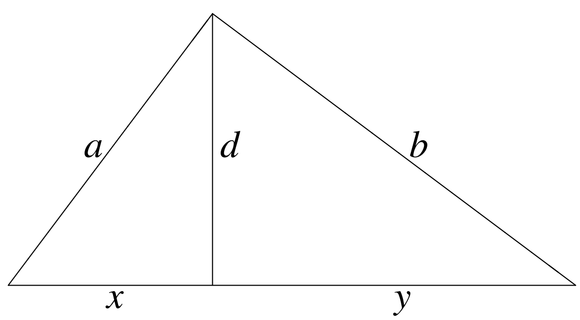
9th VTRMC, 1987, Solutions

1. The length of $\overline{P_0P_1}$ is $\sqrt{2}/2$. For $n \geq 1$, the horizontal distance from P_n to P_{n+1} is 2^{-n-1}, while the vertical distance is $3 \cdot 2^{-n-1}$. Therefore the length of $\overline{P_n, P_{n+1}}$ is $2^{-n-1}\sqrt{10}$ for $n \geq 1$, and it follows that the distance of the path is $(\sqrt{2}+\sqrt{10})/2$.

2. We want to solve in positive integers $a^2 = x^2 + d^2$, $b^2 = d^2 + y^2$, $a^2 + b^2 = (x+y)^2$. These equations yield $xy = d^2$, so we want to find positive integers x, y such that $x(x+y)$ and $y(x+y)$ are perfect squares. One way to do this is to choose positive integers x, y such that $x, y, x+y$ are perfect squares, so one possibility is $x = 9$ and $y = 16$. Thus we could have $a = 15$, $b = 20$ and $c = 25$.



3. If n is odd, then there are precisely $(n+1)/2$ odd integers in $\{1, 2, \dots\}$. Since $(n+1)/2 > n/2$, there exists an odd integer r such that a_r is odd, and then $a_r - r$ is even. It follows that $(a_1 - 1)(a_2 - 2) \cdots (a_n - n)$ is even.

4. (a) Since $|a_0| = |p(0)| \leq |0|$, we see that $a_0 = 0$.

   (b) We have $|p(x)/x| \leq 1$ for all $x \neq 0$ and $\lim_{x \to 0} p(x)/x = a_1$. Therefore $|a_1| \leq 1$.

5. (a) Set $n_1 = 2^{31}$, which has binary representation 1 followed by 31 0's. Since 31 has binary representation 11111, we see that $n_i = 31$ for all $i \geq 2$.

   (b) It is clear that $\{n_i\}$ is monotonic increasing for $i \geq 2$, so we need to prove that $\{n_i\}$ is bounded. Suppose $2^k \leq n_i < 2^{k+1}$ where $i \geq 2$. If ℓ is the number of zeros in the binary representation of n_i, then $n_i < 2^{k+1} - \ell$ and we see that $n_{i+1} < 2^{k+1}$. We deduce that $n_i < 2^{k+1}$ for all $i \geq 2$ and the result follows.

6. Of course, $\{a_n\}$ is the Fibonacci sequence (so in particular $a_n > 0$ for all n), and it is obvious that $x = -1$ is a root of $p_n(x)$ for all n (because $a_{n+2} - a_{n+1} - a_n = 0$). Since the roots of $p_n(x)$ are real and their product is $-a_n/a_{n+2}$, we see that $\lim_{n\to\infty} r_n = -1$. Finally $s_n = \lim_{n\to\infty} a_n/a_{n+2}$. If $f = \lim_{n\to\infty} a_{n+1}/a_n$, then $f^2 = f+1$, so $f = (1+\sqrt{5})/2$ because $f > 0$. Thus $f^2 = (3+\sqrt{5})/2$ and we deduce that $\lim_{n\to\infty} s_n = (3-\sqrt{5})/2$.

7. Let $D = \{d_{ij}\}$ be the diagonal matrix with $d_{nn} = t$, $d_{ii} = 1$ for $i \neq n$, and $d_{ij} = 0$ if $i \neq j$. Then $A(t) = DA$ and $B(t) = DB$. Therefore

$$A(t)^{-1}B(t) = (DA)^{-1}DB = A^{-1}D^{-1}DB = A^{-1}B$$

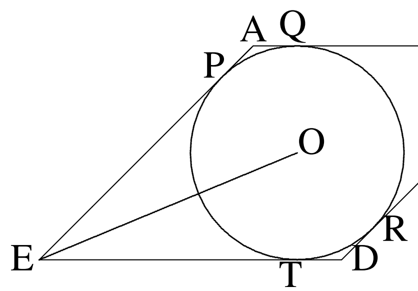
as required.

8. (a) $x'(t) = u(t) - x(t)$, $y'(t) = v(t) - y(t)$, $u'(t) = -x(t) - u(t)$, $v'(t) = y(t) - v(t)$.

   (b) Set $Y = \begin{pmatrix} u(t) \\ x(t) \end{pmatrix}$ and $A = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$. Then we want to solve $Y' = AY$. The eigenvalues of A are $-1 \pm i$, and the corresponding eigenvectors are $\begin{pmatrix} \pm i \\ 1 \end{pmatrix}$. Therefore $u(t) = e^{-t}(-A\sin t + B\cos t)$ and $x(t) = e^{-t}(A\cos t + B\sin t)$, where A, B are constants to be determined. However when $t = 0$, we have $u(t) = x(t) = 10$, so $A = B = 10$. Therefore $u(t) = 10e^{-t}(\cos t - \sin t)$ and $x(t) = 10e^{-t}(\cos t + \sin t)$. Finally the cat will hit the mirror when $u(t) = 0$, that is when $t = \pi/4$.

10th VTRMC, 1988, Solutions

1. Let $ABDE$ be the parallelogram S and let the inscribed circle C have center O. Thus $\angle AED = \theta$. Let S touch C at P, Q, R, T. It is well known that S is a rhombus; to see this, note that $EP = ET$, $AP = AQ$, $BQ = BR$ and $DR = DT$. In particular O is the intersection of AD and EB, $\angle EOD = \pi/2$ and $\angle OED = \theta/2$. Let $x =$ area of S. Then $x = 4$ times area of EOD. Since $ED = ET + TD = r\cot\theta/2 + r\tan\theta/2$, we conclude that $x = 2r^2(\cot\theta/2 + \tan\theta/2)$.



2. Let the check be for x dollars and y cents, so the original check is for $100x + y$ cents. Then $100y + x - 5 = 2(100x + y)$. Therefore $98y - 196x = 5 + 3x$. Of course, x and y are integers, and presumably $0 \le x, y \le 99$. Since 98 divides $5 + 3x$, we see that $x = 31$ and hence $y = 2x + (5 + 3x)/98 = 63$. Thus the original check was for \$31.63.

3. If we differentiate $y(x) + \int_1^x y(t)\,dt = x^2$ with respect to x, we obtain $y' + y = 2x$. This is a first order linear differential equations, and the general solution is $y = Ce^{-x} + 2x - 2$, where C is an arbitrary constant. However when $x = 1$, $y(1) = 1$, so $1 = C/e + 2 - 2$ and hence $C = e$. Therefore $y = 2x - 2 + e^{1-x}$.

4. If $a = 1$, then $a^2 + b^2 = 2 = ab + 1$ for all n, and we see that $a^2 + b^2$ is always divisible by $ab + 1$. From now on, we assume that $n \ge 2$.

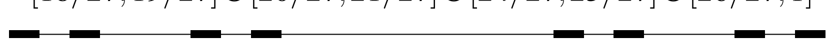
   Suppose $a^{n+1} + 1$ divides $a^2 + a^{2n}$, where n is a positive integer. Then $a^{n+1} + 1$ divides $a^{n-1} - a^2$ and hence $a^{n+1} + 1$ divides $a^4 + 1$. Thus in particular $n \le 3$. If $n = 3$, then $a^{n+1} + 1 = a^4 + 1$ divides $a^2 + a^{2n} = a^2 + a^6$. If $n = 1$, then $a^2 + 1$ divides $2a^2$ implies $a^2 + 1$ divides 2, which is not possible. Finally if $n = 2$, we obtain $a^3 + 1$ divides $a^2 + a^4$, so $a^3 + 1$ divides $a^2 - a$ which again is not possible.

We conclude that if $a > 1$, then a^2+b^2 is divisible by $ab+1$ if and only if $n = 3$.

5. Using Rolle's theorem, we see that f is either strictly monotonic increasing or strictly monotonic decreasing; without loss of generality assume that f is monotonic increasing. Then $f'(X) \geq 2$ for all x, so $|\alpha - x_0| < .00005$. Thus the smallest upper bound is .00005.

6. $f(x) = ax - bx^3$ has an extrema when $f'(x) = 0$, that is $a - 3bx^2 = 0$, so $x = \pm a/(\sqrt{3}b)$. Then $f(x) = \pm \frac{2a^2}{3\sqrt{3b}}$. Since f has 4 extrema on $[-1,1]$, two of the extrema must occur at ± 1. Thus we have $|a-b| = 1$. Thus a possible choice is $a = .1$ and $b = 1.1$.

