb{Z}/p\mathbb{Z}) = G/G'\,G^p$, $H^1(G, \mathbb{Z}) = 0$.

If $G = G'$, $H^1(G, M) = 0$ (if $M = M^G$).

<u>Proof</u> Let $\mathfrak{g}$ be the augmentation ideal of $\mathbb{Z}G$, the ideal with $\mathbb{Z}$–basis

$\{g - 1 \mid 1 \neq g \in G\}$.

Then we have an exact sequence $0 \longrightarrow \mathfrak{g} \longrightarrow \mathbb{Z}G \longrightarrow \mathbb{Z} \longrightarrow 0$

hence by Corollary 1.13 (ii) an exact sequence

$$0 \longrightarrow \mathrm{Ext}^0_{\mathbb{Z}G}(\mathbb{Z}, M) \longrightarrow \mathrm{Ext}^0_{\mathbb{Z}G}(\mathbb{Z}G, M) \longrightarrow \mathrm{Ext}^0_{\mathbb{Z}G}(\mathfrak{g}, M) \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}G}(\mathbb{Z}, M) \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}G}(\mathbb{Z}G, M).$$

Therefore we have an exact sequence

$$\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, M) \xrightarrow{\theta} \mathrm{Hom}_{\mathbb{Z}G}(\mathfrak{g}, M) \longrightarrow H^1(G, M) \longrightarrow 0$$

Note that $\mathrm{im}\,\theta = 0$ (because $M\mathfrak{g} = 0$). The result follows because $G/G' \cong \mathfrak{g}/\mathfrak{g}^2$ (as $\mathbb{Z}$–modules) via $G'g \longmapsto \mathfrak{g}^2 + g - 1$.

<u>Lemma 1.21</u> Let $A$ be a chain complex of $k$–modules and $L$ be a $k$–lattice. Then

$$H_n(A \otimes_k L) \cong H_n(A) \otimes_k L.$$

<u>Proof</u> Exercise. If $\alpha_n$ are the boundary maps of $A$ then $A \otimes_k L$ denotes the chain complex $(A \otimes_k L)_n = A_n \otimes_k L$ with boundary maps $\alpha_n \otimes 1$.

Bockstein map  Let $k = \mathbb{Z}/p\,\mathbb{Z}$. We have a short exact sequence

$$0 \longrightarrow k \xrightarrow{\theta} \mathbb{Z}/p^2\,\mathbb{Z} \xrightarrow{\varphi} k \longrightarrow 0 .$$

Therefore by Corollary 1.13 (i) there is a long exact sequence

$$\ldots \longrightarrow H^n(G, k) \xrightarrow{\theta_n} H^n(G, \mathbb{Z}/p^2\,\mathbb{Z}) \xrightarrow{\varphi_n} H^n(G, k) \xrightarrow{\beta_n} H^{n+1}(G, k) \longrightarrow \ldots$$

$\beta_n$ is the Bockstein map.

Use Remark 1.19 (iii) to define $\beta_n$ for an arbitrary field of characteristic $p$.

1.22 Description of $\beta_n$  Let

$$\ldots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

be a projective resolution of $\mathbb{Z}$. Let $u \in H^n(G, k)$. Then $u$ is represented by $f \in \mathrm{Hom}_{\mathbb{Z}G}(P_n, k)$. Lift $f$ to $\hat{f} \in \mathrm{Hom}_{\mathbb{Z}G}(P_n, \mathbb{Z}/p^2\mathbb{Z})$. Then

$\partial_{n+1}\,\hat{f} : P_{n+1} \longrightarrow \mathbb{Z}/p^2\,\mathbb{Z}$ has image contained in $p\mathbb{Z}/p^2\mathbb{Z} = k$ (because $\partial_{n+1}\, f = 0$).

Then $\partial_{n+1}\,\hat{f} \in \mathrm{Hom}_{\mathbb{Z}G}(P_{n+1}, k)$ represents $\beta_n(u)$.

Lemma 1.23 (i) $\beta_{n+1}\,\beta_n = 0$ (because $\partial_{n+2}\,\partial_{n+1} = 0$).

(ii) $\beta_0 = 0$ (exercise).



2. Künneth Formula  This will be especially important when cup products are introduced.

Definition  Let

$$A : \ldots \longrightarrow A_2 \xrightarrow{\alpha_2} A_1 \xrightarrow{\alpha_1} A_0 \xrightarrow{\alpha_0} 0$$

$$B : \ldots \longrightarrow B_2 \xrightarrow{\beta_2} B_1 \xrightarrow{\beta_1} B_0 \xrightarrow{\beta_0} 0$$

be chain complexes of kG–modules. Then $A \otimes_k B$ is the chain complex of kG–modules with

$$(A \otimes_k B)_n = \bigoplus_{r+s=n} A_r \otimes_k B_s$$

and boundary map $\partial_n$ defined by

$$(a \otimes b)\,\partial_n = a\,\alpha_r \otimes b + (-1)^r\, a \otimes b\,\beta_s \quad \text{for } a \in A_r,\ b \in B_s .$$

The $(-1)^r$ ensures $\partial_{n+1}\,\partial_n = 0$.

Similarly if $H$ is a group, $A$ is a chain complex of kG–modules and $B$ is a chain complex of kH–modules then $A \otimes_k B$ is a chain complex of $k[G \times H]$–modules.

Similarly if $\quad A : 0 \longrightarrow A_0 \xrightarrow{\alpha_1} A_1 \xrightarrow{\alpha_2} A_2 \longrightarrow \ldots$

and $\quad B : 0 \longrightarrow B_0 \xrightarrow{\beta_1} B_1 \xrightarrow{\beta_2} B_2 \longrightarrow \ldots$

are cochain complexes then $A \otimes_k B$ is a cochain complex with

$(a \otimes b)\delta_n = a\,\alpha_{r+1} \otimes b + (-1)^r a \otimes b\,\beta_{s+1}$ for $a \in A_r$, $b \in B_s$.

Theorem 2.1 (Künneth Formula)  Let $A$ be a chain complex of k–lattices, let $B$ be a complex of k–modules and let $n \in \mathbb{N}$. Define

$$\pi : \bigoplus_{r+s=n} H_r(A) \otimes_k H_s(B) \longrightarrow H_n(A \otimes_k B)$$

as follows. If $u \in H_r(A)$ and $v \in H_s(B)$ are represented by $a \in A_r$ and $b \in B_s$ respectively then $(u \otimes v)\pi$ is represented by $a \otimes b \in (A \otimes_k B)_n$. Then there is a natural short exact sequence of k–modules

$$0 \longrightarrow \bigoplus_{r+s=n} H_r(A) \otimes_k H_s(B) \xrightarrow{\pi} H_n(A \otimes_k B) \longrightarrow \bigoplus_{r+s=n-1} \mathrm{Tor}_1^k(H_r(A), H_s(B)) \longrightarrow 0$$

which splits, but not naturally.

2.2 Remarks on Tor  Let $R$ be a ring, let $0 \longrightarrow L \longrightarrow F \longrightarrow M \longrightarrow 0$ be an exact sequence of $R$–modules where $F$ is a free $R$–module, and let $N$ be an $R$–module.

(i) There is an exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^R(M, N) \longrightarrow L \otimes_R N \longrightarrow F \otimes_R N \longrightarrow M \otimes_R N \longrightarrow 0 .$$

(ii) $\mathrm{Tor}_1^R(M, N) \cong \mathrm{Tor}_1^R(N, M)$ .

(iii) $\mathrm{Tor}_1^R(M, P) = 0$ if $P$ is projective.

(iv) $\mathrm{Tor}_1^R(\mathbb{Z}/p^r\mathbb{Z} , \mathbb{Z}/p^s\mathbb{Z}) \cong \mathbb{Z}/p^{\min(r,s)}\mathbb{Z}$ . Thus for $\mathbb{Z}$–modules $A, B$ with $|A| , |B| < \infty$ , $\mathrm{Tor}_1^{\mathbb{Z}}(A, B) \cong A \otimes B$ and also $\mathrm{Ext}_{\mathbb{Z}}^1(A, B) \cong A \otimes B$ .

(v) A homomorphism $M \longrightarrow N$ induces homomorphisms $\mathrm{Tor}_1^R(L, M) \longrightarrow \mathrm{Tor}_1^R(L, N)$ and $\mathrm{Tor}_1^R(M, L) \longrightarrow \mathrm{Tor}_1^R(N, L)$ .

2.3 Remarks on Theorem 2.1 (i) If $k$ is a field, then

$$H_n(A \otimes_k B) \cong \bigoplus_{r+s=n} H_r(A) \otimes_k H_s(B) .$$

(ii) Let $M$ and $N$ be $kG$–modules with projective resolutions $(P, \epsilon)$ and $(Q, v)$ respectively. Then $(P \otimes_k Q , \epsilon \otimes v)$ is a projective resolution of the $kG$–module $M \otimes_k N$. (That $P \otimes_k Q$ is projective follows from Lemma 1.11; that $P \otimes_k Q$ is a resolution follows from the Künneth formula). This result is used in the construction of cup products.

(iii) Consider the special case $B_r = 0$ for all $r > 0$ . Write $M = B_0$ and let $n \in \mathbb{N}$ . Then we have a natural exact sequence which splits (but not naturally)

$$0 \longrightarrow H_n(A) \otimes_k M \longrightarrow H_n(A \otimes_k M) \longrightarrow \mathrm{Tor}_1^k(H_{n-1}(A) , M) \longrightarrow 0 .$$

(Remember that $M$ can be arbitrary, but $A$ needs to be a chain complex of $k$–lattices.) This is often referred to as the "Universal Coefficient Theorem".

(iv) Künneth Formula for cochain complexes  Let $A$ be a cochain complex of $k$–lattices, let



$B$ be a cochain complex of $k$–modules and let $n \in \mathbb{N}$ . Then there is a natural short exact sequence of $k$–modules which splits (but not naturally)

$$0 \longrightarrow \bigoplus_{r+s=n} H^r(A) \otimes_k H^s(B) \longrightarrow H^n(A \otimes_k B) \longrightarrow \bigoplus_{r+s=n+1} \mathrm{Tor}_1^k(H^r(A) , H^s(B)) \longrightarrow 0 .$$

2.4 Computation of $H^n(G \times H , k)$  Let $H$ be a group and let $(P, \epsilon)$ and $(Q, v)$ be projective resolutions of $k$ with $kG$ and $kH$–modules respectively. Then $(P \otimes_k Q , \epsilon \otimes v)$ is a projective resolution of $k \otimes_k k$ with $k[G \times H]$–modules by the Künneth formula and $k \otimes_k k$ is naturally isomorphic to $k$ via the map $k_1 \otimes k_2 \overset{\mu}{\longmapsto} k_1 k_2$ . Let $\pi = (\epsilon \otimes v)\mu$ so that $(P \otimes_k Q , \pi)$ is a projective resolution of $k$ with $k[G \times H]$–modules. Since $\mathrm{Hom}_{kG}(P, k)$ is a cochain complex of $kG$–lattices the Künneth formula yields a natural exact sequence of $k$–modules which splits

$$0 \longrightarrow \bigoplus_{r+s=n} H^r(\mathrm{Hom}_{kG}(P, k)) \otimes_k H^s(\mathrm{Hom}_{kH}(Q, k)) \longrightarrow$$
$$H^n(\mathrm{Hom}_{kG}(P, k) \otimes_k \mathrm{Hom}_{kH}(Q, k)) \longrightarrow$$
$$\bigoplus_{r+s=n+1} \mathrm{Tor}_1^k(H^r(\mathrm{Hom}_{kG}(P, k)) , H^s(\mathrm{Hom}_{kH}(Q, k))) \longrightarrow 0 .$$

Now we have a natural isomorphism of cochain complexes

$$\theta : \mathrm{Hom}_{kG}(P, k) \otimes_k \mathrm{Hom}_{kH}(Q, k) \longrightarrow \mathrm{Hom}_{k[G \times H]}(P \otimes_k Q, k)$$

defined by sending $f \otimes g$ to the map $u \otimes v \longmapsto u f v g$ $(f \in \mathrm{Hom}_{kG}(P_r, k), g \in \mathrm{Hom}_{kH}(Q_s, k),$ $u \in P_r, v \in Q_s)$. No sign is needed here even though it is in the definition of the tensor product of complexes.

Now $H^r(\mathrm{Hom}_{kG}(P, k)) = H^r(G, k)$ etc, hence the above exact sequence yields a natural exact sequence of $k$–modules which splits

$$0 \to \bigoplus_{r+s=n} H^r(G,k) \otimes_k H^s(H, k) \to H^n(G \times H, k) \to \bigoplus_{r+s=n+1} \mathrm{Tor}_1^k(H^r(G, k), H^s(H, k)) \to 0.$$

Thus once $H^n(G, k)$ has been calculated for $G$ cyclic it can be calculated when $G$ is any abelian group. If $k$ is a field then $H^n(G \times H, k) \cong \bigoplus_{r+s=n} H^r(G, k) \otimes_k H^s(H, k)$.

Later we will show that if $n \in \mathbb{P}$ and $G = \mathbb{Z}/n\mathbb{Z}$ then $H_0(G, \mathbb{Z}) = \mathbb{Z}$, $H^r(G, \mathbb{Z}) = 0$ if $r$ is odd and $H^r(G, \mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$ if $r$ is even and $\neq 0$.

<u>Example</u>: $H^4(\mathbb{Z}_6 \times \mathbb{Z}_3, \mathbb{Z})$. We have a split exact sequence

$$0 \to \bigoplus_{r+s=4} H^r(\mathbb{Z}_6, \mathbb{Z}) \otimes H^s(\mathbb{Z}_3, \mathbb{Z}) \to H^4(\mathbb{Z}_6 \times \mathbb{Z}_3, \mathbb{Z}) \to \bigoplus_{r+s=5} \operatorname{Tor}_1^{\mathbb{Z}}(H^r(\mathbb{Z}_6, \mathbb{Z}), H^s(\mathbb{Z}_3, \mathbb{Z})) \longrightarrow 0 .$$

Therefore $H^4(\mathbb{Z}_6 \times \mathbb{Z}_3, \mathbb{Z}) \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_6$.

<u>Exercise</u> $H^4(\mathbb{Z}_6 \times \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}) \cong \mathbb{Z}_3^6 \oplus \mathbb{Z}_6$.

<u>2.5 Universal Coefficient Theorem</u> Here we relate $H^n(G, \mathbb{Z})$ and $H^n(G, k)$. Let $(P, \epsilon)$ be a projective resolution of $\mathbb{Z}$ with $\mathbb{Z}G$–modules. Then we have a split exact sequence

$$0 \to H^n(\operatorname{Hom}_{\mathbb{Z}G}(P,\mathbb{Z})) \otimes k \to H^n(\operatorname{Hom}_{\mathbb{Z}G}(P,\mathbb{Z}) \otimes k) \to \operatorname{Tor}_1^{\mathbb{Z}}(H^{n+1}(\operatorname{Hom}_{\mathbb{Z}G}(P,\mathbb{Z})),k) \to 0$$

for all $n \in \mathbb{N}$, because $\operatorname{Hom}_{\mathbb{Z}G}(P, \mathbb{Z})$ is a $\mathbb{Z}$–lattice (see 2.3 (iii)). But $\operatorname{Hom}_{\mathbb{Z}G}(P, \mathbb{Z}) \otimes k$ is naturally isomorphic to $\operatorname{Hom}_{kG}(P \otimes k, k)$ and $(P \otimes k, \epsilon \otimes 1)$ is a projective resolution of $k$ with kG–modules. Thus $H^n(\operatorname{Hom}_{\mathbb{Z}G}(P, \mathbb{Z}) \otimes k) \cong H^n(G, k)$ (cf. 1.19) and we have a split exact sequence

$$0 \longrightarrow H^n(G, \mathbb{Z}) \otimes k \longrightarrow H^n(G, k) \longrightarrow \operatorname{Tor}_1^{\mathbb{Z}}(H^{n+1}(G, \mathbb{Z}), k) \longrightarrow 0 .$$

<u>Exercises</u> (i) Show $H^2(G, \mathbb{Z}) \cong G/G'$.

(ii) Let $M$ be a $\mathbb{Z}G$–lattice and let $n \in \mathbb{N}$. Show

$$H^n(G, M \otimes k) \cong H^n(G, M) \otimes k \oplus \operatorname{Tor}_1^{\mathbb{Z}}(H^{n+1}(G, M), k) .$$

<u>Proof of Theorem 2.1</u> Let $\alpha_r$ and $\beta_s$ denote the boundary maps of $A$ and $B$ respectively. We begin by considering a special case. Suppose $A$ is a chain complex $X$ with trivial boundary (so $X_r \cong H_r(X)$ for all $r \in \mathbb{N}$). Then $X \otimes_k B$ is the chain complex with $(X \otimes_k B)_n =$



$\bigoplus_{r+s=n} X_r \otimes_k B_s$ and boundary $\bigoplus_{r+s=n} (-1)^r i_r \otimes \beta_s$, where $i_r$ is the identity map on $X_r$. Thus $H_n(X \otimes_k B) \cong \bigoplus_{r+s=n} H_s(X_r \otimes_k B)$ and since $H_s(X_r \otimes_k B) \cong X_r \otimes_k H_s(B)$ by Lemma 1.21 we deduce that $\pi : \bigoplus_{r+s=n} H_r(X) \otimes_k H_s(B) \longrightarrow H_n(X \otimes_k B)$ is an isomorphism. In general write

$$C_n = \ker \alpha_n : A_n \longrightarrow A_{n-1}$$

$$D_n = \operatorname{im} \alpha_n : A_n \longrightarrow A_{n-1} .$$

Note that $C_n$ and $D_n$ are projective k–modules. Regard $C$ and $D$ as chain complexes with trivial boundary. Then $0 \longrightarrow C \longrightarrow A \longrightarrow D \longrightarrow 0$ is an exact sequence of chain complexes and hence so is $0 \longrightarrow C \otimes_k B \longrightarrow A \otimes_k B \longrightarrow D \otimes_k B \longrightarrow 0$ because $D$ is projective (use 2.2).