7. (a)
$$\begin{aligned} f([0,1]) &= [0,1/3] \cup [2/3,1] \\ f(f([0,1])) &= [0,1/9] \cup [2/9,3/9] \cup [6/9,7/9] \cup [8/9,1] \\ f(f(f([0,1]))) &= [0,1/27] \cup [2/27,3/27] \cup [6/27,7,27] \cup [8/27,9/27] \cup \\ &[18/27,19/27] \cup [20/27,21/27] \cup [24/27,25/27] \cup [26/27,1] \end{aligned}$$



(b) Let $T \subseteq \mathbb{R}$ be a bounded set such that $f(T) = T$. First note that T contains no negative numbers. Indeed if T contains negative numbers, let $t_0 = \inf_{t \in T} t$ and choose $t_1 \in T$ with $t_1 < t_0/3$. Then there is no $t \in T$ with $f(t) = t_1$.

Therefore we may assume that T contains no negative numbers. Now suppose $1/2 \in T$. Since $(x+2)/3 = 1/2$ implies $x = -1/6$, we see that $3/2 \in T$. Now let $t_2 = \sup_{t \in T} t$ and choose $t_3 \in T$ such that $t_3 > (t_2+2)/3$. Since $f(T) = T$, we see that there exists $s \in T$ such that $f(s) > (t_2+2)/3$, which is not possible.

We conclude that there is no bounded subset T such that $f(T) = T$ and $1/2 \in T$.

8. If we have a triangle with integer sides a, b, c, then we obtain a triangle with integer sides $a+1, b+1, c+1$. Conversely if we have a triangle with integer sides a, b, c and the perimeter $a+b+c$ is even, then none of a, b, c can be 1 (because in a triangle, the sum of the lengths of any two sides is strictly greater than the length of the third side). This means we can obtain a triangle with sides $a-1, b-1, c-1$. We conclude that $T(n) = T(n-3)$ if n is even.

11th VTRMC, 1989, Solutions

1. Let B be the area of the triangle, let A be the area of the top triangle, and set $x = b - a$. Then $B/A = (b/a)^2$, so $(B-A)/B = 1-(a/b)^2$. Since $B - A = a^2 + ax/2$, we see that $B = ab^2/(2x)$. Therefore $B - 2a^2 = (ab^2 - 4a^2x)(2x) = a(a-x)^2/(2a) \geq 0$. This proves the result.

2. It is easily checked that a 2×2 matrix with all elements 0 or 1 has determinant 0, or 1, or -1.

   (a) Suppose A has determinant ± 3. Then by expanding by the first row, we see that all entries of the first row must be 1. Similarly by expanding by the second row, we see that all entries of the second row must be 1. But this tells us that the determinant of A is 0 and the result follows.

   (b) From (a) and expanding by the first row, we now see that $\det A = 0$, ± 1, ± 2. It is very easy to see that $\det A$ can take the values 0 and ± 1. To get the value 2, set $A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, while to get the value -2, take the above matrix and interchange the first two columns. Thus the values the determinant of A can take are precisely 0, ± 1, ± 2.

3. Note that $2(1,0,1)+(2,-2,1) = (2,-1,3)$. Therefore a solution of the system when $b_1 = 2$, $b_2 = -1$, $b_3 = 3$ is $(x_1,x_2,x_3) = 2(-1,3,2)+(2,-2,1)$. This yields the solution $x_1 = 0$, $x_2 = 4$, $x_3 = 5$.

4. We have $(r-a)(r-b)(r-c)(r-d) = 9$, where $r-a$, $r-b$, $r-c$, $r-d$ are distinct integers. This means that in some order, these four numbers must take the values ± 1, ± 3, in particular $(r-a)+(r-b)+(r-c)+(r-d) = 1-1+3-3 = 0$. The result follows.

5. (i) $1+x+x^2+x^3+x^4 = (x^5-1)(x-1)$. The only real root of $x^5 - 1$ is $x = 1$. The result follows.

   (ii) As indicated in the hint, we have $d/dx(xf_n(x)) = f_{n-1}(x)$. By part (i), we can assume that $f_{n-1}(x)$ has no real zero by induction on n. However it is then clear that $f_{n-1}(x)$ is always positive and therefore $xf_n(x)$ is a strictly increasing function. We deduce that $xf_n(x)$ has only one zero, namely $x = 0$, and the result follows.

6. Since
$$f(x/(x-1)) = \frac{x/(x-1)}{x/(x-1)-1} = \frac{x}{x-(x-1)} = x,$$
we see that $f^n(x) = x$ for x even and $f^n(x) = x/(x-1)$ for x odd. Therefore $\sum_{k=0}^{\infty} 2^{-k} f^k(x) = 4f^0(x)/3 + 2f^1(x)/3 = (4x^2-2x)/(3(x-1))$.

7. Let \$$x$ be the selling price before noon, and let \$$y$ be the selling price after noon. Let the first farmer sell a chickens before noon, the second farmer b chickens before noon, and the third farmer c chickens before noon. Then we have
$$\begin{aligned} ax + (10-a)y &= 35 \\ bx + (16-b)y &= 35 \\ cx + (26-c)y &= 35 \end{aligned}$$
Thus in particular
$$\begin{vmatrix} a & 10-a & 1 \\ b & 16-b & 1 \\ c & 26-c & 1 \end{vmatrix} = 0.$$
Remember that a, b, c are positive integers, $a < 10$, $b < 16$, $c < 26$. Also $x > y > 0$. By inspection, we must have $a = 9$, $b = 8$ and $c = 9$. This yields $x = 15/4$ and $y = 5/4$. Thus the cost of a chicken before noon is \$15/4, and the cost after noon is \$5/4.

8. The number of numbers in the sequence is the number of zero's, plus the number of 1's, plus the number of two's, plus .... In other words $n = a_0 + a_1 + \ldots a_{n-1}$. Also $a_0 \neq 0$, because if $a_0 = 0$, then since $n = a_0 + \cdots + a_n$, we would obtain $a_i = 1$ for all i. Thus there are $n - a_0 - 1$ nonzero terms in $\{a_1, \ldots, a_{n-1}\}$ which sum to $n - a_0$. Thus one of these nonzero terms is 2 and the rest are 1. If $a_1 = 0$, then $a_2 = 2$, hence $a_0 = 2$ and we have a contradiction because $n \geq 6$. If $a_1 = 1$, then $a_2 = 2$, hence $a_0 = 2$ and again we have a contradiction because $n \geq 6$. We deduce that $a_1 = 2$ and hence $a_2 = 1$. We conclude that the sequence must be $n-4, 2, 1, 0, \ldots, 0, 1, 0, 0, 0$. In particular for $n = 7$, the sequence is 3,2,1,1,0,0,0.

12th VTRMC, 1990, Solutions

1. Let a be the initial thickness of the grass, let b the rate of growth of the grass, and let c be the rate at which the cows eat the grass (in the appropriate units). Let n denote the number of cows that will eat the third field bare in 18 weeks. Then we have

$$\begin{aligned} 10(a+4b)/3 &= 12*4c \\ 10(a+9b) &= 21*9c \\ 24(a+18b) &= n18c \end{aligned}$$

If we multiply the first equation by $-27/5$ and the second equation by 14/5, we obtain $10(a+18b) = 270c$, so $(a+18b)/c = 27$. We conclude that $n = 36$, so the answer is 36 happy cows.

2. The exact number N of minutes to complete the puzzle is $\sum_{x=0}^{999} 3(1000 - x)/(1000+x)$. Since $3(1000-x)/(1000+x)$ is a non-negative monotonic decreasing function for $0 \leq x \leq 1000$, we see that

$$N - 3 \leq \int_0^{1000} -3 + 6000/(1000+x)\, dx \leq N.$$

Therefore $N/60 \approx 50(2\ln 2 - 1)$. Using $\ln 2 \approx .69$, we conclude that it takes approximately 19 hours to complete the puzzle.

3. One can quickly check that $f(2) = 2$ and $f(3) = 3$, so it seems reasonable that $f(n) = n$, so let us try to prove this. Certainly if $f(n) = n$, then $f(1) = 1$, so we will prove the result by induction on n; we assume that the result is true for all integers $\leq n$. Then

$$f(n+1) = f(f(n)) + f(n+1-f(n)) = n + f(1) = n+1$$

as required and it follows that $f(n) = n$ for $n = 1, 2, \ldots$.

4. Write $P(x) = ax^3 + bx^2 + cx + d$, where $a, b, c, d \in \mathbb{Z}$. Let us suppose by way of contradiction that $a, b, c, d \geq -1$. From $P(2) = 0$, we get $8a + 4b + 2c + d = 0$, in particular d is even and hence $d \geq 0$. Since $4b + 2c + d \geq -7$, we see that $a \leq 0$. Also $a \neq 0$ because $P(x)$ has degree 3, so $a = -1$. We now have $4b + 2c + d = 8$ and $b + c + d = 1$ from $P(1) = 0$. Thus $-2c - 3d = 4$, so $-2c = 4 + 3d \geq 4$ and we conclude that $c \leq -2$. The result follows.

5. (a) For small positive x, we have $x/2 < \sin x < x$, so for positive integers n, we have $1/(2n) < \sin(1/n) < 1/n$. Since $\sum_{n=1}^{\infty} 1/n^p$ is convergent if and only if $p > 1$, it follows from the basic comparison test that $\sum_{n=1}^{\infty} (\sin 1/n)^p$ is convergent if and only if $p > 1$.

   (b) It is not difficult to show that any real number x, there exists an integer $n > x$ such that $|\sin n| > 1/2$. Thus whatever p is, $\lim_{n\to\infty} |\sin n|^p \neq 0$. Therefore $\sum_{n=1}^{\infty} |\sin n|^p$ is divergent for all p.

6. (a) If y^* is a steady-state solution, then $y^* = y^*(2-y^*)$, so $y^* = 0$ or $1 = 2-y^*$. Therefore the steady-state solutions are $y^* = 0$ or 1.

   (b) Suppose $0 < y_n < 1$. Then $y_{n+1}/y_n = 2 - y_n > 1$, so $y_{n+1} > y_n$. Also $y_{n+1} = 1-(1-y_n)^2$, so $y_{n+1} < 1$. We deduce that y_n is a monotonic positive increasing function that is bounded above by 1, in particular y_n converges to some positive number ≤ 1. It follows that y_n converges to 1.

7. Let $y \in [0,1]$ be such that $(g(y) + uf(y)) = u$. Let us suppose we do have constants A and B such that $F(x) = Ag(x)/(f(x)+B)$ is a continuous function on $[0.1]$ with $\max_{0\leq x\leq 1} F(x) = u$. We will guess that the maximum occurs when $x = y$, so $u = Ag(y)/(f(y)+B)$. Then $A = B = -1$ satisfies these equations, so $F(x) = g(x)/(1-f(x))$.

   So let us prove that $F(x) = g(x)/(1-f(x))$ has the required properties. Certainly $F(x)$ is continuous because $f(x) < 1$ for all $x \in [0,1]$, and $F(y) = u$ from above. Finally $\max_{0\leq x\leq 1}(g(x)+uf(x)) = u$, so $g(x) \leq u(1-f(x))$ for all x and we conclude that $F(x) \leq u$. The result is proven.

8. Suppose we can disconnect F by removing only 8 points. Then the resulting framework will consist of two nonempty frameworks A, B such that there is no segment joining a point of A to a point of B. Let a be the number of points in A. Then there are $9-a$ points in B, at most $a(a-1)/2$ line segments joining the points of A, and at most $(10-a)(10-a-1)/2$ line segments joining the points of B. It follows that the resulting framework has at most $45-10a+a^2$. Since $10a-a^2 > 8$ for $1 \leq a \leq 9$, the result follows.

13th VTRMC, 1991, Solutions

1. Let P denote the center of the circle. Then $\angle ACP = \angle ABP = \pi/2$ and $\angle BAP = \alpha/2$. Therefore $BP = a\tan(\alpha/2)$ and we see that $ABPC$ has area $a^2\tan(\alpha/2)$. Since $\angle BPC = \pi - \alpha$, we find that the area of the sector BPC is $(\pi/2 - \alpha/2)a^2\tan^2(\alpha/2)$. Therefore the area of the curvilinear triangle is
$$a^2(1 + \frac{\alpha}{2} - \frac{\pi}{2})\tan^2\frac{\alpha}{2}.$$

2. If we differentiate both sides with respect to x, we obtain $3f(x)^2f'(x) = f(x)^2$. Therefore $f(x) = 0$ or $f'(x) = 1/3$. In the latter case, $f(x) = x/3 + C$ where C is a constant. However $f(0)^3 = 0$ and we see that $C = 0$. We conclude that $f(x) = 0$ and $f(x) = x/3$ are the functions required.

3. We are given that α satisfies $(1+x)x^{n+1} = 1$, and we want to show that α satisfies $(1+x)x^{n+2} = x$. This is clear, by multiplying the first equation by x.

4. Set $f(x) = x^n/(x+1)^{n+1}$, the left hand side of the inequality. Then
$$f'(x) = \frac{x^{n-1}}{(x+1)^{n+2}}(n-x).$$
This shows, for $x > 0$, that $f(x)$ has its maximum value when $x = n$ and we deduce that $f(x) \leq n^n/(n+1)^{n+1}$ for all $x > 0$.

5. Clearly there exists c such that $f(x) - c$ has a root of multiplicity 1, e.g. $x = c = 0$. Suppose $f(x) - c$ has a multiple root r. Then r will also be a root of $(f(x) - c)' = 5x^4 - 15x^2 + 4$. Also if r is a triple root of $f(x) - c$, then it will be a double root of this polynomial. But the roots of $5x^4 - 15x^2 + 4$ are $\pm((15 \pm \sqrt{145})/10)^{1/2}$, and we conclude that $f(x) - c$ can have double roots, but neither triple nor quadruple roots.

6. Expand $(1-1)^n$ by the binomial theorem and divide by $n!$. We obtain for $n > 0$
$$\frac{1}{0!n!} - \frac{1}{1!(n-1)!} + \frac{1}{2!(n-2)!} - \cdots + \frac{(-1)^n}{n!0!} = 0.$$
Clearly the result is true for $n = 0$. We can now proceed by induction; we assume that the result is true for positive integers $< n$ and plug into the

above formula. We find that

$$\frac{a_0}{n!}+\frac{a_1}{(n-1)!}+\frac{a_2}{(n-2)!}+\cdots+\frac{a_{n-1}}{1!}+\frac{(-1)^n}{n!0!}=0$$

and the result follows.

7. Suppose $2/3 < a_n, b_n < 7/6$. Then $2/3 < a_{n+2}, b_{n+2} < 7/6$. Now if $c = 1.26$, then $2/3 < a_3, b_3 < 1$, so if $x_n = a_{2n+1}$ or b_{2n+1}, then $x_{n+1} = x_n/4 + 1/2$ for all $n \geq 1$. This has the general solution of the form $x_n = C(1/4)^n + 2/3$. We deduce that as $n \to \infty$, a_{2n+1}, b_{2n+1} decrease monotonically with limit 2/3, and a_{2n}, b_{2n} decrease monotonically with limit 4/3.

   On the other hand suppose $a_n > 3/2$ and $b_n < 1/2$. Then $a_{n+1} > 3/2$ and $b_{n+1} < 1/2$. Now if $c = 1.24$, then $a_3 > 3/2$ and $b_3 < 1/2$. We deduce that $a_{n+1} = a_n/2+1$ and $b_{n+1} = b_n/2$. This has general solution $a_n = C(1/2)^n + 2$, $b_n = D(1/2)^n$. We conclude that as $n \to \infty$, a_n increases monotonically to 2 and b_n decreases monotonically to 0.

8. Let A be a base campsite and let h be a hike starting and finishing at A which covers each segment exactly once. Let B be the first campsite which h visits twice (i.e. B is the earliest campsite that h reaches a second time). This could be A after all segments have been covered, and then we are finished (just choose $\mathscr{C} = \{h\}$). Otherwise let h_1 be the hike which is the part of h which starts with the first visit to B and ends with the second visit to B (so B is the base campsite for h_1). Let h' be the hike obtained from h by omitting h_1 (so h' doesn't visit all segments). Now do the same with h'; let C be the first campsite on h' (starting from A) that is visited twice and let h_2 be the hike which is the part of h' that starts with the first visit to C and ends with the second visit to C. Then $\mathscr{C}$ can be chosen to be the collection of hikes $\{h_1, h_2, \dots\}$ to do what is required.

14th VTRMC, 1992, Solutions

1. First make the substitution $y = x^3$. Then $dF/dx = (dF/dy)(dy/dx) = e^{y^2}3x^2 = 3x^2e^{x^6}$ by the chain rule. Therefore $d^2F/dx^2 = 6xe^{x^6} + 18x^7e^{x^6}$. To find the point of inflection, we set $d^2F/dx^2 = 0$; thus we need to solve $6x + 18x^7 = 0$. The only solution is $x = 0$, so this is the point of inflection (perhaps we should note that d^3F/dx^3 is $6 \neq 0$ at $x = 0$, so $x = 0$ is indeed a point inflection).

2. The shortest path will first be reflected off the x-axis, then be reflected off the y-axis. So we reflect (x_2, y_2) in the y-axis, and then in the x-axis, which yields the point $(-x_2, -y_2)$. Thus the length of the shortest path is the distance from (x_1, y_1) to $(-x_2, -y_2)$, which is $((x_1 + x_2)^2 + (y_1 + y_2)^2)^{1/2}$.

3. (i) We have $f(f(x)) = 1 + \sin(f(x) - 1) = 1 + \sin(\sin(x-1))$, so $f_2(x) = x$ if and only if $x - 1 = \sin\sin(x-1)$. If y is a real number, then $|\sin y| \leq |y|$ with equality if and only if $y = 0$. It follows that $y = \sin\sin(y)$ if and only if $y = 0$ and we deduce that there is a unique point x_0 such that $f_2(x_0) = x_0$, namely $x_0 = 1$.