Now apply Lemma 1.12 to obtain an exact sequence

$$\cdots \longrightarrow H_{n+1}(D \otimes_k B) \xrightarrow{\theta_{n+1}} H_n(C \otimes_k B) \longrightarrow H_n(A \otimes_k B) \xrightarrow{\varphi_n} H_n(D \otimes_k B) \longrightarrow \cdots .$$

We also have an exact sequence $0 \longrightarrow D_{r+1} \longrightarrow C_r \longrightarrow H_r(A) \longrightarrow 0$ for all $r \in \mathbb{N}$ and hence an exact sequence

$$0 \longrightarrow \operatorname{Tor}_1^k(H_r(A), H_s(B)) \longrightarrow D_{r+1} \otimes_k H_s(B) \longrightarrow C_r \otimes_k H_s(B) \longrightarrow H_r(A) \otimes_k H_s(B) \longrightarrow 0$$

by 2.2 (i). Therefore we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 \to \bigoplus_{r+s=n} \operatorname{Tor}_1^k(H_r(A),H_s(B)) & \to & \bigoplus_{r+s=n} D_{r+1} \otimes_k H_s(B) & \to & \bigoplus_{r+s=n} C_r \otimes_k H_s(B) & \to & \bigoplus_{r+s=n} H_r(A) \otimes_k H_s(B) \to 0 \\
 & & \downarrow \delta & & \downarrow \gamma & & \downarrow \pi \\
H_{n+1}(A \otimes_k B) & \xrightarrow{\varphi_{n+1}} & H_{n+1}(D \otimes_k B) & \xrightarrow{\theta_{n+1}} & H_n(C \otimes_k B) & \longrightarrow & H_n(A \otimes_k B) \xrightarrow{\varphi_n} \cdots
\end{array}$$

where $\delta$ and $\gamma$ are isomorphisms by the special case when $A$ has trivial boundary. A routine diagram chase shows that $\ker \pi = 0$, $\operatorname{im} \pi = \ker \varphi_n$ and $\ker \theta_{n+1} \cong \bigoplus_{r+s=n} \operatorname{Tor}_1^k(H_r(A), H_s(B))$.

But we have an exact sequence $0 \longrightarrow \ker \varphi_n \longrightarrow H_n(A \otimes_k B) \longrightarrow \ker \theta_n \longrightarrow 0$, and the required natural exact sequence follows easily.

It remains to show that the sequence splits. First consider the case when $B$ (as well as $A$) is a lattice. Write $E_n = \ker \beta_n : B_n \longrightarrow B_{n-1}$. Since submodules of k–lattices are projective we may write $A_n = C_n \oplus C'_n$ and $B_n = E_n \oplus E'_n$ for some k–sublattices $C'_n$ and $E'_n$, but not naturally. It follows that the natural epimorphisms $C_n \longrightarrow H_n(A)$ and $E_n \longrightarrow H_n(B)$ can be extended to epimorphisms $\gamma_n : A_n \longrightarrow H_n(A)$ and $\delta_n : B_n \longrightarrow H_n(B)$ respectively, and hence to an epimorphism $(\gamma \otimes \delta)_n : (A \otimes_k B)_n \longrightarrow \bigoplus_{r+s=n} H_r(A) \otimes_k H_s(B)$.

If $a \in A_r$ and $b \in B_s$ then $(a\,\alpha_r \otimes b + (-1)^r a \otimes b\,\beta_s)(\gamma \otimes \delta)_{r+s-1} = 0$, because $a\,\alpha_r\,\gamma_{r-1} = 0 = b\,\beta_s\,\delta_{s-1}$. Therefore $(\gamma \otimes \delta)_n$ induces a homomorphism

$$(\gamma \otimes \delta)_* : H_n(A \otimes_k B) \longrightarrow \bigoplus_{r+s=n} H_r(A) \otimes_k H_s(B).$$

It is clear that $\pi(\gamma \otimes \delta)_*$ is the identity on $\bigoplus_{r+s=n} H_r(A) \otimes_k H_s(B)$, i.e. the sequence splits.

When B is not a lattice we need the following result.

Lemma 2.6 Let B be a complex of k–modules. Then there exists a complex C of free k–modules and a chain map $\theta : C \longrightarrow B$ such that the induced map $\theta_* : H(C) \longrightarrow H(B)$ is an isomorphism.

(Problem: can $\theta$ always be taken to be onto?)

We prove this by establishing Lemmas 2.7 and 2.8.

Lemma 2.7 Let B be a complex of k–modules. Then there exists a complex C of free k–modules such that $H_n(C) \cong H_n(B)$ for all $n \in \mathbb{N}$.

Lemma 2.8 Let B be a complex of k–modules, let C be a complex of free k–modules and let $\theta_n : H_n(C) \longrightarrow H_n(B)$ be a homomorphism for each $n \in \mathbb{N}$. Then there exists a chain map $\varphi : C \longrightarrow B$ such that $\varphi_{n*} = \theta_n$ for all $n \in \mathbb{N}$.



Proof The proof of Lemma 2.7 is very easy so we leave it as an exercise. We prove Lemma 2.8 by induction on n. Let $\beta_r$ and $\gamma_r$ denote the boundary maps of B and C respectively. For $n \in \mathbb{N}$, having constructed $\varphi_r : C_r \longrightarrow B_r$ such that $\varphi_{r*} = \theta_r$, $\varphi_r\,\beta_r = \gamma_r\,\varphi_{r-1}$, $(\ker \gamma_r)\varphi_r \subseteq \ker \beta_r$, $(\mathrm{im}\,\gamma_{r+1})\varphi_r \subseteq \mathrm{im}\,\beta_{r+1}$ for $r < n$, we construct $\varphi_n : C_n \longrightarrow B_n$ having the same properties (where $\varphi_{-1} = 0$).

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\gamma_{n+1}} & C_n & \xrightarrow{\gamma_n} & C_{n-1} & \xrightarrow{\gamma_{n-1}} & C_{n-2} & \longrightarrow & \cdots \\
 & & & & \downarrow{\scriptstyle \varphi_{n-1}} & & \downarrow{\scriptstyle \varphi_{n-2}} & & \\
 & \xrightarrow{\beta_{n+1}} & B_n & \xrightarrow{\beta_n} & B_{n-1} & \xrightarrow{\beta_{n-1}} & B_{n-2} & \longrightarrow & \cdots
\end{array}$$

Now $\theta_n$ is a homomorphism $\theta_n : \ker \gamma_n / \mathrm{im}\,\gamma_{n+1} \longrightarrow \ker \beta_n / \mathrm{im}\,\beta_{n+1}$. Since $\ker \gamma_n$ is a free k–module, $\theta_n$ lifts to a homomorphism $\psi : \ker \gamma_n \longrightarrow \ker \beta_n$. Also $\mathrm{im}\,\gamma_n$ is a free k–module, so we may write $C_n = \ker \gamma_n \oplus D$ for some k–sublattice D of $C_n$. Since $\gamma_n\,\varphi_{n-1}$ maps D into $\mathrm{im}\,\beta_n$ there is a homomorphism $\delta : D \longrightarrow B_n$ such that $d\,\delta\,\beta_n = d\,\gamma_n\,\varphi_{n-1}$ for all $d \in D$. We may now set $\theta_n = \psi \oplus \delta : \ker \gamma_n \oplus D = C_n \longrightarrow B_n$ and the induction step is complete.

We now show that the sequence of Theorem 2.1 splits when B is an arbitrary chain complex. By Lemma 2.6 we may choose a complex C of free k–modules and a chain map $\theta : C \longrightarrow B$ such that the induced map $\theta_*$ is an isomorphism. We now have a commutative diagram with exact rows in which the top row splits and the two outside vertical maps are isomorphisms.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{r+s=n} H_r(A) \otimes_k H_s(C) & \longrightarrow & H_n(A \otimes_k C) & \longrightarrow & \bigoplus_{r+s=n-1} \mathrm{Tor}_1^k(H_r(A), H_s(C)) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1_* \otimes \theta_*} & & \downarrow{\scriptstyle (1 \otimes \theta)_*} & & \downarrow{\scriptstyle (1_*, \theta_*)} & & \\
0 & \longrightarrow & \bigoplus_{r+s=n} H_r(A) \otimes_k H_s(B) & \longrightarrow & H_n(A \otimes_k B) & \longrightarrow & \bigoplus_{r+s=n-1} \mathrm{Tor}_1^k(H_r(A), H_s(B)) & \longrightarrow & 0
\end{array}$$

The Five Lemma shows that the middle vertical map is an isomorphism and a routine diagram chase now shows that the bottom row splits, as required.

Exercise  Let A be a chain complex of k–lattices and let B, C be chain complexes of k–modules. Suppose $\theta : B \longrightarrow C$ is a chain map such that the induced map $\theta_* : H_n(B) \longrightarrow H_n(C)$ is an isomorphism for all $n \in \mathbb{N}$. Prove that $(1 \otimes \theta)_* : H_n(A \otimes_k B) \longrightarrow H_n(A \otimes_k C)$ is an isomorphism for all $n \in \mathbb{N}$.



## 3. Cup Products

Notation  If M is a kG–module, write $H^*(G, M) = \bigoplus_{r \in \mathbb{N}} H^r(G, M)$.

Aim  To make $H^*(G, k)$ into a graded anticommutative k–algebra and $H^*(G, M)$ into a graded $H^*(G, k)$–module. This means that if $u \in H^r(G, M)$, $x \in H^s(G, k)$, $y \in H^t(G, k)$ then $ux \in H^{r+s}(G, M)$ and $xy = (-1)^{st} yx \in H^{s+t}(G, k)$. (Thus if p and s are odd and k is a field of characteristic p then $x^2 = 0$.)

Let $(P, \epsilon) : \ldots \longrightarrow P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} k \longrightarrow 0$ be a projective resolution of k with kG–modules. Then $(P \otimes_k P, \epsilon \otimes \epsilon)$ is a projective resolution of $k \otimes_k k$ with kG–modules (Remark 2.3 (ii)). Also we have a natural isomorphism of kG–modules $\mu : k \otimes k \longrightarrow k$ where $(a \otimes b)\mu = ab$ for $a, b \in k$. Thus if $\pi = (\epsilon \otimes \epsilon)\mu$ then $(P \otimes_k P, \pi)$ is a projective resolution of k with kG–modules. By Lemma 1.2 there exists a chain map

3.1  $$\theta : P \longrightarrow P \otimes_k P$$

extending the identity map on k.

Suppose $u \in H^r(G, M)$, $x \in H^s(G, k)$. Choose $f \in \mathrm{Hom}_{kG}(P_r, M)$ and $g \in \mathrm{Hom}_{kG}(P_s, k)$ representing u and x respectively. Then $f \otimes g \in \mathrm{Hom}_{kG}(P_r \otimes_k P_s, M)$, where $(a \otimes b)(f \otimes g) = (af)(bg)$ for $a \in P_r$, $b \in P_s$. Therefore $\theta^*(f \otimes g) = \theta(f \otimes g) \in \mathrm{Hom}_{kG}(P_{r+s}, M)$. Since $\partial_r^* f = 0 = \partial_s^* g$, we have $\partial_r f = 0 = \partial_s g$ and hence $\partial_{r+s}^*(\theta^*(f \otimes g)) = \partial_{r+s}\theta(f \otimes g) = \theta(\partial_r f \otimes g + (-1)^r f \otimes \partial_s g) = 0$.

Therefore $\theta^*(f \otimes g)$ represents an element of $H^{r+s}(G, M)$: it is denoted ux, the cup–product of u and x. Lemma 1.2 (ii) shows that ux does not depend on $\theta$. We shall use the notation $v_i$ to denote the $i^{th}$ component of an element v in $H^*(G, M)$ : thus $v = \Sigma v_i$ where $v_i \in H^i(G, M)$. If u and x are arbitrary elements of $H^*(G, M)$ and $H^*(G, k)$ we can now define $ux \in H^*(G, M)$ by

$$(ux)_r = \sum_{i+j=r} u_i x_j .$$

Remark We could also define the cup product by letting $(P, \epsilon)$ be a projective resolution of $\mathbb{Z}$ with $\mathbb{Z}G$–modules and $\theta : P \longrightarrow P \otimes_{\mathbb{Z}} P$ be a chain map extending the identity on $\mathbb{Z}$. This would give the same result: cf. Remark 1.19 (iv).

Lemma 3.2 $H^*(G, k)$ is a graded anticommutative ring with a 1 and $H^*(G, M)$ is a graded $H^*(G, k)$–module. If $(P, \epsilon)$ is a projective resolution for $k$ then $1 \in H^*(G, k)$ is represented by $\epsilon \in \mathrm{Hom}_{kG}(P_0, k)$.

Proof All is clear except for the anticommutativity: we must prove that if $x \in H^r(G, k)$ and $y \in H^s(G, k)$ then $xy = (-1)^{rs} yx$.

Let $(P, \epsilon)$ be a projective resolution of $k$, let $f \in \mathrm{Hom}_{kG}(P_r, k)$ represent $x$ and let $g \in \mathrm{Hom}_{kG}(P_s, k)$ represent $y$. Let $\theta : P \longrightarrow P \otimes_k P$ be a chain map extending the identity map on $k$ (see 3.1). Then by definition $\theta(f \otimes g)$, $\theta(g \otimes f) \in \mathrm{Hom}_{kG}(P_{r+s}, k)$ represent $xy, yx \in H^{r+s}(G, k)$ respectively. By Lemma 1.2 (ii)

$$\theta^* : H^{r+s}(\mathrm{Hom}_{kG}(P \otimes_k P, k)) \longrightarrow H^{r+s}(\mathrm{Hom}_{kG}(P, k))$$

is an isomorphism, so we want to show that $f \otimes g$ and $(-1)^{rs} g \otimes f$ represent the same element in $H^{r+s}(\mathrm{Hom}_{kG}(P \otimes_k P, k))$.

Define a chain map $\tau : P \otimes_k P \longrightarrow P \otimes_k P$ by

$$(a \otimes b)\tau = (-1)^{rs} (b \otimes a) \quad \text{for } a \in P_r ,\ b \in P_s .$$

Then the induced map $\tau^* : H^{r+s}(\mathrm{Hom}_{kG}(P \otimes_k P, k)) \longrightarrow H^{r+s}(\mathrm{Hom}_{kG}(P \otimes_k P, k))$ is the identity by Lemma 1.2 (ii) and the result follows.

Definition 3.3 Let $A, B$ be anticommutative graded k–algebras, say $A = \bigoplus_{n=0}^{\infty} A_n$, $B = \bigoplus_{n=0}^{\infty} B_n$. Then $a \in A$ is homogeneous means $a \in A_n$ for some $n \in \mathbb{N}$ and then we write $\deg a = n$ (if $a \neq 0$). We make $A \otimes_k B$ into an anticommutative graded k–algebra by defining



$(A \otimes_k B)_n = \bigoplus_{r+s=n} A_r \otimes_k B_s$, and for homogeneous elements $a_1, a_2 \in A$ and $b_1, b_2 \in B$,

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{\deg b_1 \deg a_2} a_1 a_2 \otimes b_1 b_2 .$$

Theorem 3.4 Let $H$ be a group. Then there is a natural monomorphism of anticommutative k–algebras $\pi : H^*(G \times H, k) \longrightarrow H^*(G, k) \otimes_k H^*(H, k)$. If $k$ is a field, then $\pi$ is an epimorphism.

Proof This is just 2.4; all that needs to be checked is that $\pi$ respects multiplication as well as addition.

Lemma 3.5 Let $L, M, N$ be kG–modules, let $H$ be a group, let $u \in H^*(G, M)$ and $y \in H^*(G, k)$.

(i) If $\theta : H \longrightarrow G$ is a homomorphism then $\theta^*(u)\, \theta^*(y) = \theta^*(uy)$.

(ii) If $\varphi : M \longrightarrow N$ is a kG–homomorphism then $\varphi_*(u)y = \varphi_*(uy)$.

(iii) If $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ is exact and $\delta : H^*(G, N) \longrightarrow H^*(G, L)$ is the connecting homomorphism (cf. 1.13 (i)) then $\delta(vy) = (\delta v)y$ for $v \in H^*(G, N)$.

(iv) If $H \leq G$ then $\mathrm{tr}_{H,G}(\mathrm{res}_{G,H}(u)z) = u\, \mathrm{tr}_{H,G}\, z$ for $z \in H^*(H, k)$.

(v) If $k$ is a field, char $k = p$, and $x \in H^r(G, k)$, then $\beta(xy) = (\beta x)y + (-1)^r x\, (\beta y)$.

Proof We prove (v), leaving the other parts as exercises. We may assume that $k = \mathbb{Z}/p\mathbb{Z}$ by Remark 1.19 (iii), and $y$ is homogeneous of degree $s$ for some $s \in \mathbb{N}$.

Let $(P, \epsilon) : \ldots \longrightarrow P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$ be a projective resolution of $\mathbb{Z}$ with $\mathbb{Z}G$–modules, let $f \in \mathrm{Hom}_{\mathbb{Z}G}(P_r, k)$ and $g \in \mathrm{Hom}_{\mathbb{Z}G}(P_s, k)$ represent $x$ and $y$ respectively, and let $\theta : P \longrightarrow P \otimes_{\mathbb{Z}} P$ be a chain map extending the identity map on $\mathbb{Z}$ (cf. 3.1). Then $xy$ is represented by $\theta(f \otimes g) \in \mathrm{Hom}_{\mathbb{Z}G}(P_{r+s}, k)$.