   (ii) From (i), we have $x_n = 1$ for all n. Thus we need to find $\sum_{n=0}^{\infty} 1/3^n$. This is a geometric series with first term 1 and ratio between successive terms 1/3. Therefore this sum is $1/(1 - 1/3) = 3/2$.

4. Clearly $t_n \geq 1$ for all n. Set $T = (1 + \sqrt{5})/2$ (the positive root of $x^2 - x - 1$). Note that if $1 \leq x < T$, then $x^2 < 1 + x < T^2$. This shows that $t_n < T$ for all n, and also that t_n is an increasing sequence, because $t_{n+1}^2 - t_n^2 = 1 + +t_n - t_n^2$. Therefore this sequence converges to a number between 1 and T. Since the number must satisfy $x^2 = x + 1$, we deduce that $\lim_{n\to\infty} t_n = T = (1 + \sqrt{5})/2$.

5. First we find the eigenvalues and eigenvectors of A. The eigenvalues satisfy $x(x-3) + 2 = 0$, so the eigenvalues are 1,2. To find the eigenvectors corresponding to 1, we need to solve the matrix equation

$$\begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

   One solution is $u = 2$, v$= -1$ and we see that $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ is an eigenvector corresponding to 1.

To find the eigenvectors corresponding to 2, we need to solve

$$\begin{pmatrix} -2 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

One solution is $u = 1$, $v = -1$ and we see that $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector corresponding to 2.

We now know that if $T = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$, then $T^{-1}AT = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. We deduce that $T^{-1}A^{100}T = \begin{pmatrix} 1 & 0 \\ 0 & 2^{100} \end{pmatrix}$. Therefore

$$A^{100} = \begin{pmatrix} 2-2^{100} & 2-2^{101} \\ -1+2^{100} & -1+2^{101} \end{pmatrix}.$$

6. Since $p(r) = 0$, we may write $p(x) = q(x)(x-r)$, where $q(x)$ is of the form $x^2 + dx + e$. Then $p(x)/(x-r) - 2p(x+1)/(x+1-r) + p(x+2)/(x+2-r) = q(x) - 2q(x+1) + q(x+2) = 2$ as required.

7. Assume that log means natural log. Note that $\log x$ is a positive increasing function for $x > 1$. Therefore

$$\int_1^n x \log x \, dx \leq \sum_{t=2}^{n} t \log t \leq \int_2^{n+1} x \log x \, dx.$$

Since $\int x \log x = (x^2 \log x)/2 - x^2/4$, we see that

$$(n^2 \log n)/2 - n^2/4 \leq \sum_{t=2}^{n} t \log t \leq ((n+1)^2 \log n)/2 - (n+1)^2/4 - 2\log 2 + 1.$$

Now divide by $n^2 \log n$ and take $\lim_{n\to\infty}$. We obtain

$$1/2 \leq \lim_{n\to\infty} \frac{\sum_{t=2}^{n} t \log t}{n^2 \log n} \leq 1/2.$$

We conclude that the required limit is 1/2.

8. For $G(3)$ we have 8 possible rows of goblins, and by writing these out we see that $G(3) = 17$. Similarly for $G(4)$ we have 16 possible rows of goblins, and by writing these out we see that $G(4) = 44$.

   In general, let X be a row of N goblins. Then the rows with $N+1$ columns are of the form $X2$ or $X3$ (where $X2$ indicates adding a goblin with height $2'$ to the end of X). If X ends in 3 or 32 ($2^{N-1}+2^{N-2}$ possible rows), then $X3$ has 1 more LGG than X; on the other hand if X ends in 22, then $X3$ has the same number of LGG's as X. If X ends in 2 (2^{N-1} possible rows), then $X2$ has 1 more LGG than X, while if X ends in 3, then $X2$ has the same number of LGG's as X. We conclude that for $N \geq 2$,

$$G(N+1) = 2G(N) + 2^{N-1} + 2^{N-1} + 2^{N-2} = 2G(N) + 5 \cdot 2^{N-2}.$$

   We now solve this recurrence relation in a similar way to solving a linear differential equation. The general solution will be $G(N) = C \cdot 2^N + aN2^{N-2}$, where a is to be determined. Then $G(N+1) = 2G(N) + 5 \cdot 2^{N-2}$ yields $a = 5/2$, so $G(N) = C \cdot 2^N + 5N2^{N-3}$. The initial condition $G(2) = 6$ shows that $C = 1/4$ and we conclude that $G(N) = 2^{N-2} + 5N2^{N-3}$ for all $N \geq 2$. We also have $G(1) = 2$.

15th VTRMC, 1993, Solutions

1. We change the order of integration, so the integral becomes

$$\int_0^1 \int_0^{\sqrt{y}} e^{y^{3/2}}\, dxdy = \int_0^1 y^{1/2} e^{y^{3/2}} = \left[2e^{y^{3/2}}/3\right]_0^1 = (2e-2)/3$$

as required.

2. Since f is continuous, $\int_0^x f(t)\,dt$ is a differentiable function of x, hence f is differentiable. Differentiating with respect to x, we obtain $f'(x) = f(x)$. Therefore $f(x) = Ae^x$ where A is a constant. Since $f(0) = 0$, we see that $A = 0$ and we conclude that $f(x)$ is identically zero as required.

3. From the definition, we see immediately that $f_n(1) = 1$ for all $n \geq 1$. Taking logs, we get $\ln f_{n+1}(x) = f_n(x)\ln x$. Now differentiate both sides to obtain $f'_{n+1}(x)/f_{n+1}(x) = f'_n(x)\ln x + f_n(x)/x$. Plugging in $x = 1$, we find that $f'_{n+1}(1)/f_{n+1}(1) = f_n(1)$ for all $n \geq 1$. It follows that $f'_n(1) = 1$ for all $n \geq 1$. Differentiating

$$f'_{n+1}(x) = f_{n+1}(x) f'_n(x)\ln x + f_{n+1}(x) f_n(x)/x,$$

we obtain

$$\begin{aligned} f''_{n+1}(x) = {} & f'_{n+1}(x) f'_n(x)\ln x + f_{n+1}(x) f''_n(x)\ln x + f_{n+1}(x) f'_n(x)/x \\ & + f'_{n+1}(x) f_n(x)/x + f_{n+1}(x) f'_n(x) - f_{n+1}(x) f_n(x)/x^2. \end{aligned}$$

Plugging in $x = 1$ again, we obtain $f''_n(1) = 2$ for all $n \geq 2$ as required.

4. Suppose we have an equilateral triangle ABC with integer coordinates. Let $\mathbf{u} = \overrightarrow{AB}$ and $\mathbf{v} = \overrightarrow{AC}$. Then by expressing the cross product as a determinant, we see that $|\mathbf{u}\times\mathbf{v}|$ is an integer. Also $|\mathbf{u}|^2$ is an integer, and $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}|^2 \sin(\pi/3)$ because $\angle BAC = \pi/3$. We conclude that $\sin(\pi/3) = \sqrt{3}/2$ is a rational number, which is not the case.

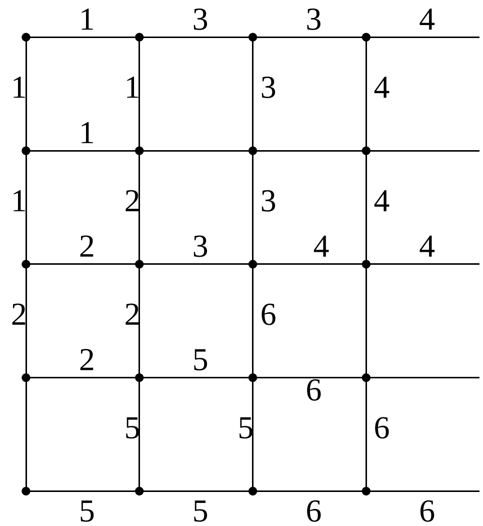
5. For $|x| < 1$, we have the geometric series $\sum_{n=0}^{\infty} x^n = 1/(1-x)$. If we integrate term by term from 0 to x, we obtain $\sum_{n=1}^{\infty} x^n/n = -\ln(1-x)$, which is also valid for $|x| < 1$. Now plug in $x = 1/3$: we obtain $\sum_{n=1}^{\infty} 3^{-n}/n = -\ln(2/3) = \ln 3 - \ln 2$.

6. Suppose f is not bijective. Since f is surjective, this means that there is a point $A_0 \in \mathbb{R}^2$ such that at least two distinct points are mapped to A_0 by f. Choose points $B_0, C_0 \in \mathbb{R}^2$ such that A_0, B_0, C_0 are not collinear. Now select points $A, B, C \in \mathbb{R}^2$ such that $f(A) = A_0$, $f(B) = B_0$ and $f(C) = C_0$. Since f maps collinear points to collinear points, we see that A, B, C are not collinear. Now given two sets each with three non-collinear points, there is a bijective affine transformation (i.e. a linear map composed with a translation) sending the first set of points to the second set. This means that there are bijective affine transformations $g, h \colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $g(0,0) = A$, $g(0,1) = B$, $g(1,0) = C$, $h(A_0) = (0,0)$, $h(B_0) = (0,1)$, $h(C_0) = (1,0)$. Then $k := hfg \colon \mathbb{R}^2 \to \mathbb{R}^2$ fixes $(0,0), (0,1), (1,0)$, and has the property that if P, Q, R are collinear, the so are $k(P), k(Q), k(R)$. Also there is a point $(a,b) \neq (0,0)$ such that $k(a,b) = (0,0)$. We want to show that this situation cannot happen.