Lift $f$ and $g$ to $\hat{f}$ and $\hat{g}$, elements of $\mathrm{Hom}_{\mathbb{Z}G}(P_r, \mathbb{Z}/p^2\mathbb{Z})$ and $\mathrm{Hom}_{\mathbb{Z}G}(P_s, \mathbb{Z}/p^2\mathbb{Z})$ respectively. Then $\theta(\hat{f} \otimes \hat{g}) \in \mathrm{Hom}_{\mathbb{Z}G}(P_{r+s}, \mathbb{Z}/p^2\mathbb{Z})$ lifts $\theta(f \otimes g)$ and so $\beta(xy)$ is represented by

$\partial_{r+s+1}\theta(\hat{f}\otimes\hat{g}) \in \mathrm{Hom}_{\mathbb{Z}G}(P_{r+s+1}, \mathbb{Z}/p^2\mathbb{Z})$ (see 1.22). But this is

$$\theta\partial_{r+s+1}(\hat{f}\otimes\hat{g}) = \theta(\partial_{r+1}\hat{f}\otimes g) + (-1)^r\,\theta(\hat{f}\otimes\partial_{s+1}\hat{g}).$$

Since $\partial_{r+1}\hat{f}$ represents $\beta x$ and $\partial_{s+1}\hat{g}$ represents $\beta y$ the result follows.

<u>3.6 Cohomology of the Cyclic Group</u>

Let $G = \langle g \rangle$ be a cyclic group and let $\mathfrak{g}$ be the augmentation ideal of $kG$, so $\mathfrak{g}$ is a free $k$–module with basis $\{g-1 \mid g \in G\backslash 1\}$. Define $kG$–homomorphisms $\epsilon : kG \longrightarrow k$ and $\nu : kG \longrightarrow \mathfrak{g}$ by $1\epsilon = 1$ and $1\nu = g-1$. Then we have the exact sequences

$$0 \longrightarrow \mathfrak{g} \longrightarrow kG \longrightarrow k \longrightarrow 0$$
$$0 \longrightarrow k \longrightarrow kG \longrightarrow \mathfrak{g} \longrightarrow 0.$$

Since $H^n(G, kG) = 0$ for all $n \in \mathbb{P}$ by Corollary 1.10, the long exact sequences for cohomology (Corollary 1.13 (i)) show that the connecting homomorphisms give isomorphisms

$\gamma : H^n(G, k) \xrightarrow{\cong} H^{n+1}(G, \mathfrak{g})$ and $\delta : H^n(G, \mathfrak{g}) \xrightarrow{\cong} H^{n+1}(G, k)$ for all $n \in \mathbb{P}$. (i)

Thus $H^{n+2}(G, k) \cong H^n(G, k)$ for $n \in \mathbb{P}$. Let us consider two special cases.

<u>Case 1</u> $k$ is a field of characteristic $p$ and $p \mid |G|$ (if $p$ does not divide $|G|$ then $H^n(G, k) = 0$ for all $n \in \mathbb{P}$ – exercise). The exact sequence $0 \longrightarrow \mathfrak{g} \longrightarrow kG \longrightarrow k \longrightarrow 0$ yields an exact sequence

$$0 \longrightarrow H^0(G, \mathfrak{g}) \longrightarrow H^0(G, kG) \longrightarrow H^0(G, k) \xrightarrow{\gamma} H^1(G, \mathfrak{g}) \longrightarrow 0.$$

Since $H^0(G, \mathfrak{g}) \cong H^0(G, kG) \cong H^0(G, k) \cong k$ it follows that

$$\gamma : H^0(G, k) \xrightarrow{\cong} H^1(G, \mathfrak{g}) \quad \text{(ii)}$$

is an isomorphism and $H^1(G, \mathfrak{g}) \cong k$. Also $H^1(G, k) \cong \mathrm{Hom}(G, k) \cong k$ by Proposition 1.20. It now follows from (i) that $H^n(G, k) \cong k$ for all $n \in \mathbb{N}$. Thus we have the additive structure of $H^*(G, k)$ and we now calculate the multiplicative structure.

By (i) and (ii) $\delta\gamma : H^n(G, k) \longrightarrow H^{n+2}(G, k)$ is an isomorphism for all $n \in \mathbb{N}$. Also if $x \in H^n(G, k)$ and $y \in H^m(G, k)$, $m \in \mathbb{N}$, then $\delta\gamma(xy) = (\delta\gamma x)y$ by Lemma 3.5 (iii). Now $1y = y$ where $1 \in H^0(G, k)$ is the identity. It follows that if $n$ is even and $x \neq 0$ then $y \longmapsto xy$ is a bijective map from $H^m(G, k)$ to $H^{m+n}(G, k)$. This shows that

$$\bigoplus_{n\ \text{even}} H^n(G, k) \cong k[u],$$



a polynomial ring where $u$ can be taken to be any nonzero element of $H^2(G, k)$. If $p$ is odd and $v \in H^1(G, k)$ then $v^2 = 0$ because $H^*(G, k)$ is anticommutative and

$$H^*(G, k) \cong k[u, v] \,/\, (v^2, uv - vu) \quad \text{(iii)}$$

where $\deg u = 2$, $v \neq 0$, $\deg v = 1$.

On the other hand if $p = 2$ we need a further subdivision of cases. First suppose $|G| = 2$. Then $k \cong \mathfrak{g}$ as $kG$–modules, hence from (i) and (ii) we have an isomorphism $\gamma : H^n(G, k) \longrightarrow H^{n+1}(G, k)$ for all $n \in \mathbb{N}$. It follows that $H^*(G, k) \cong k[v]$, a polynomial ring where $v$ can be taken to be any nonzero element of $H^1(G, k)$.

In general let $H = \langle h \rangle$ be the subgroup of order 2 in $G$. Identifying $H^0(H, \mathfrak{g})$ and $H^0(G, \mathfrak{g})$ with the fixed points of $\mathfrak{g}$ under the action of $H$ and $G$ respectively, $1 + h \in H^0(H, \mathfrak{g})$ and $\mathrm{tr}_{H,G}(1+h) = \sum_{g\in G} g \in H^0(G, \mathfrak{g}) \cong k$, so $\mathrm{tr}_{H,G} : H^0(H, \mathfrak{g}) \longrightarrow H^0(G, \mathfrak{g})$ is onto. Moreover the exact sequence $0 \longrightarrow k \longrightarrow kG \longrightarrow \mathfrak{g} \longrightarrow 0$ yields (by Corollary 1.13 (i)) a commutative diagram with exact rows

$$\begin{array}{ccccc}
H^0(H, \mathfrak{g}) & \longrightarrow & H^1(H, k) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \mathrm{tr}_{H,G}} & & \downarrow{\scriptstyle \mathrm{tr}_{H,G}} & & \\
H^0(G, \mathfrak{g}) & \longrightarrow & H^1(G, k) & \longrightarrow & 0
\end{array}$$

and we deduce that $\mathrm{tr}_{H,G} : H^1(H, k) \longrightarrow H^1(G, k)$ is an isomorphism. Let $\ell = [G : H]$. Using $\mathrm{tr}_{H,G}\,\mathrm{res}_{G,H} = \ell$ (Lemma 1.14) we see that $\mathrm{tr}_{H,G} : H^2(H, k) \longrightarrow H^2(G, k)$ is an isomorphism if 2 does not divide $\ell$ and $\mathrm{res}_{G,H} : H^1(G, k) \longrightarrow H^1(H, k)$ is zero if $2 \mid \ell$. Now let $0 \neq u \in H^1(G, k)$ and $z \in H^1(H, k)$ such that $\mathrm{tr}_{H,G}(z) = u$. Then

$$u^2 = u\,\mathrm{tr}_{H,G}(z) = \mathrm{tr}_{H,G}(\mathrm{res}_{G,H}\,u)z \quad \text{by Lemma 3.5 (iv)}$$
$$= 0 \quad \text{if and only if } 2 \mid \ell.$$

We conclude that

$H^*(G, k) \cong k[v]$ if 4 does not divide $|G|$ (a polynomial ring where $v \in H^1(G, k)$,

$H^*(G, k) \cong k[u, v] \,/\, (v^2, uv - vu)$ if $4 \mid |G|$ (where $v \in H^1(G, k)$ and $u \in H^2(G, k)$).

Next we calculate the Bockstein map $\beta_n : H^n(G, k) \longrightarrow H^{n+1}(G, k)$ .

As above,

$$H^i(G, \mathbb{Z}/p^2\mathbb{Z}) \cong \mathbb{Z}/p^2\mathbb{Z} \text{ for all } i \in \mathbb{N} \text{ if } p^2 \mid |G| ,$$
$$H^i(G, \mathbb{Z}/p^2\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z} \text{ for all } i \in \mathbb{P} \text{ if } p^2 \text{ does not divide } |G| . \quad \text{(iv)}$$

The exact sequence $0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{Z}/p^2\mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$ yields (see Corollary 1.13 (i) and 1.22) an exact sequence

$$0 \longrightarrow H^0(G, \mathbb{Z}/p\mathbb{Z}) \longrightarrow H^0(G, \mathbb{Z}/p^2\mathbb{Z}) \longrightarrow H^0(G, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\beta_0} H^1(G, \mathbb{Z}/p\mathbb{Z}) \longrightarrow \dots .$$

Using (iv) we deduce

$$\beta_i = 0 \text{ for all } i \in \mathbb{N} \text{ if } p^2 \mid |G|$$
$$\beta_{2i} = 0 , \beta_{2i+1} \text{ is an isomorphism for all } i \in \mathbb{N} \text{ if } p^2 \text{ does not divide } |G| .$$

Thus if $p^2$ does not divide $|G|$ we can rewrite (iii) as ($p$ odd, $p \mid |G|$)

$$H^*(G, k) \cong k[v, \beta v] / (v^2, v\beta v - (\beta v)v)$$

where $v$ is any nonzero element of $H^1(G, k)$ .

Case 2 $k = \mathbb{Z}$ . Let $\ell = |G|$ . By Proposition 1.20 and Exercise 2.5 (i), $H^1(G, \mathbb{Z}) = 0$ and $H^2(G, \mathbb{Z}) \cong G/G'$ and it now follows from (i) that

$$H^0(G, \mathbb{Z}) \cong \mathbb{Z} , H^{2n}(G, \mathbb{Z}) \cong \mathbb{Z}/\ell\mathbb{Z} , H^{2n-1}(G, \mathbb{Z}) = 0 \ (n \in \mathbb{P}) .$$

Also $\gamma : H^0(G, \mathbb{Z}) \longrightarrow H^1(G, \mathfrak{g})$ is onto because $H^1(G, \mathbb{Z}G) = 0$ . By a similar argument to Case 1 we now see that if $m, n \in \mathbb{P}$ and $x$ is a generator of $H^{2m}(G, \mathbb{Z})$ then $y \longmapsto xy$ is a bijective map from $H^n(G, \mathbb{Z})$ to $H^{2m+n}(G, \mathbb{Z})$ . Therefore

$$H^*(G, \mathbb{Z}) \cong \mathbb{Z}[u]/(\ell u)$$

where $u$ is any generator of $H^2(G, \mathbb{Z})$ .

Notation Let $E_k[u_1,\dots,u_d]$ denote the exterior algebra on $d$ generators, an anticommutative graded $k$–algebra which as a $k$–module is free of rank $2^d$. Thus $E_k[u] \cong k[u]/(u^2) = k \oplus k\,u$ where $u$ has degree 1 and $u^2 = 0$ , and

$$E_k[u_1,\dots, u_d] \cong E_k[u_1] \otimes_k E_k[u_2] \otimes_k \dots \otimes_k E_k[u_d] .$$

We can now state

Lemma 3.7 Let $k$ be a field of characteristic $p$ , let $|G| = p$ and let $0 \neq u \in H^1(G, k)$.



Then

(i) If $p$ is odd then $H^*(G, k) \cong k[\beta u] \otimes_k E_k[u]$ .

(ii) If $p = 2$ then $H^*(G, k) \cong k[u]$ .

Cohomology of an elementary abelian p–group Let $k$ be a field of characteristic $p$ , let $d \in \mathbb{P}$ and let $G$ be the elementary abelian p–group of rank $d$ (so $|G| = p^d$) . Let $(u_1,\dots,u_d)$ be a $k$–basis for $H^1(G, k)$ (= Hom(G, k) by Proposition 1.20). By Theorem 3.4 and Lemma 3.7 we now have

Theorem 3.8 (i) If $p$ is odd then $H^*(G, k) \cong k[\beta u_1,\dots, \beta u_d] \otimes_k E_k[u_1,\dots, u_d]$ .

(ii) If $p = 2$ then $H^*(G, k) \cong k[u_1,\dots, u_d]$ .

Cohomology with coefficients in $\mathbb{Z}$ Let $G$ be an elementary abelian p–group. Then we need

Lemma 3.9 If $n \in \mathbb{P}$ and $x \in H^n(G, \mathbb{Z})$, then $px = 0$ .

Proof Exercise using 2.4 (Künneth formula) and 3.6.

Let $k = \mathbb{Z}/p\mathbb{Z}$ . Then we have an exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\mu} \mathbb{Z} \xrightarrow{\varphi} k \longrightarrow 0$$

where $\mu$ is "multiplication by p", and hence an exact sequence

$$\dots \longrightarrow H^n(G, \mathbb{Z}) \xrightarrow{\mu_*} H^n(G, \mathbb{Z}) \xrightarrow{\varphi_*} H^n(G, k) \xrightarrow{\delta} H^{n+1}(G, \mathbb{Z}) \xrightarrow{\mu_*} H^{n+1}(G, \mathbb{Z}) \longrightarrow \dots$$

by Corollary 1.13 (i), and $\mu_*$ is "multiplication by p". Using Lemma 3.9, im $\mu_* = 0$ for all $n \in \mathbb{P}$ so we have an exact sequence

$$0 \longrightarrow H^n(G, \mathbb{Z}) \xrightarrow{\varphi_*} H^n(G, k) \xrightarrow{\delta} H^{n+1}(G, \mathbb{Z}) \longrightarrow 0 .$$

Define $\tilde{H}^n(G, \mathbb{Z}) = H^n(G, \mathbb{Z})$ $n > 0$

$\tilde{H}^0(G, \mathbb{Z}) = k$ ,

so $\tilde{H}^*(G, \mathbb{Z}) \cong H^*(G, \mathbb{Z})/(p)$ as anticommutative graded rings and $\varphi_*$ induces a ring mono–

morphism $\tilde{H}^*(G, \mathbb{Z}) \longrightarrow H^*(G, k)$. Therefore $\tilde{H}^*(G, \mathbb{Z}) \cong \ker \delta = \ker \varphi_* \delta$. Now $\varphi_* \delta = \beta :$ $H^n(G, k) \longrightarrow H^{n+1}(G, k)$ (exercise), so

$$\tilde{H}^*(G, \mathbb{Z}) \cong \ker \beta : H^*(G, k) \longrightarrow H^*(G, k) .$$

<u>Example</u> $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, $p$ odd. Then (Theorem 3.8) $H^*(G, k) \cong k[x, y] \otimes_k E_k[u, v]$, $\beta u = x$, $\beta v = y$.

Now $\beta(f_1 + f_2 u + f_3 v + f_4 uv) = 0 \quad (f_i \in k[x, y])$

$\Leftrightarrow$ (using 3.5 (v) and 3.6) $f_2 x + f_3 y$ and $f_4(xv - yu) = 0$

$\Leftrightarrow$ $f_4 = 0$, $f_2 = yf$, $f_3 = -xf$ some $f \in k[x, y]$.

Therefore $\tilde{H}^*(G, \mathbb{Z}) \cong k[x, y] \otimes_k E_k[uy - vx]$.

<u>Exercise</u> If $p = 2$ show $\tilde{H}^*(G, \mathbb{Z}) \cong k[x^2, y^2, x^2y + xy^2]$.



4. <u>The Evens Norm Map</u> Let $H \leq G$. Recall the transfer map

$$tr_{H,G} : H^*(H, k) \longrightarrow H^*(G, k)$$

is a map satisfying $tr_{H,G}(x + y) = tr_{H,G}(x) + tr_{H,G}(y)$ i.e. $tr_{H,G}$ respects the additive structure. The Evens norm map is a map

$$norm_{H,G} : H^*(H, k) \longrightarrow H^*(G, k)$$

which respects the multiplicative structure. To define this map, we need to consider tensor induction. Write $G = x_1 H \cup \ldots \cup x_\ell H$ and let $M$ be a kH–module. For $g \in G$ write

$$g\, x_i = x_{\hat{g}i}\, g_i \qquad (1)$$

where $g_i \in H$ ($i = 1, \ldots, \ell$) and $\hat{g} \in \Sigma_\ell$. Define a kG–module by

$$M^\ell = M \otimes_k \ldots \otimes_k M \quad (\ell \text{ times}),$$

$$(m_1 \otimes \ldots \otimes m_\ell) g = m_{\hat{g}1}\, g_1 \otimes \ldots \otimes m_{\hat{g}\ell}\, g_\ell . \qquad (2)$$

It is easy to check that this gives a well defined kG–module whose isomorphism type is independant of the choice of transversal $\{x_1, \ldots, x_\ell\}$, and $k^\ell \cong k$ (naturally).

However $P^\ell$ is not a projective kG–module in general when $P$ is a projective kH–module.

Similarly if $P$ is a chain complex of kH–modules then $P^\ell$ is a chain complex of kG–modules, but we need a sign in (2) (so that the G–action commutes with the boundary maps), namely (when the $m_i$ are homogeneous)

$$\prod_{\substack{i<j \\ \hat{g}^{-1}i > \hat{g}^{-1}j}} (-1)^{\deg m_i \deg m_j} \qquad (3)$$

However we must check that (3) gives a G–action, and that the action commutes with the boundary map: i.e. for $f, g \in G$ and $u \in P^\ell$, $u$ homogeneous,

$$(uf)g = u(fg) \quad \text{and} \quad (u\partial)g = (ug)\partial .$$

To do this directly is technically unpleasant, especially the sign in the latter equality, so we proceed differently and first use (1) to embed G in the Wreath product $\Sigma_\ell \wr H$. Recall that $\Sigma_\ell \wr H$ consists of elements $(\pi; h_1,\dots, h_\ell)$ $(\pi \in \Sigma_\ell, h_i \in H)$ with multiplication

$$(\pi; h_1,\dots, h_\ell)\,(\sigma; e_1,\dots, e_\ell) = (\pi\,\sigma; h_{\sigma 1}\, e_1,\dots, h_{\sigma \ell}\, e_\ell)\,.$$

Clearly $(\pi; 1,\dots, 1)^{-1} = (\pi^{-1}; 1,\dots, 1)$ and $(1; h_1,\dots, h_\ell)^{-1} = (1; h_1^{-1},\dots, h_\ell^{-1})$. For convenience we shall let sign $(\pi; h_1,\dots, h_\ell)$ denote the sign of the permutation $\pi$. Using the notation of (1), define $\theta : G \longrightarrow \Sigma_\ell \wr H$ by

$$g\,\theta = (\hat{g}; g_1,\dots, g_\ell)\,.$$

Then we have

<u>Lemma 4.1</u> (i) $\theta$ is a monomorphism.

(ii) Suppose $\{y_1,\dots, y_\ell\}$ is another left transversal for H in G and $\varphi : G \longrightarrow \Sigma_\ell \wr H$ is the corresponding monomorphism. Then there exists $w \in \Sigma_\ell \wr H$ such that $g\,\varphi = w^{-1}(g\,\theta)w$ for all $g \in G$, and sign $(w)$ = sign of the permutation $x_iH \longmapsto y_iH$ on the left cosets of H in G.

<u>Proof</u> (i) This is routine checking.

(ii) It will be sufficient to consider the following two cases:

Case 1 There exist $h_1,\dots, h_\ell \in H$ such that $y_i = x_ih_i$. Here we choose $w = (1; h_1,\dots, h_\ell)$.

Case 2 There exists $\sigma \in \Sigma_\ell$ such that $y_i = x_{\sigma i}$. Here we choose $w = (\sigma; 1,\dots, 1)$.

We now need to discuss differential graded algebras as described in VI 7 of [S. MacLane, Homology, Springer–Verlag, Berlin–New York 1975]. Section 4.2 is no more than a summary of portions of Chapter VI of MacLane's book.

4.2 <u>Definitions</u> Let K be a commutative ring with a 1, and let $A = \oplus_{i=0}^{\infty} A_i$ be a graded K–module. An element a in A is homogeneous means $a \in A_i$ for some $i \in \mathbb{N}$.

(i) Suppose A is a K–algebra. Then A is a graded K–algebra means $A_iA_j \subseteq A_{i+j}$.



(ii) If there is a K–module homomorphism $\partial : A \longrightarrow A$ such that $A_i\partial \subseteq A_{i-1}$ for all $i \in \mathbb{P}$, $A_0\partial = 0$ and $\partial^2 = 0$, then A is called a DG–module.

(iii) Suppose A is a graded K–algebra which is also a DG–module. Then A is a DG–algebra means $(ab)\partial = (a\partial)b + (-1)^{\deg a}\, a(b\partial)$ for all homogeneous elements a, b in A.

(iv) If A and B are graded K–algebras, then $A \otimes_k B$ is a graded K–algebra with multiplication and degree

$$\begin{aligned}(a \otimes b)(a' \otimes b') &= aa' \otimes bb'(-1)^{\deg b \deg a'}\\ \deg(a \otimes b) &= \deg a + \deg b\end{aligned}$$

$(a, a' \in A,\ b, b' \in B$ homogeneous). Note that forming tensor products of graded algebras is associative: i.e. we get the same sign in the above whether we consider $A \otimes_K (B \otimes_K C)$ or $(A \otimes_K B) \otimes_K C$, and in both cases

$$(a \otimes b \otimes c)(a' \otimes b' \otimes c') = aa' \otimes bb' \otimes cc'\ (-1)^{\sigma}$$

where $\sigma = \deg a' \deg b + \deg a' \deg c + \deg b' \deg c$. Thus we can write unambiguously $A \otimes_K B \otimes_K C$.

(v) If A and B are DG–modules, then $A \otimes_K B$ is a DG–module with $\deg(a \otimes b) = \deg a + \deg b$ and

$$(a \otimes b)\partial = a\partial \otimes b + (-1)^{\deg a}\, a \otimes b\partial$$

for homogeneous $a \in A$, $b \in B$. As in (iv) forming tensor products is associative i.e we get the same sign in the above whether we consider $A \otimes_K (B \otimes_K C)$ or $(A \otimes_K B) \otimes_K C$, and in both cases

$$(a \otimes b \otimes c)\partial = a\partial \otimes b \otimes c + (-1)^{\deg a}\, a \otimes b\partial \otimes c + (-1)^{\deg a + \deg b}\, a \otimes b \otimes c\partial\,.$$

Thus again we can write unambiguously $A \otimes_K B \otimes_K C$.

(vi) Suppose A is a DG–module which is also a graded K–algebra. Then A is a DG–algebra means

$$(ab)\partial = (a\partial)b + (-1)^{\deg a}\, a(b\partial)$$

for all homogeneous $a, b \in A$. If A, B, C are DG–algebras, then $A \otimes_K B$ is a DG–algebra and by parts (iv) and (v) and we can write unambiguously $A \otimes_K B \otimes_K C$.

(vii) The tensor algebra

$$T(A) = K \oplus A \oplus A \otimes_K A \oplus \ldots$$

is a graded K–algebra with $\deg(a_1 \otimes \ldots \otimes a_\ell) = \deg a_1 + \ldots + \deg a_\ell$ and

$$(a_1 \otimes \ldots \otimes a_\ell)(a'_1 \otimes \ldots \otimes a'_{\ell'}) = a_1 \otimes \ldots \otimes a_\ell \otimes a'_1 \otimes \ldots \otimes a'_{\ell'} .$$

If $A$ is a DG–module, then $T(A)$ becomes a DG–algebra with

$$(a_1 \otimes \ldots \otimes a_\ell)\partial = \sum_{i=1}^{\ell} (-1)^{\sigma_i} a_1 \otimes \ldots \otimes a_i \partial \otimes \ldots \otimes a_\ell$$

where $\sigma_i = \deg a_1 + \ldots + \deg a_{i-1}$ . Note that the natural injection $A \longrightarrow T(A)$ is a chain map.

Recall the following elementary result:

Lemma 4.3 Let $A, R$ be K–algebras, let $\theta : A \longrightarrow R$ be a K–module homomorphism, and let $X \subseteq A$ such that $X$ generates $A$ as a K–module. If $(xy)\theta = x\theta\, y\theta$ for all $x, y \in X$ , then $\theta$ is a K–algebra homomorphism.

We now have

Proposition 4.4 Let $\alpha : A \longrightarrow R$ , $\beta : B \longrightarrow R$ be homomorphisms of graded K–algebras, let $X, Y$ be the homogeneous elements of $A, B$ respectively, and let $X' \subseteq X$ be a subset which generates $A$ as a K–algebra. If

$$x\alpha\, y\beta = (-1)^{\deg x \deg y}\, y\beta\, x\alpha$$

for all $x \in X'$ , $y \in Y$ , then there is a unique graded K–algebra homomorphism $\theta : A \otimes_K B \longrightarrow R$ such that $(a \otimes b)\theta = a\alpha\, b\beta$ for all $a \in A$ , $b \in B$ .

Proof Certainly there is a unique K–module homomorphism $\theta : A \otimes_K B \longrightarrow R$ such that $(a \otimes b)\theta = a\alpha\, b\beta$ for all $a \in A$ , $b \in B$ , so we need to prove that $\theta$ respects multiplication. Let $X''$ be the multiplicative semigroup generated by $X'$ . If $x = x_1 x_2$ with $x, x_1, x_2 \in X''$ and

$$x_i\alpha\, y\beta = (-1)^{\deg x_i \deg y}\, y\beta\, x_i\alpha \qquad (i = 1, 2),$$



then

$$\begin{aligned} x\alpha\, y\beta = (x_1 x_2)\alpha\, y\beta &= x_1\alpha\, x_2\alpha\, y\beta \\ &= (-1)^{\deg x_2 \deg y}\, x_1\alpha\, y\beta\, x_2\alpha \\ &= (-1)^{\deg x_2 \deg y} (-1)^{\deg x_1 \deg y}\, y\beta\, x_1\alpha\, x_2\alpha \\ &= (-1)^{\deg x \deg y}\, y\beta(x_1 x_2)\alpha = (-1)^{\deg x \deg y}\, y\beta\, x\alpha \end{aligned}$$

and we deduce that

$$x\alpha\, y\beta = (-1)^{\deg x \deg y}\, y\beta\, x\alpha$$

for all $x \in X''$, $y \in Y$ . An easy calculation now shows that $((u_1 \otimes v_1)(u_2 \otimes v_2))\theta = (u_1 \otimes v_1)\theta\,(u_2 \otimes v_2)\theta$ for all $u_1, u_2 \in X''$, $v_1, v_2 \in Y$ . Since the elements $\{u \otimes v \mid u \in X'', v \in Y\}$ generate $A \otimes_K B$ as a K–module, the result follows from Lemma 4.3.

Corollary 4.5 Let $\alpha_i : A_i \longrightarrow R$ $(i = 1,\ldots, n)$ be homomorphisms of graded K–algebras such that

$$a_i\alpha_i\, a_j\alpha_j = (-1)^{\deg a_i \deg a_j}\, a_j\alpha_j\, a_i\alpha_i \quad \text{for all } i \neq j$$

($a_i \in A_i$, homogeneous). Then there is a unique graded K–algebra homomorphism $\theta : A_1 \otimes_K \ldots \otimes_K A_n \longrightarrow R$ such that $(a_1 \otimes \ldots \otimes a_n)\theta = a_1\alpha_1 \ldots a_n\alpha_n$ .

Proof Certainly there is a unique K–module homomorphism $\theta : A_1 \otimes_K \ldots \otimes_K A_n \longrightarrow R$ such that $(a_1 \otimes \ldots \otimes a_n)\theta = a_1\alpha_1 \ldots a_n\alpha_n$ , so we need to prove that $\theta$ respects multiplication.

We shall use induction on $n$ , so if $\varphi : A_1 \otimes_K \ldots \otimes_K A_{n-1} \longrightarrow R$ is defined by $(a_1 \otimes \ldots \otimes a_{n-1})\varphi = a_1\alpha_1 \ldots a_{n-1}\alpha_{n-1}$ , we may assume that $\varphi$ is a K–algebra homomorphism. In view of Proposition 4.4 we need to prove

$$(1 \otimes \ldots \otimes a_i \otimes \ldots \otimes 1)\,\varphi\, a_n\alpha_n = (-1)^{\deg a_i \deg a_n}\, a_n\alpha_n(1 \otimes \ldots \otimes a_i \otimes \ldots \otimes 1)\varphi$$

which is true because $(1 \otimes \ldots \otimes a_i \otimes \ldots \otimes 1)\varphi = a_i\alpha_i$ .

(4.6) Let A be a graded K–algebra, let $\ell \in \mathbb{P}$, let $\pi \in \Sigma_\ell$, and let $A^\ell = A \otimes_K \dots \otimes_K A$ ($\ell$ factors). For $i = 1, \dots, \ell$ define $\alpha_i : A \longrightarrow A^\ell$ by

$$a\,\alpha_i = 1 \otimes \dots \otimes a \otimes \dots \otimes 1$$

where the a on the right is in the $\pi^{-1}i$–position. Since

$$a\,\alpha_i\, b\,\alpha_j = (-1)^{\deg a \deg b}\, b\,\alpha_j\, a\,\alpha_i \quad \text{for all } i \neq j,$$

it follows from Corollary 4.5 that $\pi$ induces a unique graded K–algebra homomorphism $\pi : A^\ell \longrightarrow A^\ell$. Clearly this defines an action of $\Sigma_\ell$ on $A^\ell$ (i.e. $\alpha(\pi\,\sigma) = (\alpha\,\pi)\sigma$ for $\alpha \in A^\ell$, $\pi$, $\sigma \in \Sigma_\ell$), and $\pi$ satisfies for homogeneous $a_i \in A_i$

$$(a_1 \otimes \dots \otimes a_\ell)\pi = a_{\pi 1} \otimes \dots \otimes a_{\pi \ell}\,\chi$$

where $\chi$ is a sign (depending on $\pi$ and the degrees of the $a_i$).

(4.7) Let A be a DG–algebra and let $\Sigma_\ell$ act on $A^\ell$ by the rule

$$(a_1 \otimes \dots \otimes a_\ell)\pi = a_{\pi 1} \otimes \dots \otimes a_{\pi \ell}\,\chi$$

as described in (4.6). We want to show that the action commutes with the boundary map, i.e.

$$\alpha\,\partial\,\pi = \alpha\,\pi\,\partial \quad \text{for all } \alpha \in A^\ell,\ \pi \in \Sigma_\ell . \qquad (4)$$

Note that if $\alpha, \beta$ are homogeneous elements of $A^\ell$ and $\alpha\,\partial\,\pi = \alpha\,\pi\,\partial$, $\beta\,\partial\,\pi = \beta\,\pi\,\partial$, then $(\alpha + \beta)\partial\,\pi = (\alpha + \beta)\pi\,\partial$ and

$$\begin{aligned}(\alpha\,\beta)\partial\,\pi &= (\alpha\,\partial\,\beta + (-1)^{\deg \alpha}\,\alpha\,\beta\,\partial)\pi \\ &= \alpha\,\partial\,\pi\,\beta\,\pi + (-1)^{\deg \alpha\,\pi}\,\alpha\,\pi\,\beta\,\partial\,\pi \\ &= \alpha\,\pi\,\partial\,\beta\,\pi + (-1)^{\deg \alpha\,\pi}\,\alpha\,\pi\,\beta\,\pi\,\partial \\ &= (\alpha\,\pi\,\beta\,\pi)\partial = (\alpha\,\beta)\pi\,\partial .\end{aligned}$$

It follows that we need only check (4) when $\alpha$ is of the form $1 \otimes \dots \otimes a \otimes \dots \otimes 1$, and this is obvious.

(4.8) Let $P = \oplus_{i=0}^{\infty} P_i$ be a DG–module and let $\Sigma_\ell$ act on $P^\ell$ according to the formula

$$(p_1 \otimes \dots \otimes p_\ell)\pi = p_{\pi 1} \otimes \dots \otimes p_{\pi \ell}\,\chi$$



as described in (4.6). We want to show that this <u>is</u> an action and that it commutes with the boundary map i.e.

$$\alpha\,\pi\,\rho = \alpha\,\rho\,\pi \quad \text{and} \quad \alpha\,\pi\,\partial = \alpha\,\partial\,\pi$$

for all $\alpha \in P^\ell$ and $\pi, \rho \in \Sigma_\ell$. By (4.7) this is certainly true if $\alpha \in T(P)^\ell$ (where T(P) is the tensor algebra of Definition 4.2 (vii)). But the natural injection $P \longrightarrow T(P)$ is a chain map, and the natural injection $P^\ell \longrightarrow T(P)^\ell$ commutes with the action of $\Sigma_\ell$, and the result follows.

Note that in the special case $\pi$ is a transposition $(n\ \ n+1)$, it is easy to see that $\chi = (-1)^{\deg p_n \deg p_{n+1}}$. Consequently $\chi = \text{sign}(\pi)$ when all the $\deg p_i$ are equal and odd.

(4.9) Let H be a group, let P be a complex of KH–modules, let $\ell \in \mathbb{P}$, and let $W = \Sigma_\ell \wr H$ denote the Wreath product. We make $P^\ell$ into a complex of $KH^\ell$–modules by defining

$$(p_1 \otimes \dots \otimes p_\ell)(h_1, \dots, h_\ell) = p_1\,h_1 \otimes \dots \otimes p_\ell\,h_\ell$$

and into a complex of $K\Sigma_\ell$–modules (using (4.8)) by defining for homogeneous $p_i \in P_i$

$$(p_1 \otimes \dots \otimes p_\ell)\pi = p_{\pi 1} \otimes \dots \otimes p_{\pi \ell}\,\chi .$$

We claim that $P^\ell$ is a complex of KW–modules with

$$(p_1 \otimes \dots \otimes p_\ell)(\pi\,h) = ((p_1 \otimes \dots \otimes p_\ell)\pi)h$$

($\pi \in \Sigma_\ell$, $h \in H^\ell$). To establish this claim, we must verify

(i) $(p_1 \otimes \dots \otimes p_\ell)(g_1\,g_2) = ((p_1 \otimes \dots \otimes p_\ell)g_1)g_2$

(ii) $((p_1 \otimes \dots \otimes p_\ell)g)\partial = ((p_1 \otimes \dots \otimes p_\ell)\partial)g$

for all $g_1, g_2, g \in W$. Since W is generated by $\Sigma_\ell$ and $H^\ell$ we need only check (i) and for this we use

<u>Lemma</u> Let G be a semidirect product of A and H, so $H \triangleleft G$ and every element of G can be written uniquely in the form $a\,h$ ($a \in A$, $h \in H$), and let K be a commutative ring with

a 1 . Suppose M is both a KA–module and a KH–module. Define $m(a\,h) = (m\,a)h$ for $m \in M$.