   Without loss of generality, we may assume that $b \neq 0$ Let ℓ denote the line joining $(1,0)$ to $(0,b)$. Then $k(\ell)$ is contained in the x-axis. We claim that k maps the horizontal line through $(0,1)$ into itself. For if this was not the case, there would be a point with coordinates (c,d) with $d \neq 1$ such that $k(c,d) = (1,1)$. Then if m was the line joining (c,d) to $(0,1)$, we would have $k(m)$ contained in the horizontal line through $(0,1)$. Since m intersects the x-axis and the x-axis is mapped into itself by k, this is not possible and so our claim is established. Now let ℓ meet this horizontal line at the point P. Then we have that $k(P)$ is both on this horizontal line and also the x-axis, a contradiction and the result follows.

7. The problem is equivalent to the following. Consider a grid in the xy-plane with horizontal lines at $y = 2n+1$ and vertical lines at $x = 2n+1$, where n is an arbitrary integer. A ball starts at the origin and travels in a straight line until it reaches a point of intersection of a horizontal line and a vertical line on the grid. Then we want to show that the distance d ft travelled by the ball is not an integer number of feet. However $d^2 = (2m+1)^2 + (2n+1)^2$ for some integers m, n and hence $d^2 \equiv 2 \mod 4$. Since $d^2 = 0$ or $1 \bmod 4$, we have a contradiction and the result is proven.

8. The answer is 6; here is one way to get 6.

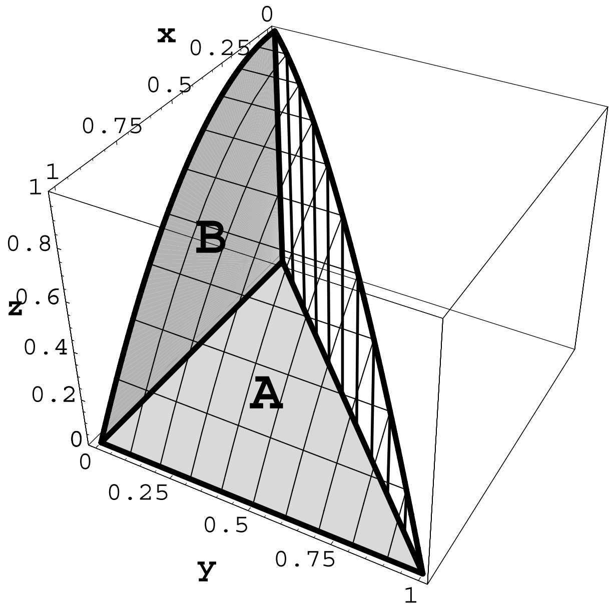


In this diagram, pieces of wire with the same corresponding number belong to the same logo. Also one needs to check that a welded point is not contained in more than one logo.

On the other hand each logo has three welded points, yet the whole frame has only 20 welded points. Thus we cannot cut more than 20/3 logos and it follows that the maximum number of logos that can be cut is 6.

16th VTRMC, 1994, Solutions

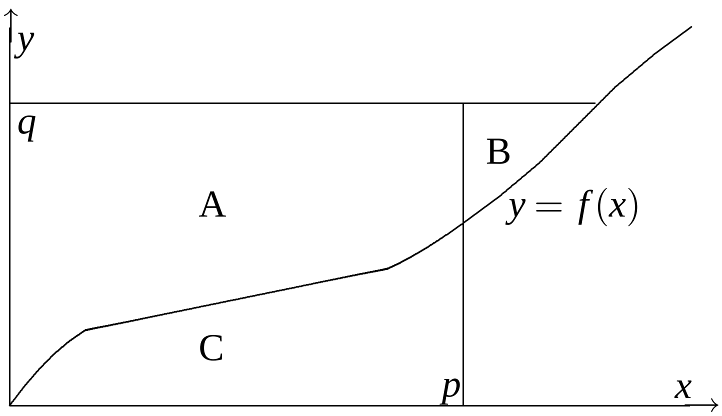
1. Let $I = \int_0^1 \int_0^x \int_0^{1-x^2} e^{(1-z)^2}\, dzdydx$. We change the order of integration, so we write $I = \iiint_V e^{(1-z)^2}\, dV$, where V is the region of integration.



It can be described as the cylinder with axis parallel to the z-axis and cross-section A, bounded below by $z = 0$ and bounded above by $z = 1 - x^2$. This region can also be described as the cylinder with axis parallel to the y-axis and cross-section B, bounded on the left by $y = 0$ and on the right by $y = x$. Therefore

$$\begin{aligned} I &= \int_0^1 \int_0^{\sqrt{1-z}} \int_0^x e^{(1-z)^2}\, dydxdz \\ &= \int_0^1 \int_0^{\sqrt{1-z}} xe^{(1-z)^2}\, dxdz = \int_0^1 (1-z)e^{(1-z)^2}/2\, dz \\ &= [-e^{(1-z)^2}/4]_0^1 = (e-1)/4. \end{aligned}$$

2. We need to prove that $pq \le \int_0^p f(t)\, dt + \int_0^q g(t)\, dt$. Either $q \le f(p)$ or $q \ge f(p)$ and without loss of generality we may assume that $q \ge f(p)$ (if $q \le f(p)$, then we interchange x and y; alternatively just follow a similar argument to what is given below). Then we have the following diagram.



We now interpret the quantities in terms of areas: pq is the area of $A \cup C$, $\int_0^p f(t)\,dt$ is the area of C, and $\int_0^q g(t)\,dt$ is the area of $A \cup B$. The result follows.

3. Differentiating both sides with respect to x, we obtain $2ff' = f^2 - f^4 + (f')^2$. Thus $f^4 = (f - f')^2$, hence $f - f' = \pm f^2$ and we deduce that $dx/df = \frac{1}{f \pm f^2}$. We have two cases to consider; first we consider the + sign, that is $dx/df = 1/f - 1/(f+1)$ and we obtain $x = \ln|f| - \ln|f+1| + C$, where C is an arbitrary constant. Now we have the initial condition $f(0) = \pm 10$. If $f(0) = 10$, we find that $C = \ln(11/10)$ and consequently $x = \ln(11/10) - \ln|(f+1)|/f|$. Solving this for x, we see that $f(x) = 10/(11e^{-x} - 10)$. On the other hand if $f(0) = -10$, then $C = \ln(9/10)$, consequently $x = \ln(9/10) - \ln|(f+1)/f|$. Solving this for x, we conclude that $f(x) = 10/(9e^{-x} - 10)$.

   Now we consider the – sign, that is $dx/df = 1/f - 1/(f-1)$ and we obtain $x = \ln|f| - \ln|f-1| + D$, where D is an arbitrary constant. If the initial condition $f(0) = 10$, we find that $D = \ln(9/10)$ and consequently $x = \ln|f/(f-1)| + \ln(9/10)$. Solving this for x, we deduce that $f(x) = 10/(10 - 9e^{-x})$. On the other hand if the initial condition is $f(0) = -10$, then $D = \ln(11/10)$ and hence $x = \ln|f/(f-1)| + \ln(11/10)$. Solving for x, we conclude that $f(x) = 10/(10 - 11e^{-x})$.

   Summing up, we have

$$f(x) = \frac{\pm 10}{10 - 9e^{-x}} \quad \text{or} \quad \frac{\pm 10}{10 - 11e^{-x}}.$$

4. Set $f(x) = ax^4 + bx^3 + x^2 + bx + a = 0$. We will show that the maximum value of $a + b$ is $-1/2$; certainly $-1/2$ can be obtained, e.g. with $a = 1$

corresponding first order differential equation $y' = 4y + 4t$. The solution to $a_{n+1} = 4a_n$ is $a_n = C4^n$ for some constant n. Then we look for a solution to $a_{n+1} = 4a_n + 4n$ in the form $a_n = An + B$, where A, B are constants to be determined. Plugging this into the recurrence relation, we obtain $A(n+1) + B = 4An + 4B + 4n$, and then equating the coefficients of n and the constant term, we find that $A = -4/3$, $B = -4/9$. Therefore $a_n = C4^n - 4n/3 - 4/9$, and then plugging in $a_1 = 1$, we see that $C = 25/36$ and we conclude that $a_n = 25 \cdot 4^n/36 - 4n/3 - 4/9$. We now need to calculate $\sum_{n=1}^{N} a_n$. This is

$$25(4^N - 1)/27 - 2N^2/3 - 10N/9.$$

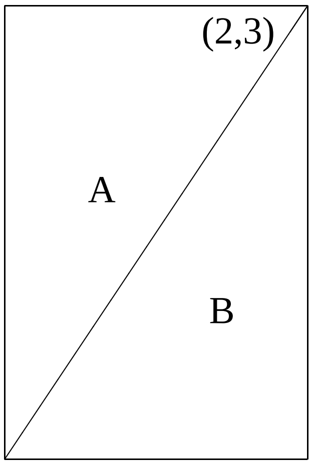
8. We have

$$x_{n+3} = \frac{19x_{n+2}}{94_{n+1}} = \frac{19^2}{94^2 x_n}$$

and we deduce that $x_{n+6} = x_n$ for all nonnegative integers n. It follows that $\sum_{n=0}^{\infty} x_{6n}/2^n = \sum_{n=0}^{\infty} 10/2^n = 20$.