(1) If $(m\,h)a = m\,a(a^{-1}h\,a)$ for all $m \in M$, $a \in A$, $h \in H$, then M is a KG–module.

(2) If $A = \langle A_0 \rangle$, $H = \langle H_0 \rangle$, M is generated as a K–module by $M_0$, and

$$(m\,h)a = m\,a(a^{-1}h\,a) \text{ for all } m \in M_0,\ a \in A_0,\ h \in H_0,$$

then M is a KG–module.

We omit the elementary proof. Thus to verify (i), we need only show

$$((p_1 \otimes \dots \otimes p_\ell)(h_1, \dots h_\ell))\pi = (p_1 \otimes \dots \otimes p_\ell)\pi(\pi^{-1}(h_1, \dots, h_\ell)\pi)$$

for $h_i \in H$ and $\pi$ a transposition $(n\ n+1)$, which is obvious (recall $\pi^{-1}(h_1, \dots, h_\ell)\pi = (h_1, \dots, h_{n+1}, h_n, \dots, h_\ell))$.

(4.10) Now let us return to the situation at the beginning of this chapter, so $H \leq G$, $G = x_1 H \uplus \dots \uplus x_\ell H$, $g\,x_i = x_{\hat{g}i}\, g_i$, and P is a chain complex of kH–modules. Let W be the Wreath product $\Sigma_\ell \wr H$, and let $\theta : G \longrightarrow W$ be the monomorphism of Lemma 4.1. Then (4.9) shows that $P^\ell$ is a complex of kW–modules, hence $P^\ell$ becomes a complex of kG–modules with G–action given by $q\,g = q(g\,\theta)$ for $q \in P^\ell$ and $g \in G$. Explicitly the G–action is given by

$$(m_1 \otimes \dots \otimes m_\ell)g = m_{\hat{g}1}\, g_1 \otimes \dots \otimes m_{\hat{g}\ell}\, g_\ell\, \chi$$

for homogeneous $m_i \in P_i$ where $\chi$ is a sign (depending on g and the degrees of the $m_i$): it is easy to see that when all the deg $m_i$ are equal $\chi$ is given by (3) (see 4.8); we leave it as an exercise to check that $\chi$ is always given by (3), since we do not need the general case in the sequel.

Suppose $\{y_1, \dots, y_\ell\}$ is another set of left coset representatives, so that $G = y_1 H \uplus \dots \uplus y_\ell H$, and let $\varphi : G \longrightarrow W$ be the corresponding monomorphism. Then Lemma 4.1 shows that there exists $w \in W$ such that $g\,\varphi = w^{-1}(g\,\theta)w$ and sign(w) = sign of the permutation $x_i H \longmapsto y_i H$,



and we now have a chain isomorphism $\psi : P^\ell \longrightarrow P^\ell$ extending the identity defined by $q\,\psi = q\,w$ which satisfies $q(g\,\theta)\psi = q\,\psi(g\,\varphi)$ for all $q \in P^\ell$. In particular the different chain complexes arising from different left coset representatives of H in G are chain isomorphic.

We can now define the Evens norm map. Let

$P: \dots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow k \longrightarrow 0$ be a projective resolution with kH–modules,

$V: \dots \longrightarrow V_1 \longrightarrow V_0 \xrightarrow{\epsilon} k \longrightarrow 0$ be a resolution with kG–modules.

Then $P^\ell \otimes_k V$ is a resolution of k with kG–modules (by the Künneth formula), not in general projective. So we choose V to make $P^\ell \otimes_k V$ projective (eg. if V is projective, then $P^\ell \otimes_k V$ is projective by Lemma 1.11). Let $k_n$ denote the kG–module which is the sign of the permutation representation of G on $\{x_1 H, \dots, x_\ell H\}$ for n odd, and is the trivial module k for n even. Thus $k_n = k$ as k–modules and for $\lambda \in k_n$, $g \in G$

$$\lambda\, g = \lambda \text{ if } n \text{ is even,}$$

$$\lambda\, g = \prod_{\substack{i<j \\ \hat{g}^{-1}i > \hat{g}^{-1}j}} (-1).$$

Write

$$H(G) = \bigoplus_{i \in \mathbb{N}} H^i(G, k) \text{ if } k \text{ is a field of characteristic two,}$$

$$= \bigoplus_{i \in \mathbb{N}} H^{2i}(G, k) \text{ otherwise.}$$

Let $u \in H^*(H, k)$ and let $f \in \mathrm{Hom}_{kH}(P, k)$ represent u.

(i) If $u \in H(H)$, then $f \otimes \dots \otimes f \otimes \epsilon \in \mathrm{Hom}_{kG}(P^\ell \otimes_k W, k)$ represents an element $\mathrm{norm}_{H,G}(u) \in H(G)$. If u is homogeneous with degree n, then $\mathrm{norm}_{H,G}(u)$ is homogeneous with degree $n\ell$.

(ii) If $f \in \mathrm{Hom}_{kH}(P_n, k)$ (so u is homogeneous with degree n, n possibly odd), then

$$f \otimes \dots \otimes f \otimes \epsilon \in \mathrm{Hom}_{kG}((P^\ell \otimes_k W)_{n\ell}, k_n)$$

and represents an element $\mathrm{norm}_{H,G}(u) \in H^{n\ell}(G, k_n)$.

Note If $n$ is odd, we need $k_n$ (not $k$). Also if $g \in \mathrm{Hom}_{kH}(P_n, k)$ represents $u$, then $g \otimes \ldots \otimes g \otimes \epsilon$ represents $\mathrm{norm}_{H,G}(u)$ (i.e. $\mathrm{norm}_{H,G}(u)$ does not depend on the choice of f). To see this, write $f = g + \delta h$ where $h \in \mathrm{Hom}_{kH}(P_{n-1}, k)$ (so $\delta$ is the coboundary map and $\delta g = 0$). Then $f \otimes \ldots \otimes f \otimes \epsilon - g \otimes \ldots \otimes g \otimes \epsilon$ is a sum of elements of the form $g_1 \otimes \ldots \otimes g_{i-1} \otimes \delta h \otimes g_{i+1} \otimes \ldots \otimes g_\ell \otimes \epsilon$ where each $g_i = g$ or $\delta h$, which up to sign is

$$\delta(g_1 \otimes \ldots \otimes g_{i-1} \otimes h \otimes g_{i+1} \otimes \ldots \otimes g_\ell \otimes \epsilon)$$

because $\delta g_i = 0$ for all $i$.

Lemma 4.11 Let $H \leq G$ and $\ell = [G : H]$.

(i) If $\lambda \in k = H^0(H, k)$, then $\mathrm{norm}_{H,G}(\lambda) = \lambda^\ell$.

(ii) If $u, v \in H^*(H, k)$ are homogeneous, then

$$\mathrm{norm}_{H,G}(u\, v) = \mathrm{norm}_{H,G}(u)\, \mathrm{norm}_{H,G}(v)\, (-1)^{\deg u \deg v \frac{\ell(\ell-1)}{2}}.$$

(iii) If $u, v \in H(H)$, then $\mathrm{norm}_{H,G}(u\, v) = \mathrm{norm}_{H,G}(u)\, \mathrm{norm}_{H,G}(v)$.

Proof (i) is obvious. (ii) and (iii) are very similar, so we will prove just (ii).

Let $P$ be a projective resolution of $k$ with kH–modules, let $(V, \epsilon)$ be a projective resolution of $k$ with kG–modules, and let

$$\theta : P \longrightarrow P \otimes_k P,\ \varphi : V \longrightarrow V \otimes_k V$$

be chain maps extending the identity map on $k$ (cf. 3.1).

Define $$\tau : P^\ell \otimes_k P^\ell \otimes_k V \otimes_k V \longrightarrow P^\ell \otimes_k V \otimes_k P^\ell \otimes_k V$$

by $(\bar{p} \otimes \bar{q} \otimes u \otimes v)\tau = \bar{p} \otimes u \otimes \bar{q} \otimes v\, (-1)^{\deg \bar{q} \deg u}$ where $\bar{p}, \bar{q} \in P^\ell$, $u, v \in V$ and $\bar{q}, u$ are homogeneous. Then $\tau$ is a G–map which is a chain map extending the identity. Now use (4.8) to define a chain map $\pi : (P \otimes_k P)^\ell \longrightarrow P^\ell \otimes_k P^\ell$ extending the identity by

$$(p_1 \otimes q_1 \otimes \ldots \otimes p_\ell \otimes q_\ell)\pi = p_1 \otimes \ldots \otimes p_\ell \otimes q_1 \otimes \ldots \otimes q_\ell\, \chi$$



where $\chi$ is a sign. Let $\hat{\pi} \in \Sigma_{2\ell}$ be the permutation corresponding to $\pi$. Then $\hat{\pi}$ can be written as the product of $\ell(\ell-1)/2$ transpositions of the form $(n\ \ n+1)$, each interchanging a $p_i$ and a $q_j$. So if all the $p_i$ have the same degree, and all the $q_j$ have the same degree, then

$$\chi = (-1)^{\deg p_1 \deg q_1\, \ell(\ell-1)/2}$$

by (4.8).

Finally we claim that $\pi$ is a G–map. By embedding $G$ in $\Sigma_\ell \wr H$ as in Lemma 4.1, this amounts to showing that $\pi$ commutes with the action of $\Sigma_\ell$. This is a consequence of the following Lemma, whose proof we omit.

Lemma Let $\sigma \in \Sigma_\ell$ and define $\alpha, \beta \in \Sigma_{2\ell}$ by

$$\left.\begin{aligned} \alpha(2i-1) &= 2\sigma i - 1,\ \alpha(2i) = 2\sigma i \\ \beta i &= \sigma i,\ \beta(i+\ell) = \sigma i + \ell. \end{aligned}\right\} \quad (1 \leq i \leq \ell)$$

If $\pi \in \Sigma_{2\ell}$ is defined by

$$\pi(2i-1) = i,\ \pi(2i) = i + \ell \qquad (1 \leq i \leq \ell)$$

then $\pi\alpha = \beta\pi$.

Let $r = \deg u$, $s = \deg v$, and let $f \in \mathrm{Hom}_{kH}(P_r, k)$, $g \in \mathrm{Hom}_{kH}(P_s, k)$ represent $u, v$ respectively. Then

$$\theta(f \otimes g) \in \mathrm{Hom}_{kH}(P_{r+s}, k)$$

represents $u\, v \in H^{r+s}(H, k)$,

$$\theta(f \otimes g) \otimes \ldots \otimes \theta(f \otimes g) \otimes \epsilon = (\theta^\ell \otimes \varphi)^*(f \otimes g \otimes \ldots \otimes f \otimes g \otimes \epsilon \otimes \epsilon) \in \mathrm{Hom}_{kG}((P^\ell \otimes_k V)_{\ell r+\ell s}, k_{r+s})$$

represents $\mathrm{norm}_{H,G}(u\, v) \in H^{\ell(r+s)}(G, k_{r+s})$,

$$f \otimes \ldots \otimes f \otimes \epsilon \in \mathrm{Hom}_{kG}((P^\ell \otimes_k V)_{\ell r}, k_r)$$

represents $\mathrm{norm}_{H,G}(u)$,

$$g \otimes \ldots \otimes g \otimes \epsilon \in \mathrm{Hom}_{kG}((P^\ell \otimes_k V)_{\ell s}, k_s)$$

represents $\mathrm{norm}_{H,G}(v)$, and

$$(\theta^\ell \otimes \varphi)^*(\pi \otimes id)^*\tau^*(f \otimes \dots \otimes f \otimes \epsilon \otimes g \otimes \dots \otimes g \otimes \epsilon) \in \mathrm{Hom}_{kG}((P^\ell \otimes_k V)_{\ell r+\ell s}, k_{r+s})$$

represents $\mathrm{norm}_{H,G}(uv) \in H^{\ell(r+s)}(G, k_{r+s})$.

Therefore $\mathrm{norm}_{H,G}(u\ v) = \mathrm{norm}_{H,G}(u)\ \mathrm{norm}_{H,G}(v)$ unless both r and s are odd, in which case they differ by a sign $(-1)^{\ell(\ell-1)/2}$.

4.12 Change of coset representatives Let $H \le G$, let $r \in \mathbb{N}$, and suppose

$$G = x_1H \uplus \dots \uplus x_\ell H = y_1H \uplus \dots \uplus y_\ell H .$$

Define

$$N_1 : H^r(H, k) \longrightarrow H^{r\ell}(G, k_r) \text{ to be } \mathrm{norm}_{H,G} \text{ with respect to } \{x_1, \dots, x_\ell\}$$

$$N_2 : H^r(H, k) \longrightarrow H^{r\ell}(G, k_r) \text{ to be } \mathrm{norm}_{H,G} \text{ with respect to } \{y_1, \dots, y_\ell\} .$$

Then for $u \in H^r(H, k)$,

$$N_1(u) = N_2(u)\sigma$$

where $\sigma = 1$ if r is even, and $\sigma =$ sign of the permutation $x_iH \longmapsto y_iH$ on the left cosets of H in G.

Proof Let

$P : \dots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow k \longrightarrow 0$ be a projective resolution with kH–modules

$V : \dots \longrightarrow V_1 \longrightarrow V_0 \xrightarrow{\epsilon} k \longrightarrow 0$ be a projective resolution with kH–modules.

Let Q(1) denote $P^\ell$ with kG–module structure with respect to $\{x_1, \dots, x_\ell\}$, let Q(2) denote $P^\ell$ with kG–module with respect to $\{y_1, \dots, y_\ell\}$, and let $f \in \mathrm{Hom}_{kH}(P_r, k)$ represent u. Then

$$f \otimes \dots \otimes f \otimes \epsilon \in \mathrm{Hom}_{kG}(Q(i) \otimes_k V, k_r)$$

represents $N_i(u)$ (i = 1, 2). Using (4.10) there is a chain isomorphism $\psi : Q(1) \longrightarrow Q(2)$ extending the identity: in the notation of (4.10) $q\psi = q\,w$ where $w \in \Sigma_\ell \wr H$ and $\mathrm{sign}(w) =$ sign of the permutation $x_iH \longmapsto y_iH$. Clearly $\psi(f \otimes \dots \otimes f) = (f \otimes \dots \otimes f)\sigma$ (use 4.10) and the result follows.



Remark If $v \in H(H)$, then similarly in the above $N_1(v) = N_2(v)$.

Lemma 4.13 Let $\theta : G \longrightarrow H$ be a group homomorphism, let $B \le H$, let $A = B\theta^{-1}$, and let $u \in H^*(B, k)$. Suppose u is homogeneous or $u \in H(B)$, and $G : A = H : B$. Then

$$\mathrm{norm}_{A,G}(u\,\theta^*) = (\mathrm{norm}_{B,H} u)\theta^* .$$

Note Write $G = x_1A \uplus \dots \uplus x_\ell A$. Then the hypothesis implies $H = (x_1\theta)B \uplus \dots \uplus (x_\ell\theta)B$ and we have calculated $\mathrm{norm}_{A,G}$ with respect to $\{x_1, \dots, x_\ell\}$, and $\mathrm{norm}_{B,H}$ with respect to $\{x_1\theta, \dots, x_\ell\theta\}$.

Proof Let P be a projective resolution of k with kB–modules, and let $(V, \epsilon)$ be a projective resolution of k with kH–modules. We shall just consider the case u is homogeneous, so let $u \in H^r(B, k)$ and let $f \in \mathrm{Hom}_{kH}(P_r, k)$ represent u. Then $f \in \mathrm{Hom}_{kA}(P_r, k)$ represents $u\,\theta^*$, where P is regarded as a kA–module via $q\,a = q(a\,\theta)$ for $q \in P$ and $a \in A$. Also $f \otimes \dots \otimes f \otimes \epsilon \in \mathrm{Hom}_{kH}(P_r^\ell \otimes_k V, k_r)$ represents $\mathrm{norm}_{B,H} u$, and $f \otimes \dots \otimes f \otimes \epsilon \in \mathrm{Hom}_{kG}(P_r^\ell \otimes_k V, k_r)$ represents $\mathrm{norm}_{A,G}\ u\,\theta^*$. Regard the kH–module $P_r^\ell \otimes_k V$ as a kG–module via $y\,g = y(g\,\theta)$ for $y \in P_r^\ell \otimes_k V$ and $g \in G$. Then $f \otimes \dots \otimes f \otimes \epsilon \in \mathrm{Hom}_{kG}(P_r^\ell \otimes_k V, k_r)$ represents $(\mathrm{norm}_{B,H} u)\theta^*$ with respect to this new kG–module structure on $P_r^\ell \otimes_k V$. Since the two kG–module structures on $P_r^\ell \otimes_k V$ agree, we deduce $\mathrm{norm}_{A,G}(u\,\theta^*) = (\mathrm{norm}_{B,H}\ u)\theta^*$ as required.

Combining 4.12 and 4.13 we obtain

Corollary 4.14 Let $H \triangleleft G$, let $\theta$ be an automorphism of G such that $H\theta = H$, and let $u \in H^*(H, k)$.

(i) If $u \in H(H)$, then $\mathrm{norm}_{H,G}(u\,\theta^*) = (\mathrm{norm}_{H,G}\ u)\theta^*$.

(ii) If $u \in H^r(H, k)$, then $\mathrm{norm}_{H,G}(u\,\theta^*) = (\mathrm{norm}_{H,G}\, u)\theta^*\sigma$ where $\sigma = 1$ if $r$ is even and $\sigma$ = sign of the permutation of $\theta$ on the cosets of $H$ in $G$ if $r$ is odd.

Note: we use the <u>same</u> set of coset representatives of $H$ in $G$ to calculate $\mathrm{norm}_{H,G}\, u$ and $\mathrm{norm}_{H,G}\, u\,\theta^*$.