17th VTRMC, 1995, Solutions

1. Let $A = \{(x,y) \mid 0 \le x \le 2,\ 0 \le y \le 3,\ 3x \le 2y\}$ and $B = \{(x,y) \mid 0 \le x \le 2,\ 0 \le y \le 3,\ 3x \le 2y\}$. Let $I = \int_0^3 \int_0^2 1/(1+\max(3x,2y))^2\,dxdy$. Then $\max(3x,2y) = 2y$ for $(x,y) \in A$ and $\max(3x,2y) = 3x$ for $x \in B$.



Therefore

$$\begin{aligned}
I &= \iint_A 1/(1+2y)^2\,dA + \iint_B 1/(1+3x)^2\,dA \\
&= \int_0^3 \int_0^{2y/3} 1/(1+2y)^2\,dxdy + \int_0^2 \int_0^{3x/2} 1/(1+3x)^2\,dydx \\
&= \int_0^3 2y/(3(1+2y)^2)\,dy + \int_0^2 3x/(2(1+3x)^2)\,dx \\
&= \int_0^3 1/(3(1+2y)) - 1/(3(1+2y)^2)\,dy \\
&\qquad\qquad + \int_0^2 1/(2(1+3x)) - 1/(2(1+3x)^2)\,dx \\
&= [(\ln(1+2y))/6 + 1/(6(1+2y))]_0^3 + [(\ln(1+3x))/6 + 1/(6(1+3x))]_0^2 \\
&= (\ln 7)/6 + 1/42 - 1/6 + (\ln 7)/6 + 1/42 - 1/6 - (7\ln 7 - 6)/21.
\end{aligned}$$

2. Let $A = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}$. We want to calculate powers of A, and to do this it is useful to find the Jordan canonical form of A. The characteristic polynomial of A is $\det(xI - A) = (x-4)(x+1) + 6 = x^2 - 3x + 2$ which has roots 1,2. Set $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. An eigenvector corresponding to 1 is $\mathbf{u}$ and an eigenvector

corresponding to 2 is $\begin{pmatrix}3\\2\end{pmatrix}$. Set $P = \begin{pmatrix}1 & 3\\1 & 2\end{pmatrix}$ and $D = \operatorname{diag}(1,2)$ (diagonal matrix with 1,2 on the main diagonal). Then $P^{-1} = \begin{pmatrix}-2 & 3\\1 & -1\end{pmatrix}$ and $P^{-1}AP = D$. Let $\mathbf{v} = \begin{pmatrix}1\\0\end{pmatrix}$ and let ^{T} denote transpose. Since $A = PDP^{-1}$, we find that

$$\begin{aligned}
(\theta^{100}\mathbf{v})^T &= A^{100}\mathbf{v} + (A^{99} + A^{98} + \cdots + A + A^0)\mathbf{u}\\
&= PD^{100}P^{-1}\mathbf{v} + P(D^{99} + D^{98} + \cdots + D + D^0)P^{-1}\mathbf{u}\\
&= P\begin{pmatrix}1 & 0\\0 & 2^{100}\end{pmatrix}P^{-1}\mathbf{v} + P\begin{pmatrix}100 & 0\\0 & 2^{100}-1\end{pmatrix}P^{-1}\mathbf{u}\\
&= \begin{pmatrix}98 + 3\cdot 2^{100}\\98 + 2\cdot 2^{100}\end{pmatrix}.
\end{aligned}$$

Thus $\theta^{100}(1,0) = (98 + 3\cdot 2^{100}, 98 + 2\cdot 2^{100})$.

3. Let $g(x)$ denote the power series in x

$$1 - (x + x^2 + \cdots + x^n) + (x + x^2 + \cdots + x^n)^2 - \cdots + (-1)^n(x + x^2 + \cdots + x^n)^n + \cdots.$$

Then for $2 \le r \le n$, the coefficient of x^r in $f(x)$ is the same as the coefficient of x^r in $g(x)$. Since $x + x^2 + \cdots + x^n = x(1-x^n)/(1-x)$, we see that $g(x)$ is a geometric series with ratio between successive terms $-x(1-x^n)/(1-x)$, hence its sum is

$$\frac{1}{1 + x(1-x^n)/(1-x)} = \frac{1-x}{1-x^{n+1}} = (1-x)(1 + x^{n+1} + x^{2n+2} + \cdots).$$

clearly the coefficient of x^r in the above is 0 for $2 \le r \le n$, which proves the result.

4. Write $[\tau n] = p$. Then p is the unique integer satisfying $p < \tau n < p+1$ because $p \ne \tau n$ (otherwise $\tau = p/n$, a rational number), that is $p/\tau < n < p/\tau + 1$. Since $1/\tau = \tau - 1$, we see that $p\tau - p < n < p\tau - p + 1$ and we deduce that $n + p - 1 < p\tau < n + p$. Therefore $[p\tau] = n + p - 1$ and hence $[\tau[\tau n] + 1] = n + p$. But $\tau^2 n = \tau n + n$, consequently $[\tau^2 n] = p + n$ and the result follows.

5. Suppose $x \in \mathbb{R}$ and $\theta(x) \leq -1$. Fix $y \in \mathbb{R}$ with $y < x$. Then if n is a positive integer and $x > p_1 > \cdots > p_n > y$, we have for $1 \leq i \leq n$

$$\begin{aligned} \theta(x) \geq \theta(x)^3 &> \theta(p_1), \\ \theta(p_i) > \theta(p_i)^3 &> \theta(p_{i+1}), \\ \theta(p_n) > \theta(p_n)^3 &> \theta(y), \end{aligned}$$

and we deduce that $\theta(x)\theta(p_1)^{2n-2} > \theta(y)$, for all n. this is not possible, so $\theta(x) > -1$ for all $x \in \mathbb{R}$. The same argument works if $0 \leq \theta(y) < \theta(x) \leq 1$.

6. We will concentrate on the bottom left hand corner of the square and determine the area A of that portion of the square that can be painted by the brush, and then multiply that by 4. We make the bottom of the square the x-axis and the left hand side of the square the y-axis. The equation of a line of length 4 from $(a,0)$ to the y-axis is $x/a + y/\sqrt{16-a^2} = 1$, that is $y = (1-x/a)\sqrt{16-a^2}$. For fixed x, we want to know the maximum value y can take by varying a. To do this, we differentiate y with respect to a and then set the resulting expression to 0. Thus we need to solve

$$(x/a^2)\sqrt{16-a^2} - a(1-x/a)/\sqrt{16-a^2} = 0.$$

On multiplying by $\sqrt{16-a^2}$ and simplifying, we obtain $16x = a^3$ and hence $dx/da = 3a^2/16$. Therefore

$$\begin{aligned} A &= \int_{x=0}^{x=4} (1-x/a)\sqrt{16-a^2}\,dx = \int_{a=0}^{a=4} (1-x/a)\sqrt{16-a^2}\,\frac{dx}{da}da \\ &= \int_{a=0}^{a=4} 3a^2(1-a^2/16)\sqrt{16-a^2}/16\,da = \int_0^4 3a^2(16-a^2)^{3/2}/256\,da. \end{aligned}$$

This ia a standard integral which can be evaluated by a trigonometric substitution. Specifically we set $a = 4\sin t$, so $da/dt = 4\cos t$ and we find that

$$\begin{aligned} A &= \int_0^{\pi/2} 48\cos^4 t \sin^2 t\,dt = \int_0^{\pi/2} 6\sin^2 2t(1+\cos 2t)\,dt \\ &= \int_0^{\pi/2} 3(1-\cos 4t)\,dt = 3\pi/2. \end{aligned}$$

We conclude that the total area that can be painted by the brush is $6\pi\,\mathrm{in}^2$.

7. Note that if p is a prime, then $f(p) = p$. Thus $f(100) = f(2^2 \cdot 5^2) = 4 + 10 = 14$, $f(2 \cdot 7) = 2 + 7 = 9$, $f(3^2) = 3 \cdot 2 = 6$. Therefore $g(100) = 6$. Next $f(10^{10}) = f(2^{10} \cdot 5^{10}) = 20 + 50 = 70$, $f(2 \cdot 5 \cdot 7) = 14$, $f(2 \cdot 7) = 2 + 7 = 9$, $f(3^2) = 3 \cdot 2 = 6$. Therefore $g(10^{10}) = 6$.