<u>Mackey Decomposition</u> Let $A, B \leq G$, let $M$ be a $kA$–module, and let $x \in G$. We define $A^x = x^{-1}A\,x$ and $M^x$ to be the $kA^x$–module by $M = M^x$ as $k$–modules and action $m\,a^x = m\,a$ where $a^x = x^{-1}\,a\,x$ (so $M^x \cong M \otimes x$). If $N$ is a $kG$–module, then $N{\downarrow_B}$ denotes the $kB$–module obtained by restricting the action to $B$. Then

$$M \otimes_{kA} kG \cong \bigoplus_{A\times B} M^x{\downarrow_{A^x \cap B}} \otimes_{k[A^x \cap B]} kB$$

where $\bigoplus_{A\times B}$ means the sum is over a set of $(A - B)$ double coset representatives (in the following $\sum_{A\times B}$ and $\prod_{A\times B}$ will likewise mean the sum and product over a set of $(A - B)$ double coset representatives). There are similar formulae involving res, tr and norm.

We have a homomorphism $i_x : A^x \longrightarrow A$ defined by $c\, i_x = x\,c\,x^{-1}$ $(c \in A^x)$, hence a homomorphism $i_x^* : H^*(A, k) \longrightarrow H^*(A^x, k)$. For $u \in H^*(A, k)$, we define $u^x = i_x^*(u)$.

<u>Lemma 4.15</u>

(i) $\mathrm{res}_{G,B}\, \mathrm{tr}_{A,G}(u) = \sum_{A\times B} \mathrm{tr}_{A^x\cap B,B}(\mathrm{res}_{A^x,A^x\cap B}\, u^x)$.

(ii) Suppose $u$ is homogeneous or $u \in H(A, k)$. Then

$$\mathrm{res}_{G,B}\, \mathrm{norm}_{A,G}(u) = \prod_{A\times B} \mathrm{norm}_{A^x\cap B,B}(\mathrm{res}_{A^x,A^x\cap B}\, u^x).$$

<u>Remarks</u> If $k', k''$ are $kB$–modules, then we have a well defined cup product

$$H^i(B, k') \otimes_k H^j(B, k'') \longrightarrow H^{i+j}(B, k' \otimes_k k'')$$

where if $f \in \mathrm{Hom}_{kB}(P_i, k')$, $g \in \mathrm{Hom}_{kB}(P_j, k'')$ represent $u, v$, then $f \otimes g \in \mathrm{Hom}_{kB}(P_{i+j}, k' \otimes_k k'')$ represents $uv$. This applies when $u \in H^r(A, k)$ in (ii), with $k' = k'' = k_r$. Also when calculating $\mathrm{norm}_{A,G}$ and $\mathrm{norm}_{A^x\cap B,B}$ we must choose the coset representative "consistently", otherwise (ii) in the case $u$ is homogeneous will be correct only up to sign (cf. 4.12); a consistent choice of coset representatives will appear in the proof.



<u>Proof</u> Let $P$ be a projective resolution of $k$ with $kG$–modules, and let $f \in \mathrm{Hom}_{kA}(P, k)$ represent $u$. If $x \in G$, the map $q \longmapsto q\,x^{-1}$ $(q \in P)$ is a $kA^x$–module homomorphism from $P{\downarrow_{A^x}}$ to $P{\downarrow_A}$ regarded as a $kA^x$–module via $i_x$. Clearly this is a chain map extending the identity on $k$, so $x^{-1}f \in \mathrm{Hom}_{kA^x}(P, k)$ represents $u^x$, and $x^{-1}f = x^{-1}f\,x$ because $x$ acts trivially on $k$. Write

$$G = A\,x_1\,B \cup \ldots \cup A\,x_r\,B$$

$$B = (A^{x_i} \cap B)y_{i1} \cup \ldots \cup (A^{x_i} \cap B)\,y_{in_i} \quad (i = 1, \ldots, r).$$

Then $G = \bigcup_{i,j} A\,x_i\,y_{ij}$.

(i) $\mathrm{tr}_{A,G}(u)$ is represented by $\sum_{i=1}^{r}(\sum_{j=1}^{n_i} y_{ij}^{-1}(x_i^{-1} f\, x_i)y_{ij}$ and $\mathrm{tr}_{C\cap B,B}(\mathrm{res}_{C,C\cap B}\, u^{x_i})$ is represented by $\sum_{j=1}^{n_i} y_{ij}^{-1}(x_i^{-1}\, f\, x_i)y_{ij}$ where $C = A^{x_i}$.

(ii) We will just do the case $u$ is homogeneous. Let $(V, v)$ be a projective resolution of $k$ with $kG$–modules and let $t = G : A$. Suppose $u \in H^s(A, k)$ and $f \in \mathrm{Hom}_{kA}(P_s, k)$ represents $u$. Since $(V^r, v^r)$ is a projective resolution of $k^r \cong k$ with $kG$–modules, $f^t \otimes v^r \in \mathrm{Hom}_{kG}(P^t \otimes_k V^r)_{st}, k_s)$ represents $\mathrm{norm}_{A,G}(u)$, hence so does

$$(f^{n_1} \otimes v) \otimes \ldots \otimes (f^{n_r} \otimes v) \in \mathrm{Hom}_{kG}((P^{n_1} \otimes_k W) \otimes_k \cdots \otimes_k (P^{n_r} \otimes_k W), k_s).$$

We calculate $\mathrm{norm}_{A,G}$ with respect to the right transversal

$$\{x_1\,y_{11}, \ldots, x_1\,y_{1n_1}\,; \ldots ; x_r\,y_{rn_r}, \ldots, x_r\,y_{rn_r}\}.$$

We need to show $f^{n_i} \otimes v \in \mathrm{Hom}_{kB}(P^{n_i} \otimes_k W, k_s)$ represents $\mathrm{norm}_{C\cap B,B}(\mathrm{res}_{C,C\cap B}\, u^{x_i})$ where

$C = A^{x_i}$. By a similar argument to the first paragraph, $f \in \mathrm{Hom}_{kC}(P^{x_i}, k)$ represents $u^{x_i} \in H^s(A^{x_i}, k)$ and the result follows.

<u>Consequences of Mackey decomposition</u>

<u>Proposition 4.16</u> Let $A \triangleleft G$, let $x_1,\dots, x_n$ be a transversal for $A$ in $G$, and let $u \in H(A)$ or homogeneous in $H^*(A, k)$. Then $\mathrm{res}_{G,A}\, \mathrm{norm}_{A,G}\, u = \prod_{i=1}^{n} u^{x_i}$. In particular if the $x_i$ centralize $A$ (i.e. $a x_i = x_i a$ for all $a \in A$ and $i$), then $\mathrm{res}_{G,A}\, \mathrm{norm}_{A,G}\, u = u^n$.

We shall use the notation $N_G(A)$ for the normalizer of $A$ in $G$.

<u>Proposition 4.17</u> Let $A \leq G$ with $|A| = p$, let $r = N_G(A) : A$, and let $0 \neq u \in H^2(A, k)$. Then $H^{2r}(G, k) \neq 0$.

<u>Proof</u> Lemma 4.15 (ii) yields

$$\mathrm{res}_{G,A}\, \mathrm{norm}_{A,G}(1 + u) = \prod_{AxA} \mathrm{norm}_{A\cap A^x,A}\, \mathrm{res}_{A^x,A\cap A^x}(1 + u)^x$$

Since

$$\mathrm{norm}_{A\cap A^x,A}\, \mathrm{res}_{A^x,A\cap A^x}(1 + u)^x = 1 \text{ if } A \cap A^x = 1 \text{ (use Lemma 4.11 (i))},$$

$$= 1 + u \text{ if } A \cap A^x = A,$$

we see that

$$\mathrm{res}_{G,A}\, \mathrm{norm}_{A,G}(1 + u) = (1 + u)^r = 1 + u^r + \text{terms of intermediate degree.}$$

Thus if $v$ is the homogeneous part of $\mathrm{norm}_{A,G}(1 + u)$ of degree $2r$, $\mathrm{res}_{G,A} v = u^r \neq 0$, in particular $H^{2r}(G, k) \neq 0$.

For the rest of § 4, the following notation will be in force: $C = \mathbb{Z}/p\mathbb{Z}$ (the cyclic group of order $p$), $C = \langle c \rangle$, $k = \mathbb{Z}/p\mathbb{Z}$, $N = \mathrm{norm}_{C\times G}$, and we shall calculate $N$ with respect to the coset representatives $\{1, c,\dots, c^{p-1}\}$. Note in this situation $k_r \cong k$ for all $r \in \mathbb{N}$. Also to construct



$N$, we may assume that $W$ is a projective resolution with $kC$–modules and then let $G$ act trivially on $W$ (use Lemma 1.11). The next result is like the formula $(x + y)^p = x^p + y^p$ in a commutative ring of characteristic $p$.

<u>Lemma 4.18</u> If $u, v \in H(G)$ or $H^r(G, k)$ for some $r \in \mathbb{N}$, then $N(u + v) = N(u) + N(v)$.

<u>Proof</u> Let

$P : \dots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow k \longrightarrow 0$ be a projective resolution with $kG$–modules

$W : \dots \longrightarrow W_1 \longrightarrow W_0 \longrightarrow k \xrightarrow{\epsilon} 0$ be a projective resolution with $kC$–modules.

Let $\theta, \varphi \in \mathrm{Hom}_{kG}(P, k)$ represent $u, v$ respectively. Then $N(u + v) - N(u) - N(v)$ is represented by $(\theta + \varphi)^p \otimes \epsilon - \theta^p \otimes \epsilon - \varphi^p \otimes \epsilon \in \mathrm{Hom}_{k[C\times G]}(P^p \otimes_k W, k)$. This is a sum of elements of the form

$$\psi = \psi_1 \otimes \dots \otimes \psi_p \otimes \epsilon + \psi_2 \otimes \dots \otimes \psi_p \otimes \psi_1 \otimes \epsilon + \dots + \psi_p \otimes \psi_1 \otimes \dots \otimes \psi_{p-1} \otimes \epsilon$$

where $\psi_i = \theta$ or $\varphi$ ($i = 1,\dots, p$). Since $\delta\theta = \delta\varphi = \delta\epsilon = 0$ (where $\delta$ is the coboundary map), $\delta(\psi_1 \otimes \dots \otimes \psi_p \otimes \epsilon) = 0$ so $\psi_1 \otimes \dots \otimes \psi_p \otimes \epsilon$ represents an element $x \in H(C \times G)$ or $H^{pr}(C \times G, k)$. Let $\gamma : P^p \otimes_k W \longrightarrow P^p \otimes_k W$ denote "multiplication by c" (i.e. $(p_1 \otimes \dots \otimes p_p \otimes w)\gamma = (p_1 \otimes \dots \otimes p_p)c \otimes w$). Then $\gamma$ is a $k[C \times G]$–map extending the identity (because $c$ is central in $C \times G$), so $\gamma \circ (\psi_1 \otimes \dots \otimes \psi_p \otimes \epsilon)$ also represents $x \in H(C \times G)$ or $H^{pr}(C \times G, k)$. But $\gamma \circ (\psi_1 \otimes \dots \otimes \psi_p \otimes \epsilon) = \psi_2 \otimes \psi_3 \otimes \dots \otimes \psi_p \otimes \psi_1 \otimes \epsilon$, hence $\psi_2 \otimes \psi_3 \otimes \dots \otimes \psi_p \otimes \psi_1$ represents $x \in H(C \times G)$ or $H^{pr}(C \times G, k)$ and we deduce that $\psi$ represents $p\,x = 0$. Therefore $N(u + v) - N(u) - N(v) = 0$ and the result follows.

<u>Lemma 4.19</u> Let $u \in H^*(G, k)$ be homogeneous. If $p \neq 2$, then $\beta N(u) = 0$.

<u>Proof</u> Let $P$ be a projective resolution of $\mathbb{Z}$ with $\mathbb{Z}G$–modules, and let $(W, \epsilon)$ be a projective resolution of $\mathbb{Z}$ with $\mathbb{Z}C$–modules. Let $f \in \mathrm{Hom}_{\mathbb{Z}G}(P_r, k)$ represent $u$ where $r = \deg u$. Then $N(u)$ is represented by

$$f \otimes \dots \otimes f \otimes \epsilon \in \mathrm{Hom}_{\mathbb{Z}[C\times G]}((P^p \otimes_{\mathbb{Z}} W)_{pr}, k).$$

Lift $f$ to a $\mathbb{Z}G$–map $h : P_r \longrightarrow \mathbb{Z}/p^2\mathbb{Z}$, and $\epsilon$ to a $\mathbb{Z}C$–map $\nu : W_0 \longrightarrow \mathbb{Z}/p^2\mathbb{Z}$. Then

$$h \otimes \ldots \otimes h \otimes v \in \mathrm{Hom}_{\mathbb{Z}[C\times G]}((P^p \otimes_{\mathbb{Z}} W)_{pr}, \mathbb{Z}/p^2\mathbb{Z})$$

lifts $f \otimes \ldots \otimes f \otimes \epsilon$ (note we have used $p \neq 2$ here : if $p = 2$, then $h \otimes h$ commutes with the action of $c$ only up to sign). Let $\gamma : P^p \otimes_{\mathbb{Z}} W \longrightarrow P^p \otimes_{\mathbb{Z}} W$ denote "multiplication by c" and let $\partial$ denote the boundary map (on $P$ or $P^p \otimes_{\mathbb{Z}} W$). Then $(\partial \circ h) \otimes h \otimes \ldots \otimes h \otimes v$ represents an element $x \in H^{pr+1}(C \times G, k)$, $\partial \circ (h \otimes h \otimes \ldots \otimes h \otimes v)$ represents $\beta u \in H^{pr+1}(C \times G, k)$, and

$$\partial \circ (h \otimes \ldots \otimes h \otimes v) = (1 + \gamma + \ldots + \gamma^{p-1}) \circ ((\partial \circ h) \otimes \ldots \otimes h \otimes v))$$

(where care is needed over the sign when $r$ is odd).

As in the proof of Lemma 4.18, $\gamma \circ ((\partial \circ h) \otimes \ldots \otimes h \otimes v))$ also represents $x$, hence $\beta u = p x = 0$ as required.

<u>4.20 Remarks</u> If $u \in H(G)$ and $p \neq 2$, then $\beta N(u) = 0$. When $p = 2$, let $\beta'$ be the Bockstein (i.e. connecting homomorphism – see Corollary 1.13) associated to

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

where the action of $c$ on $\mathbb{Z}/4\mathbb{Z}$ is multiplication by $-1$. Thus $G$ acts trivially on $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/4\mathbb{Z}$, $c$ acts trivially on $\mathbb{Z}/2\mathbb{Z}$, and we have a long exact sequence

$$\ldots \longrightarrow H^n(C \times G, \mathbb{Z}/4\mathbb{Z}) \longrightarrow H^n(C \times G, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\beta'} H^{n+1}(C \times G, \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^{n+1}(C \times G, \mathbb{Z}/4\mathbb{Z}) \longrightarrow \ldots .$$

As with the ordinary Bockstein map, we use Remark 1.19 (iii) to define $\beta' : H^n(C \times G, k) \longrightarrow H^n(C \times G, k)$ for an arbitrary field $k$ of characteristic two. Then $\beta' N(u) = 0$ if $u \in H^*(G, k)$ is homogeneous of odd degree, while $\beta N(u) = 0$ if $u \in H(G)$ by a similar argument to that of Lemma 4.19. Also $\beta' : H^{2n}(C, k) \longrightarrow H^{2n+1}(C, k)$ is an isomorphism and $\beta' : H^{2n+1}(C, k) \longrightarrow H^{2n+2}(C, k)$ is the zero map $\forall\, n \in \mathbb{N}$; this can be seen by using induction on $n$ and the long exact sequence of Corollary 1.13 (i).

Recall from Proposition 1.20 that $H^1(C, k) \cong \mathrm{Hom}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$ naturally, so let $w \in H^1(C, k)$ correspond to the identity endomorphism of $\mathbb{Z}/p\mathbb{Z}$. For $\ell \in \mathbb{N}$ define

$$w_{2\ell} = (\beta w)^{\ell},\ w_{2\ell+1} = (\beta w)^{\ell} w .$$



(Thus if $p = 2$, $w_{\ell} = w^{\ell}$ by 3.6; also $\beta' w_{2\ell} = w_{2\ell+1}$, $\beta' w_{2\ell+1} = 0$). Let

$$\ldots \longrightarrow v_2 kC \longrightarrow v_1 kC \longrightarrow v_0 kC \longrightarrow k \longrightarrow 0$$

be a free resolution such that for $\ell \in \mathbb{N}$

$$v_0 \longmapsto 1,\ v_{2\ell+1} \longmapsto v_{2\ell}(c-1),\ v_{2\ell+2} \longmapsto v_{2\ell+1}(1 + c + \ldots + c^{p-1}) .$$

For $i \in \mathbb{N}$, let $x_i \in H^i(C, k)$ be represented by $f_i \in \mathrm{Hom}_{kC}(v_i kC, k)$ defined by $v_i f_i = 1$. Then we have

<u>Lemma</u> $w_i = x_i$ for all $i \in \mathbb{N}$.

<u>Proof</u> We shall use the notation of 3.6, so we have exact sequences

$$0 \longrightarrow \mathfrak{g} \longrightarrow kC \xrightarrow{\epsilon} k \longrightarrow 0$$

$$0 \longrightarrow k \longrightarrow kC \xrightarrow{v} \mathfrak{g} \longrightarrow 0$$

where $1\epsilon = 1$ and $1v = g - 1$. Also $\gamma : H^n(C, k) \longrightarrow H^{n+1}(C, \mathfrak{g})$ and $\delta : H^n(C, \mathfrak{g}) \longrightarrow H^{n+1}(G, k)$ are the corresponding connecting homomorphisms. For $i \in \mathbb{N}$, let $y_i \in H^i(G, k)$ be represented by the element $h_i \in \mathrm{Hom}_{kC}(v_i kC, \mathfrak{g})$ defined by $v_i h_i = g - 1$. Then by definition of $\gamma$ and $\delta$ (see Lemma 1.12), a straightforward calculation shows that $\gamma x_i = y_{i+1}$ and $\delta y_i = x_{i+1}$, hence $\delta \gamma x_i = x_{i+2}$ for all $i \in \mathbb{N}$. Also $x_0 = 1$ and the description of the Bockstein map given in 1.22 shows that $x_2 = w_2$. Therefore for $i \in \mathbb{N}$,

$$\begin{aligned} w_{i+2} = x_2 w_i &= (\delta \gamma x_0) w_i \\ &= \delta \gamma (x_0 w_i) \quad \text{by Lemma 3.5 (iii)} \\ &= \delta \gamma w_i . \end{aligned}$$

Since $w_0 = x_0$ and $w_1 = x_1$, an easy induction argument completes the proof.

By the Künneth formula (Theorem 3.4)

$$H^*(C \times G, k) \cong H^*(C, k) \otimes_k H^*(G, k) ,$$

so for $q \in \mathbb{N}$ and $u \in H^q(G, k)$ we can write $N(u) = \Sigma\, w_{\ell} \otimes D_{\ell} u$ for some maps $D_{\ell} : H^q(G, k) \longrightarrow H^{pq-\ell}(G, k)$. The Steenrod operations are closely related to these maps $D_{\ell}$. First we obtain some properties of the $D_\ell$'s .

Lemma 4.21 If $r, \ell \in \mathbb{N}$ and $u, v \in H(G)$ or $H^r(G)$, then $D_\ell(u+v) = D_\ell(u) + D_\ell(v)$.

Proof We have

$$\begin{aligned} \Sigma\, w_\ell \otimes D_\ell(u+v) &= N(u+v) \\ &= N(u) + N(v) \text{ by Lemma 4.18}, \\ &= \Sigma\, w_\ell \otimes (D_\ell u + D_\ell v) \end{aligned}$$

and the result follows by comparing the coefficient of $w_\ell$.

Lemma 4.22 Let $\ell, r, s \in \mathbb{N}$, let $u \in H^r(G,k)$ and let $v \in H^s(G,k)$.
If $p = 2$ then

$$D_\ell(u\,v) = \sum_{i+j=\ell} D_i\, u\, D_j\, v\,,$$

while if $p > 2$ and $\epsilon = (p-1)r\,s/2$, then

$$D_{2\ell}(u\,v) = (-1)^\epsilon \sum_{i+j=\ell} D_{2i}\, u\, D_{2j}\, v\,.$$

Proof We will assume that $p > 2$, since the proof for the case $p = 2$ is very similar. Then

$$\begin{aligned} \Sigma\, w_\ell \otimes D_\ell(u\,v) &= N(u\,v) \\ &= (-1)^\epsilon\, N\, u\, N\, v \text{ by Lemma 4.11 (ii)} \\ &= (-1)^\epsilon \sum_{i,j} (w_i \otimes D_i\, u)\, (w_j \otimes D_j\, v)\,. \end{aligned}$$

By definition of the $w_i$ and 3.6, if i and j are odd then $w_i\, w_j = 0$, while if i or j is even, then $w_i\, w_j = w_{i+j}$. The result follows by taking the coefficient of $w_{2\ell}$.

Lemma 4.23 Let $\ell \in \mathbb{N}$ and let $u \in H^*(G,k)$.

(i) Suppose p is odd and u is homogeneous. Then

$$\beta\, D_{2\ell+2}\, u = -D_{2\ell+1}\, u\,,\ \beta\, D_{2\ell+1}\, u = 0\,,\ \beta\, D_0\, u = 0\,.$$



(ii) Suppose $p = 2$. Then $\beta\, D_{2\ell+1}\, u = D_{2\ell}\, u$, $\beta\, D_{2\ell}\, u = 0$ if u is homogeneous of odd degree, while $\beta\, D_{2\ell+2}\, u = -D_{2\ell+1}\, u$, $\beta\, D_{2\ell+1}\, u = 0$, $\beta\, D_0\, u = 0$ if $u \in H(G)$.

Proof (i) Since $\beta\, N u = 0$ by Lemma 4.19, application of Lemma 3.5 (v) yields

$$0 = \beta \sum_{\ell \in \mathbb{N}} w_\ell \otimes D_\ell\, u = \sum_{\ell \in \mathbb{N}} (\beta\, w_\ell \otimes D_\ell\, u + (-1)^\ell\, w_\ell \otimes \beta\, D_\ell\, u)\,.$$

Equating the coefficients of $w_{\ell+1}$ shows that $\beta\, D_0\, u = 0$ and

$$\beta\, w_\ell \otimes D_\ell\, u + (-1)^{\ell+1}\, w_{\ell+1} \otimes \beta\, D_{\ell+1}\, u = 0 \quad \forall\, \ell \in \mathbb{N}\,.$$

But $\beta\, w_{2\ell} = 0$, $\beta\, w_{2\ell+1} = w_{2\ell+2}$ $\forall\, \ell \in \mathbb{N}$ by Lemma 3.5 (v) again and the result follows.

(ii) If u has even degree then the proof proceeds exactly as in (i), so assume that u has odd degree. The proof of Lemma 3.5 (v) shows that

$$\beta'(x\,y) = \beta'(x)y + x\,\beta'(y)$$

$\forall$ homogeneous $x, y \in H^*(C \times G, k)$. Using $\beta'\, N(u) = 0$ (see 4.20)

$$\begin{aligned} 0 = \beta' \sum_{\ell \in \mathbb{N}} w_\ell \otimes D_\ell\, u &= \sum_{\ell \in \mathbb{N}} (\beta'\, w_\ell \otimes D_\ell\, u + w_\ell \otimes \beta'\, D_\ell\, u) \\ &= \sum_{\ell \in \mathbb{N}} (\beta'\, w_\ell \otimes D_\ell\, u + w_\ell \otimes \beta\, D_\ell\, u) \end{aligned}$$

because $\beta'(1 \otimes v) = \beta(1 \otimes v)$ for $v \in H^*(G,k)$, so equating coefficients of $w_{\ell+1}$ yields $\beta\, D_0\, u = 0$ and

$$\beta'\, w_\ell \otimes D_\ell\, u + w_{\ell+1} \otimes \beta\, D_\ell\, u = 0 \quad \forall\, \ell \in \mathbb{N}\,.$$

But $\beta'\, w_{2\ell} = w_{2\ell+1}$ and $\beta'\, w_{2\ell+1} = 0$ (see 4.20) from which the result follows.

Lemma 4.24 If $r \in \mathbb{N}$ and $u \in H^r(G,k)$, then $D_0\, u = u^p$.

Proof Since $\text{res}_{C\times G,G}\, N\, u = \sum_{\ell \in \mathbb{N}} \text{res}_{C\times G}\, w_\ell \otimes D_\ell\, u$, we see that $\text{res}_{C\times G,G}\, N\, u = D_0\, u$. The result follows from Proposition 4.16.

Lemma 4.25 Let $r, \ell \in \mathbb{N}$ and let $u \in H^r(G, k)$. Then

(i) If $r$ is even, $D_\ell u = 0$ unless $\ell = 2m(p-1)$ or $2m(p-1)-1$ for some $m \in \mathbb{N}$.

(ii) If $r$ is odd, $D_\ell u = 0$ unless $\ell = (2m+1)(p-1)$ or $(2m+1)(p-1)-1$ for some $m \in \mathbb{N}$.

Proof The lemma is vacuous if $p = 2$, so we may assume that $p > 2$. Let $A$ be the subgroup of index two in Aut C and let $\alpha \in$ Aut C. Then $\alpha$ is an even permutation on C if and only if $\alpha \in A$. Let $\alpha_1$ be the automorphism of $C \times G$ which is $\alpha$ on C and the identity on G. Then Corollary 4.14 (ii) shows that $(Nu)\alpha_1^*\sigma = Nu$ where $\sigma = 1$ if $r$ is even or $\alpha \in A$, and $\sigma = -1$ if $r$ is odd and $\alpha \notin A$.

Now Aut C induces automorphisms on $H^*(C, k)$ and we have

Aut C fixes $w_\ell \Leftrightarrow \ell = 2m(p-1)$ or $2m(p-1)-1$ for some $m \in \mathbb{N}$,

A fixes $w_\ell$ and Aut C does not $\Leftrightarrow \ell = (2m+1)(p-1)$ or $(2m+1)(p-1)-1$ for some $m \in \mathbb{N}$:

this can be seen using Proposition 1.20 and 3.6. Note that Aut C fixes $w_\ell$ means that $\alpha^* w_\ell = w_\ell$ $\forall \alpha \in$ Aut C, while A fixes $w_\ell$ and Aut C does not means that $\alpha^* w_\ell = \epsilon w_\ell$ where $\epsilon$ is the sign of the permutation $\alpha$ on C. The result now follows by using $(Nu)\alpha_1^*\sigma = Nu$ from above.

Lemma 4.26 Let $\theta : H \to G$ be a homomorphism, let $u \in H^*(H, k)$ be homogeneous and let $\ell \in \mathbb{N}$. Then $D_\ell(u\,\theta^*) = (D_\ell u)\theta^*$.

Proof Apply Lemma 4.13 with $G = C \times H$ and $H = C \times G$.



Lemma 4.27 Let $r \in \mathbb{P}$ and let $u \in H^r(G, k)$. Then

(i) $D_\ell u = 0$ if $\ell > (p-1)r$,

$D_{(p-1)r} u = a_r u$

where $a_r \in k$ and is independent of G and u.

(ii) The exact value of $a_r$ is

$$a_r = \begin{cases} \left(\left(\frac{p-1}{2}\right)!\right)^r (-1)^{(p-1)r(r+1)/4} & \text{if } p \neq 2, \\ 1 & \text{if } p = 2 \end{cases}$$

To establish this, we use the following topological theorem of [D.M. Kan and W.P. Thurston, "Every connected space has the homology of a $K(\pi, 1)$", Topology 15 (1976), 253–258].

Theorem 4.28 For every path connected space X, there exists a space TX and a map $t : TX \longrightarrow X$, natural for maps of X, such that

(i) $t^* : H^*(X, k) \longrightarrow H^*(TX, k)$ is an isomorphism.

(ii) $\pi_i(TX) = 0$ if $i \neq 1$, and $t_* : \pi_1(TX) \longrightarrow \pi_1(X)$ is onto.

A proof of this is given in [C.R.F Maunder, "A short proof of a theorem of Kan and Thurston", Bull. London Math. Soc. 13 (1981), 325–327].

Now let X be a K(G, 1), so X is a connected CW–complex with $\pi_1(X) = G$ and $\pi_i(X) = 0$ for $i > 1$, and let Y be the r skeleton of X. Thus $H^*(G, k) \cong H^*(X, k)$. If $H = \pi_1(TY)$, then Theorem 4.28 shows that $H^i(H, k) = 0$ for $i > r$, and there exists a homomorphism $\theta : H \longrightarrow G$ such that

$$\theta^* : H^i(G, k) \longrightarrow H^i(H, k)$$

is an isomorphism for $i < r$, and a monomorphism for $i = r$ (note that even if G is finite, H may be infinite.). Let $v \in H^r(TY, k)$ correspond to $u\theta^*$ and write $w = v(t^*)^{-1} \in H^r(Y, k)$.

Let $Y_1$ denote the $(r-1)$–skeleton of Y, let $\pi : Y \longrightarrow Y/Y_1$ denote the natural surjection, and let $\pi^* : H^r(Y/Y_1, k) \longrightarrow H^r(Y, k)$ denote the homomorphism induced by $\pi$. Then we have an exact sequence.

$$\dots \longrightarrow H^{r-1}(Y_1, k) \longrightarrow H^r(Y/Y_1, k) \longrightarrow H^r(Y, k) \longrightarrow 0$$

because $H^r(Y_1, k) = 0$, so we can choose $f \in H^r(Y/Y_1, k)$ such that $\pi^*(f) = w$. Let $\{e_\alpha \mid \alpha \in \mathscr{A}\}$ denote the r–cells of $Y/Y_1$, let $S^r$ denote an r–sphere with basepoint $b$, and for each $\alpha \in \mathscr{A}$ let $S^r_\alpha$ denote an r–sphere with base point $b_\alpha$. Since $H^r(Y/Y_1, k)$ can be identified with $\mathrm{Hom}(C_r(Y/Y_1), k)$ where $C_r$ denotes the $r^{th}$ cellular chain group, we can view $f$ as an element of $\mathrm{Hom}(C_r(Y/Y_1), k)$. Furthermore $C_r(Y/Y_1) = \bigoplus_{\alpha \in \mathscr{A}} (i_\alpha)_* C_r(S^r_\alpha)$ where

$$(i_\alpha)_* : C_r(S^r_\alpha) \longrightarrow C_r(Y/Y_1)$$

denotes the homomorphism induced by $i_\alpha$. For $\alpha \in \mathscr{A}$ let $z_\alpha$ be a generator for $C_r(S^r_\alpha) \cong \mathbb{Z}$, and let $z$ be a generator for $C_r(S^r)$. Also choose maps $v_\alpha : S^r_\alpha \longrightarrow S^r$ such that $v_\alpha(b_\alpha) = b$ and $(v_\alpha)_*(z_\alpha) = f((i_\alpha)_* z_\alpha)z$. Then the $v_\alpha$ induce a map $v : Y/Y_1 \longrightarrow S^r$ such that $v\, i_\alpha = v_\alpha$ (maps written on left). Define $x \in \mathrm{Hom}(C_r(S^r), k)$ by $x(z) = 1$, and

$$v^* : \mathrm{Hom}(C_r(S^r), k) \longrightarrow \mathrm{Hom}(C_r(Y/Y_1), k)$$

to be the map induced by $v$. Then

$$(v^*(x))\,((i_\alpha)_* z_\alpha) = x(v_*(i_{\alpha *} z_\alpha)) = x(v_{\alpha *} z_\alpha) = x(f(i_{\alpha *} z_\alpha)z) = f((i_\alpha)_* z_\alpha)$$

so $v^*(x) = f$, hence $(v\,\pi)^*(x) = w$. Since we can identify $\mathrm{Hom}(C_r(S^r), k)$ with $H^r(S^r, k)$, this means there exists $\varphi : Y \longrightarrow S^r$ such that $\varphi^*(x) = w$.

Write $F = \pi_1(T\,S^r)$. Then $H^i(F, k) = H^i(T\,S^r, k) = H^i(S^r, k)$. Also $\varphi$ yields by naturality a map $t\,\varphi : TY \longrightarrow T\,S^r$, hence it induces a map $\psi : H = \pi_1(TY) \longrightarrow \pi_1(T\,S^r) = F$. If $y \in H^r(F, k)$ corresponds to $t^*x \in H^r(T\,S^r, k)$, then $y\,\psi^*$ corresponds to

$$(t\,\varphi)^* t^* x = t^* \varphi^* x = t^* w = v$$



and we see that $y\,\psi^* = u\,\theta^*$. Using Lemma 4.26 we have a commutative diagram

$$\begin{array}{ccccc} H^r(G, k) & \xrightarrow{\theta^*} & H^r(H, k) & \xleftarrow{\psi^*} & H^r(F, k) \\ \downarrow D_\ell & & \downarrow D_\ell & & \downarrow D_\ell \\ H^{pr-\ell}(G, k) & \xrightarrow{\theta^*} & H^{pr-\ell}(H, k) & \xleftarrow{\psi^*} & H^{pr-\ell}(F, k) \end{array}$$

Since $H^i(F, k) = 0$ for $i \neq 0, r$ and $H^r(F, k) \cong k$, we see that $D_\ell\, y = 0$ when $\ell > (p-1)r$ and $D_{(p-1)r}y = a_r y$ for some $a_r \in k$; of course $a_r$ does not depend on $G$ or $u$. Examination of the commutative diagram now yields (i).

To prove (ii), we can choose $G$ to suit our needs best, so we begin with $G = \mathbb{Z}/p\mathbb{Z}$. If $r = 2$, then (i) and Lemma 4.25 (i) show that $D_\ell u = 0$ unless $\ell = 0$, $2(p-1)$ or $2(p-1)-1$. Since $\beta$ is zero on $H^2(G, k)$ by 3.6, we see that $D_{2(p-1)-1}\, u = 0$ by Lemma 4.23. Thus we can write

$$N(u) = w_0 \otimes u^p + a_2\, w_{2p-2} \otimes u\,.$$

Let $g$ be a generator for $G$ and identify $H^1(G, k)$ with $\mathrm{Hom}(G, k)$ (Proposition 1.20). Define $\hat{g} \in H^1(G, k)$ by $\hat{g}(g) = 1$ and let $u = \beta\,\hat{g}$. Using Corollary 4.14 with $H = G$, $G = C \times G$ and $\theta$ the automorphism of $C \times G$ which is the identity on $G$ and sends $(c, 1)$ to $(c, g)$, we deduce that $N(u)\theta^* = N(u)$. It is not difficult to see that $(w_0 \otimes u)\theta^* = w_0 \otimes u + w_2 \otimes 1$ and $(w_2 \otimes 1)\theta^* = w_2 \otimes 1$, so we have

$$\begin{aligned} w_0 \otimes u^p + a_2\, w_{2p-2} \otimes u &= (w_0 \otimes u + w_2 \otimes 1)^p + a_2\, w_{2p-2} \otimes u + a_2\, w_{2p} \otimes 1 \\ &= w_0 \otimes u^p + w_{2p} \otimes 1 + a_2\, w_{2p-2} \otimes u + a_2\, w_{2p} \otimes 1\,, \end{aligned}$$

hence $a_2 = -1$ and $N(u) = w_0 \otimes u^p - w_{2p-2} \otimes u$.