   Since $f(p) = p$ if p is prime, we see that $g(p) = p$ also and thus primes cannot have property H. Note that if r, s are coprime, then $g(rs) \leq f(r)s$. Suppose n has property H and let p be a prime such that p^2 divides n, so $n = p^k r$ where $k \geq 2$ and r is prime to p. It is easy to check that if $p^k > 9$, then $p^k > 2pk$, that is $f(p^k) < p^k/2$, thus $f(n) < n/2$ and we see that n cannot have property H. Also if p, q are distinct odd primes and $pq > 15$, then $f(pq) < pq/2$ and so if $n = pqr$ with r prime to pq, then we see again that n cannot have property H.

   The only cases to be considered now are $n = 9, 15, 45$. By direct calculation, 9 has property H, but 15 and 45 do not. So the only positive odd integer larger than 1 that has property H is 9.

18th VTRMC, 1996, Solutions

1. Let $I = \int_0^1 \int_{\sqrt{y-y^2}}^{\sqrt{1-y^2}} xe^{(x^4+2x^2y^2+y^4)}\,dxdy$. We change to polar coordinates to obtain
$$I = \int_0^{\pi/2}\int_{\sin\theta}^1 r\cos\theta\, e^{r^4}\, rdrd\theta = \int_0^{\pi/2}\int_{\sin\theta}^1 r^2 e^{r^4}\cos\theta\, drd\theta.$$
Now we reverse the order of integration; also we shall write $t = \theta$. This yields
$$\begin{aligned} I &= \int_0^1 \int_0^{\sin^{-1} r} r^2 e^{r^4} \cos t\, dtdr = \int_0^1 [r^2 e^{r^4} \sin t]_0^{\sin^{-1} r}\, dr \\ &= \int_0^1 r^3 e^{r^4}\, dr = [e^{r^4}/4]_0^1 = (e-1)/4. \end{aligned}$$

2. Write $r_1 = m_1/n_1$ and $r_2 = m_2/n_2$, where m_1, n_1, m_2, n_2 are positive integers and $\gcd(m_1, n_1) = 1 = \gcd(m_2, n_2)$. Set $Q = ((m_1+m_2)/(n_1+n_2), 1/(n_1+n_2))$. We note that Q is on the line joining $(r_1, 0)$ with $P(r_2)$, that is the line joining $(m_1/n_1, 0)$ with $(m_2/n_2, 1/n_2)$. This is because
$$(m_1+m_2)/(n_1+n_2) = m_1/n_1 + (m_2/n_2 - m_1/n_1)(n_2/(n_1+n_2)).$$
Similarly Q is on the line joining $P(r_1)$ with $(r_2, 0)$. It follows that $(m_1+m_2)/(n_1+n_2), 1/(n_1+n_2)$ is the intersection of the line joining $(r_1, 0)$ to $P(r_2)$ and the line joining $P(r_1)$ and $(r_2, 0)$. Set
$$P = P((r_1 f(r_1) + r_2 f(r_2))/(f(r_1) + f(r_2))).$$
Since
$$P = ((m_1+m_2)/(n_1+n_2), /f((m_1+m_2)/(n_1+n_2))),$$
we find that P is the point of intersection of the two given lines if and only if $f((m_1+m_2)/(n_1+n_2)) = n_1+n_2$. We conclude that the necessary and sufficient condition required is that $\gcd(m_1+m_2, n_1+n_2) = 1$.

3. Taking logs, we get $dy/dx = y\ln y$, hence $dx/dy = 1/(y\ln y)$. Integrating both sides, we obtain $x = \ln(\ln y) + C$ where C is an arbitrary constant. Plugging in the initial condition $y = e$ when $x = 1$, we find that $C = 1$. Thus $\ln(\ln y) = x - 1$ and we conclude that $y = e^{(e^{x-1})}$.

4. Set $g(x) = x^2 f(x)$. Then the given limit says $\lim_{x\to\infty} g''(x) = 1$. Therefore $\lim_{x\to\infty} g'(x) = \lim_{x\to\infty} g(x) = \infty$. Thus by l'Hôpital's rule,

$$\lim_{x\to\infty} g(x)/x^2 = \lim_{x\to\infty} g'(x)/(2x) = \lim_{x\to\infty} g(x)/2 = 1/2.$$

We deduce that $\lim_{x\to\infty} f(x) = 1/2$ and $\lim_{x\to\infty}(xf'(x)/2 + f(x)) = 1/2$, and the result follows.

5. Set

$$\begin{aligned} f(x) &= a_1 + b_1x + 3a_2x^2 + b_2x^3 + 5a_3x^4 + b_3x^5 + 7a_4x^6, \\ g(x) &= a_1x + b_1x^2/2 + a_2x^3 + b_2x^4/4 + a_3x^5 + b_3x^6/6 + a_4x^7. \end{aligned}$$

Then $g(1) = g(-1)$ because $a_1 + a_2 + a_3 + a_4 = 0$, hence there exists $t \in (-1,1)$ such that $g'(t) = 0$. But $g'(x) = f(x)$ and the result follows.

6. We choose the n line segments so that the sum of their lengths is as small as possible. We claim that no two line segments intersect. Indeed suppose A, B are red balls and C, D are green balls, and AC intersects BD at the point P. Since the length of one side of a triangle is less than the sum of the lengths of the two other sides, we have $AD < AP + PD$ and $BC < BP + PC$, consequently

$$AD + BC < AP + PC + BP + PD = AC + BD,$$

and we have obtained a setup with the sum of the lengths of the line segments strictly smaller. This proves that the line segments can be chosen so that no two intersect.

7. We have $f_{n,j+1}(x) - f_{n,j}(x) = \sqrt{x}/n$, hence

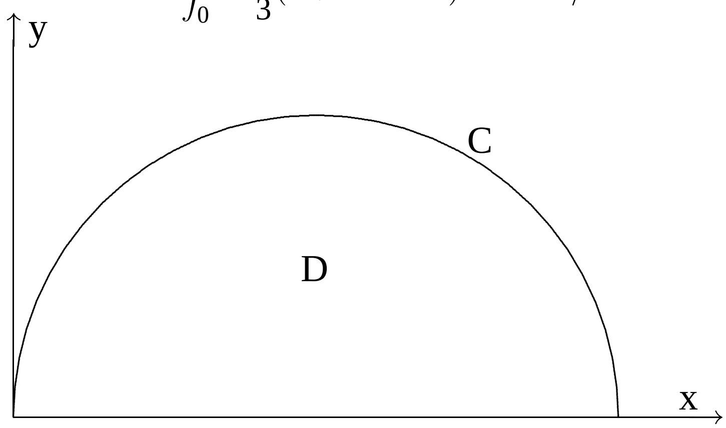
$$f_{n,j}(x) = f_{0,j}(x) + j\sqrt{x}/n = x + (j+1)\sqrt{x}/n.$$

Thus in particular $f_{n,n}(x) = x + (n+1)\sqrt{x}/n$ and we see that $\lim_{n\to\infty} f_{n,n}(x) = x + \sqrt{x}$.

19th VTRMC, 1997, Solutions

1. We change to polar coordinates. Thus $x = r\cos\theta$, $y = r\sin\theta$, and $dA = r\,dr d\theta$. The circle $(x-1)^2+y^2 = 1$ becomes $r^2 - 2r\cos\theta = 0$, which simplifies to $r = 2\cos\theta$. Also as one moves from $(2,0)$ to $(0,0)$ on the semicircle C (see diagram below), θ moves from 0 to $\pi/2$. Therefore

$$\iint_D \frac{x^3}{x^2+y^2}\,dA = \int_0^{\pi/2}\int_0^{2\cos\theta} \frac{r^3\cos^3\theta}{r^2} r\,dr d\theta = \int_0^{\pi/2}\int_0^{2\cos\theta} r^2\cos^3\theta\, dr d\theta$$
$$= \int_0^{\pi/2} \frac{8}{3}\cos^6\theta\, d\theta = \int_0^{\pi/2} \frac{1}{3}(1+\cos 2\theta)^3\, d\theta$$
$$= \int_0^{\pi/2} \frac{1}{3}(1+3\cos^2 2\theta)\, d\theta = 5\pi/12.$$



2. Since $r_1 r_2 = 2$, the roots r_1, r_2 will satisfy a quadratic equation of the form $x^2 + px + 2 = 0$, where $p \in \mathbb{C}$. Therefore we may factor

$$x^4 - x^3 + ax^2 - 8x - 8 = (x^2 + px + 2)(x^2 + qx - 4)$$

where $q \in \mathbb{C}$. Equating the coefficients of x^3 and x, we obtain $p + q = -1$ and $2q - 4p = -8$. Therefore $p = 1$ and $q = -2$. We conclude that $a = -4$ and r_1, r_2 are the roots of $x^2 + x + 2$, so r_1 and r_2 are $(-1 \pm i\sqrt{7})/2$.