Lemma 4.11 now shows that for $s \in \mathbb{P}$,

$$\begin{aligned} N(u^s) &= (w_0 \otimes u^p - w_{2p-2} \otimes u)^s \\ &= (-1)^s\, w_{2s(p-1)} \otimes u^s + \text{terms of the form } w_{s'} \otimes u' \text{ where } s' < 2s(p-1) \qquad (1) \end{aligned}$$

and we conclude that $a_{2s} = (-1)^s$. From elementary number theory, $\left[\frac{p-1}{2}\right]!^2 = -(-1)^{(p-1)/2}$ ($p$ odd) and so (ii) is proven for even $r$.

Now let us suppose r is odd. If $r = 1$, then (i) and Lemma 4.25 (ii) show that

$$N(u) = \lambda\, w_{p-2} \otimes \beta u + a_1\, w_{p-1} \otimes u \qquad (2)$$

for some $\lambda \in k$.

Using (1) and Lemma 4.11, we see that for $s \in \mathbb{P}$,

$$N(u^{2s+1}) = a_1(-1)^s\, w_{(2s+1)(p-1)} \otimes u + \text{terms of the form } w_{s'} \otimes u'$$

where $s' < (2s+1)(p-1)$ and we deduce that $a_{2s+1} = a_1(-1)^s$.

Let us now choose G and $u_1, u_2 \in H^1(G, k)$ such that $u_1 u_2 \neq 0$. Then (2) and Lemma 4.11 (ii) show that $a_1^2 = a_2(-1)^{p(p-1)/2}$, and it follows that $a_1 = \pm \left[\frac{p-1}{2}\right]! \,(-1)^{(p-1)/2}$ for p odd (because $\left[\frac{p-1}{2}\right]!^2 = -(-1)^{(p-1)/2}$ and $a_2 = -1$) and $a_1 = 1$ for $p = 2$. The + sign yields the result. A proof that the + sign holds is given in VII § 5 of [Cohomology Operations by N.E. Steenrod, written by D.B.A Epstein, Annals of Math. Studies no. 50, Princeton Univ. Press 1962], and we assume this. Unfortunately there does not seem to be an easy way to establish this. Alternatively one could use a different set of coset representatives (i.e. $\{c, 1, c^2, \ldots, c^{p-1}\}$) if necessary when calculating N which in view of Lemma 4.12 would give the correct result.



<u>5. Steenrod Operations</u> In this section $k = \mathbb{Z}/p\mathbb{Z}$. For $i, r \in \mathbb{N}$ and $u \in H^r(G, k)$, define

$$Sq^i u = D_{r-i} u \qquad (p = 2)$$

$$P^i u = (-1)^{i+(p-1)r(r+1)/4} \left[\frac{p-1}{2}\right]!^{-r} D_{(r-2i)(p-1)} u \qquad (p \neq 2)$$

(where $D_j = 0$ for $j < 0$). The $Sq^i$ and $P^i$ are called the <u>Steenrod</u> operations. We use the results of section 4 to obtain

<u>Theorem 5.1</u>

(i) $Sq^i : H^r(G, k) \longrightarrow H^{r+i}(G, k)$ is a natural homomorphism.

(ii) $Sq^0 = 1$.

(iii) $Sq^r u = u^2$.

(iv) $Sq^i u = 0$ unless $0 \leq i \leq r$.

(v) $Sq^\ell(u\,v) = \sum_{i+j=\ell} Sq^i u\, Sq^j v$.

(vi) $Sq^{2i+1} = \beta\, Sq^{2i}$ and $Sq^1 = \beta$.

<u>Theorem 5.2</u>

(i) $P^i : H^r(G, k) \longrightarrow H^{r+2i(p-1)}(G, k)$ is a natural homomorphism.

(ii) $P^0 = 1$

(iii) If r is even, say $r = 2q$, then $P^q u = u^p$.

(iv) $P^i u = 0$ unless $0 \leq 2i \leq r$.

(v) $P^\ell(u\,v) = \sum_{i+j=\ell} P^i u\, P^j v$.

<u>Proof of Theorems 5.1 and 5.2</u> In both Theorems, use Lemmas 4.21 and 4.26 for (i), Lemma 4.27 for (ii), Lemma 4.24 for (iii), Lemma 4.27 (i) for (iv) and Lemma 4.22 for (v). Finally use Lemma 4.23 (ii) for Theorem 5.1 (vi).

The Steenrod operations also satisfy the Adem relations of Theorems 5.3 and 5.4 below. To state these theorems, we let [x] denote the greatest integer $\leq x$, and the binomial coefficients are taken modulo p.

Theorem 5.3 If $a, b \in \mathbb{P}$ and $a < 2b$, then

$$Sq^a Sq^b = \sum_{j=0}^{[a/2]} \binom{b-1-j}{a-2j} Sq^{a+b-j} Sq^j .$$

Theorem 5.4 Let $a, b \in \mathbb{N}$. If $a < p^b$, then

$$p^a p^b = \sum_{t=0}^{[a/p]} (-1)^{a+t} \binom{(p-1)(b-t)-1}{a-pt} p^{a+b-t} p^t .$$

If $a \leq b$, then

$$p^a \beta p^b = \sum_{t=0}^{[a/p]} (-1)^{a+t} \binom{(p-1)(b-t)}{a-pt} \beta p^{a+b-t} p^t$$

$$+ \sum_{t=0}^{[(a-1)/p]} (-1)^{a+t-1} \binom{(p-1)(b-t)-1}{a-pt-1} p^{a+b-t} \beta p^t .$$

The Adem relations are proved by obtaining further properties of the norm map:

Lemma 5.5 Let $H \leq E \leq G$ and write $E = \biguplus_{i=1}^{m} x_i H$, $G = \biguplus_{i=1}^{n} y_i E$. Suppose the kE–module $k_m$ (as defined in the Evens norm map) is isomorphic to k. If $r \in \mathbb{N}$ and $u \in H^r(G, k)$, then

$$\mathrm{norm}_{E,G}\, \mathrm{norm}_{H,E}\, u = \mathrm{norm}_{H,G}\, u ,$$

where we have calculated $\mathrm{norm}_{H,E}$, $\mathrm{norm}_{E,G}$ and $\mathrm{norm}_{H,G}$ with respect to $\{x_1,\dots, x_m\}$, $\{y_1,\dots, y_n\}$ and $\{y_1 x_1,\dots, y_1 x_m, y_2 x_1,\dots, y_n x_{m-1}, y_n x_m\}$ respectively.

We omit the easy poof.

Lemma 5.6 Let $\theta$ be the automorphism of $G \times G$ defined by $(h, g)\theta = (g, h)$, let $r, s \in \mathbb{N}$, and let $u \in H^r(G, k)$, $v \in H^s(G, k)$. Then by Theorem 3.4 we may view $u \otimes v \in H^{r+s}(G \times G, k)$ and we have $(u \otimes v)\theta^* = (-1)^{rs} v \otimes u$.



The proof of this is very similar to Lemma 3.2: we omit the details.

Now let $B = C = \mathbb{Z}/p\mathbb{Z}$, and define $v_i \in H^i(B, k)$, $w_i \in H^i(C, k)$ in the same way as the $w_i$ in Section 4. Let b and c be generators for B and C respectively. By the Künneth formula (Theorem 3.4)

$$H^*(B \times C \times G, k) \cong H^*(B, k) \otimes_k H^*(C, k) \otimes_k H^*(G, k),$$

so for $q \in \mathbb{N}$ and $u \in H^q(G, k)$, we can imitate Section 4 and write

$$\mathrm{norm}_{G,B\times C\times G}\, u = \sum_{i,j} v_i \otimes w_j \otimes D_{ij}\, u$$

for some maps $D_{ij} : H^q(G, k) \longrightarrow H^{p^2 q-i-j}(G, k)$, where we have calculated norm with respect to $\{1, c,\dots, c^{p-1}, b, bc,\dots, b^{p-1} c^{p-2}, b^{p-1} c^{p-1}\}$ (this choice of coset representatives is to conform with Lemma 5.5: see the proof of Theorem 5.3). We now have

Lemma 5.7 If $u \in H^q(G, k)$, then

$$D_{ij} u = D_{ji} u \,.\, (-1)^{ij+p(p-1)q/2}.$$

Proof Define an automorphism $\theta$ of $B \times C \times G$ by $(b^r, c^s, g)\theta = (c^s, b^r, g)$. Then Lemma 4.13 shows

$$\mathrm{norm}_{G,B\times C\times G}\, u\, \theta^* = (\mathrm{norm}_{G,B\times C\times G}\, u)\theta^* \sigma$$

where $\sigma = 1$ if q is even, and $\sigma$ = sign of the permutation of $\theta$ on $B \times C$ if q is odd, i.e. $(-1)^{p(p-1)/2}$. Therefore

$$\mathrm{norm}_{G,B\times C\times G}\, u = \sum_{i,j} (v_i \otimes w_j \otimes D_{ij}\, u)\theta^* . (-1)^{p(p-1)q/2}$$

$$= \sum_{i,j} (v_i \otimes w_j)\theta^* \otimes D_{ij}\, u \,.\, (-1)^{p(p-1)q/2}$$

Now use Lemma 5.6.

Lemma 5.8 Let $r \in \mathbb{N}$ and let $u \in H^r(G, k)$.

(i) If $p = 2$, then $\mathrm{norm}_{G,C\times G}\, u = \sum_i w_{r-i} \otimes Sq^i u$.

(ii) If $p > 2$, then $(-1)^{(p-1)r(r+1)/4} \left[\frac{p-1}{2}\right]!^{r} \, \mathrm{norm}_{G,C\times G} =$

$$\sum_i (-1)^i (w_{(r-2i)(p-1)} \otimes P^i u - w_{(r-2i)(p-1)-1} \otimes \beta P^i u).$$

<u>Proof</u> (i) This follows immediately from the definition of $Sq^i$.

(ii) By definition $\mathrm{norm}_{G,C\times G}\, u = \sum_\ell w_\ell \otimes D_\ell u$. But $D_\ell u = 0$ unless $\ell = (r-2i)(p-1)$ or $(r-2i)(p-1)-1$ for some $i \in \mathbb{Z}$ by Lemma 4.25, and

$$D_{(r-2i)(p-1)-1}\, u = -\beta D_{(r-2i)(p-1)}\, u$$

by Lemma 4.23(i). The result follows from the definition of $P^i$.

The Adem relations are no more than interpreting Lemma 5.7 (correctly!) in terms of the Steenrod operations. However this is not easy and we shall only deal with the case $p = 2$; the case $p > 2$ is similar but more complicated.

Assume that $p = 2$. Let $x = v_1 \otimes 1$ and $y = 1 \otimes w_1$. Note that $\mathrm{norm}_{C,B\times C}\, w_1 = x y + y^2$ by Lemmas 4.24 and 4.27. If $u \in H^r(G, k)$, then

$$\sum_{i,j} v_i \otimes w_j \otimes D_{ij}\, u$$

| | |
|---|---|
| $= \mathrm{norm}_{G,B\times C\times G}\, u$ | by definition |
| $= \mathrm{norm}_{C\times G,B\times C\times G}\, \mathrm{norm}_{G,C\times G}\, u$ | by Lemma 5.5 |
| $= \mathrm{norm}_{C\times G,B\times C\times G} \sum_{j\in\mathbb{N}} w_1^{r-j} \otimes Sq^j u$ | by Lemma 5.8 (i) |
| $= \sum_{j\in\mathbb{N}} \mathrm{norm}_{C\times G,B\times C\times G}\, w_1^{r-j} \otimes Sq^j u$ | by Lemma 4.18 |
| $= \sum_{j\in\mathbb{N}} (x y + y^2)^{r-j} \otimes 1\, \mathrm{norm}_{C\times G,B\times C\times G}\, 1 \otimes Sq^j u$ | by Lemma 4.11 and Corollary 4.14 |
| $= \sum_{i,j\in\mathbb{N}} (x y + y^2)^{r-j} \otimes 1\, v_1^{r+j-i} \otimes 1 \otimes Sq^i Sq^j u$ | by Lemma 5.8 (i) and Corollary 4.14 |
| $= \sum_{i,j\in\mathbb{N}} (x y + y^2)^{r-j} x^{r+j-i} \otimes Sq^i Sq^j u.$ | (1) |



By Lemma 5.7 this expression is symmetric in $x$ and $y$, and the resulting equality is the Adem relations. However the combinatorics involved to get it in the form of Theorem 5.3 is difficult. We shall follow the treatment of [S.R. Bullett and I.G. Macdonald, On the Adem relations, Topology 21 (1982), 329–332].

Let $k(s, t)$ denote the field of fractions of the polynomial ring $k[s, t]$ in the indeterminants $s$ and $t$. Let $F(t)$ denote the formal power series

$$\sum_{i\in\mathbb{N}} t^i Sq^i .$$

One can view $F(t)$ as an element of $k(t)[[Sq^0, Sq^1, \ldots]]$, the power series ring in the non-commuting variables $Sq^i$ quotiented out by all the relations satisfied by the $Sq^i$. Similarly one can view expression (1) as an element of $k(x, y)[[Sq^0, Sq^1, \ldots]]$.

We rewrite expression (1) as

$$x^r y^r (x + y)^r \sum_{i,j} x^{-i}(y + x^{-1} y^2)^{-j} \otimes Sq^i Sq^j u = x^r y^r (x + y)^r F(x^{-1}) F((y + x^{-1} y^2)^{-1}) u .$$

Since this is symmetric in $x$ and $y$, we see that $F(x^{-1})F((y + x^{-1} y^2)^{-1})u = F(y^{-1})F((x + y^{-1} x^2)^{-1})u$ $\forall r$ and $\forall u$, hence $F(x^{-1})F((y + x^{-1} y^2)^{-1}) = F(y^{-1})F((x + y^{-1} x^2)^{-1})$. If we perform the endomorphism

$$x \longmapsto x^{-1}(x + y)^{-1}, \; y \longmapsto y^{-1}(x + y)^{-1}$$

of $k(x, y)$, then $y + x^{-1} y^2 \longrightarrow y^{-2}$ and we deduce that

$$F(x(x + y))F(y^2) = F(y(x + y))F(x^2).$$

Setting $y = 1$ yields $F(x(x+1))F(1) = F(x + 1)F(x^2)$. Equating the terms which increase the cohomological degree by $n$ (in other words the terms involving $Sq^a Sq^b$ where $a + b = n$) yields

$$\sum_{a+b=n} (x^2 + x)^a Sq^a Sq^b = \sum_{j=0}^{n} (x + 1)^{n-j} x^{2j} Sq^{n-j} Sq^j.$$

Now $Sq^a Sq^b$ is the coefficient of $(x^2 + x)^{-1}$ in

$$(x^2 + x)^{-a-1} \sum_{j=0}^{n} (x + 1)^{n-j} x^{2j} Sq^{n-j} Sq^j,$$

which is the same as the coefficient of $x^{-1}$ in

$$\sum_{j=0}^{a+b} (x+1)^{b-j-1} x^{2j-a-1} Sq^{a+b-j} Sq^j.$$

Therefore $Sq^a Sq^b = \sum_{j=0}^{a+b} \binom{b-j-1}{a-2j} Sq^{a+b-j} Sq^j$. This is Theorem 5.3: note that $\binom{i}{j} = 0$ if $j$ or $i-j<0$.



<u>6. Further Reading</u> The classic books [5], [8] and [9] are recommended for nonrecent work on homological algebra. Presently the best account of the cohomology of finite groups is [2]; this is very comprehensive and up–to–date, and is an outgrowth of [1] (though [2] does not completely supercede [1]). Less comprehensive, though more detailed, is [6]. The classic work [11] remains an excellent exposition of the Steenrod operations. For the important topic of spectral sequences, not covered in these notes, [10] is recommended. The books [3], [4] and [7] contain much valuable information and are similar in spirit to these notes, but with the emphasis on infinite groups.

## References

1. D.J. Benson, Modular Representation Theory: New Trends and Methods, LNM 1081, Springer–Verlag, Berlin–New York, 1984.
2. D.J. Benson, Representations and Cohomology I & II, Cambridge Studies in Advanced Math. nos. 30 & 31, Cambridge University Press, Cambridge–New York, 1991.
3. R. Bieri, Homological Dimension of Discrete Groups, Queen Mary College Mathematics Notes, Queen Mary College, London, 1976.
4. K.S. Brown, Cohomology of Groups, Graduate Texts in Math. no. 87, Springer–Verlag, Berlin–New York, 1982.
5. H. Cartan and S. Eilenberg, Homological Algebra, Princeton University Press, Princeton NJ, 1956.
6. L. Evens, The Cohomology of Groups, Oxford University Press, Oxford–New York, 1991.
7. K.W. Gruenberg, Cohomological Topics in Group Theory, LNM 143, Springer–Verlag, Berlin–New York, 1970.
8. P.J. Hilton and U. Stammbach, A Course in Homological Algebra, Graduate Texts in Math. no. 4, Springer–Verlag, Berlin–New York, 1971.
9. S. MacClane, Homology, Springer–Verlag, Berlin–New York, 1975.
10. J. McCleary, User's Guide to Spectral Sequences, Mathematics Lecture Series no. 12, Publish or Perish, Inc., Wilmington DE, 1985.
11. N.E. Steenrod, Cohomology Operations (written by D.B.A. Epstein), Annals of Math. Studies no. 50, Princeton University Press, Princeton NJ, 1962.