3. The number of different combinations of possible flavors is the same as the coefficient of x^{100} in

$$(1 + x + x^2 + \dots)^4$$

This is the coefficient of x^{100} in $(1-x)^{-4}$, that is $103!/(3!100!) = 176851$.

4. We can represent the possible itineraries with a matrix. Thus we let

$$A = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}$$

and let a_{ij} indicate the (i, j)th entry of A. Then for a one day period, a_{11} is the number of itineraries from New York to New York, a_{12} is the number of itineraries from New York to Los Angeles, a_{21} is the number of itineraries from Los Angeles to New York, and a_{22} is the number of itineraries from Los Angeles to Los Angeles. The number of itineraries for an n day period will be given by A^n; in particular the $(1,1)$ entry of A^{100} will be the number of itineraries starting and finishing at New York for a 100 day period.

To calculate A^{100}, we diagonalize it. Then the eigenvalues of A are $1 \pm \sqrt{3}$ and the corresponding eigenvectors (vectors $\mathbf{u}$ satisfying $A\mathbf{u} = \lambda\mathbf{u}$ where $\lambda = 1 \pm \sqrt{3}$) are $(\pm\sqrt{3}, 1)$. Therefore if $P = \begin{pmatrix} \sqrt{3} & -\sqrt{3} \\ 1 & 1 \end{pmatrix}$, then

$$P^{-1}AP = \begin{pmatrix} 1+\sqrt{3} & 0 \\ 0 & 1-\sqrt{3} \end{pmatrix}$$

Thus

$$A^{100} = P \begin{pmatrix} (1+\sqrt{3})^{100} & 0 \\ 0 & (1-\sqrt{3})^{100} \end{pmatrix} P^{-1}.$$

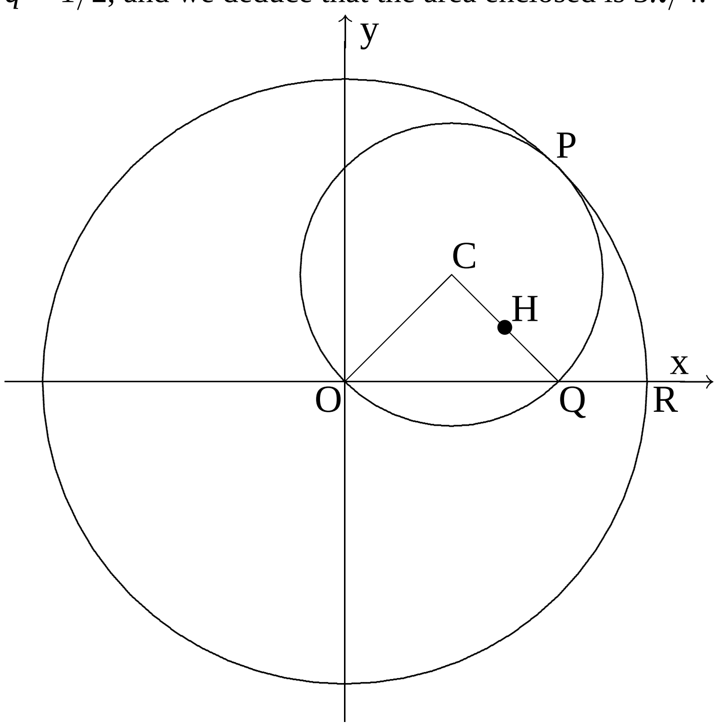
We conclude that the $(1,1)$ entry of A^{100} is $((1+\sqrt{3})^{100} + (1-\sqrt{3})^{100})/2$, which is the number of itineraries required.

5. For each city x in S, let $G_x \subset S$ denote all the cities which you can travel from x (this includes x). Clearly G_x is well served and $|G_x| \geq 3$ (where $|G_x|$ is the number of cities in G_x). Choose x so that $|G_x|$ is minimal. We need to show that if $y, z \in G_x$, then one can travel from y to z stopping only at cities in G_x; clearly we need only prove this in the case $z = x$. So suppose by way of contradiction $y \in G_x$ and we cannot travel from y to x stopping only at cities in G_x. Since $G_y \subseteq G_x$ and $x \notin G_y$, we have $|G_y| < |G_x|$, contradicting the minimality of $|G_x|$ and the result follows.

6. Let O denote the center of the circle with radius 2 cm., let C denote the center of the disk with radius 1 cm., and let H denote the hole in the center

of the disk. Choose axes so that the origin is at O, and then let the initial position have C and H on the positive x-axis with H furthest from O. The diagram below is in general position (i.e. after the disk has been moved round the inside of the circle). Let P be the point of contact of the circle and the disk, (so OCP will be a straight line), let Q be where CH meets the circumference of the disk (on the x-axis, though we need to prove that), and let R be where the circle meets the positive x-axis. Since the arc lengths PQ and PR are equal and the circle has twice the radius of that of the disk, we see that $\angle PCQ = 2\angle POR$ and it follows that Q does indeed lie on the x-axis.

Let (a,b) be the coordinates of C. Then $a^2+b^2=1$ because the disk has radius 1, and the coordinates of H are $(3a/2, b/2)$. It follows that the curve H traces out is the ellipse $4x^2+36y^2=9$. We now use the formula that the area of an ellipse with axes of length $2p$ and $2q$ is πpq. Here $p=3/2$, $q=1/2$, and we deduce that the area enclosed is $3\pi/4$.



7. Let $x = \{x_0, x_1, \ldots, x_n\} \in J$. Then

$$\begin{aligned} Tx &= LA(\{x_0, x_0+x_1, x_0+x_1+x_2, \ldots\}) \\ &= L(\{1+x_0, 1+x_0+x_1, 1+x_0+x_1+x_2, \ldots\}) \\ &= \{1, 1+x_0, 1+x_0+x_1, 1+x_0+x_1+x_2, \ldots\}. \end{aligned}$$

Therefore $T^2y = T(\{1,2,3,\ldots\}) = \{1, 1+1, 1+1+2, 1+1+2+3, \ldots\}$. We deduce that $T^2y = \{1,2,4,7,11,16,22,29,\ldots\}$ and in general $(T^2y)_n = n(n+1)/2+1$.

Suppose $z = \lim_{i\to\infty} T^i y$ exists. Then $Tz = z$, so $1 = z_0$, $1+z_0 = z_1$, $1+z_0+z_1 = z_2$, $1+z_0+z_1+z_2 = z_3$, etc. We now see that $z_n = 2^n$. To verify this, we use induction on n, the case $n = 0$ already having been established. Assume true for n; then

$$z_{n+1} = 1 + z_0 + z_1 + \cdots + z_n = 1 + 1 + 2 + \cdots + 2^n = 2^{n+1},$$

so the induction step is complete and the result is proven.

20th VTRMC, 1998, Solutions

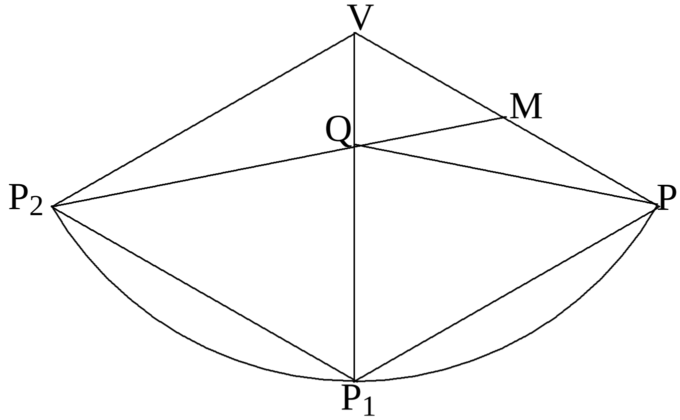
1. Set $r = x^2 + y^2$. Then $f(x,y) = \ln(1-r) - 1/(2r - (x+y)^2)$, so for given r, we see that f is maximized when $x + y = 0$. Therefore we need to maximize $\ln(1-r) - 1/(2r)$ where $0 < r < 1$. The derivative of this function is

$$\frac{1}{r-1} + \frac{1}{2r^2} = \frac{2r^2 + r - 1}{2(r-1)r^2}$$

which is positive when $r < 1/2$, 0 when r=1/2, and negative when $r > 1/2$. It follows that the maximum value of this function occurs when $r = 1/2$ and we deduce that $M = -1 - \ln 2$.

2. We cut the cone along PV and then open it out flat, so in the picture below P and P_1 are the same point. We want to find Q on VP_1 so that the length of MQP is minimal. To do this we reflect in VP_1 so P_2 is the image of P under this reflection, and then MQP_2 will be a straight line and the problem is to find the length of MP_2.

   Since the radius of the base of the cone is 1, we see that the length from P to P_1 along the circular arc is 2π, hence the angle $\angle PVP_1$ is $\pi/3$ because $VP = 6$. We deduce that $\angle PVP_2 = 2\pi/3$, and since $VM = 3$ and $VP_2 = VP = 6$, we conclude that $MP_2 = \sqrt{3^2 + 6^2 - 2 \cdot 3 \cdot 6 \cdot \cos(2\pi/3)} = 3\sqrt{7}$.

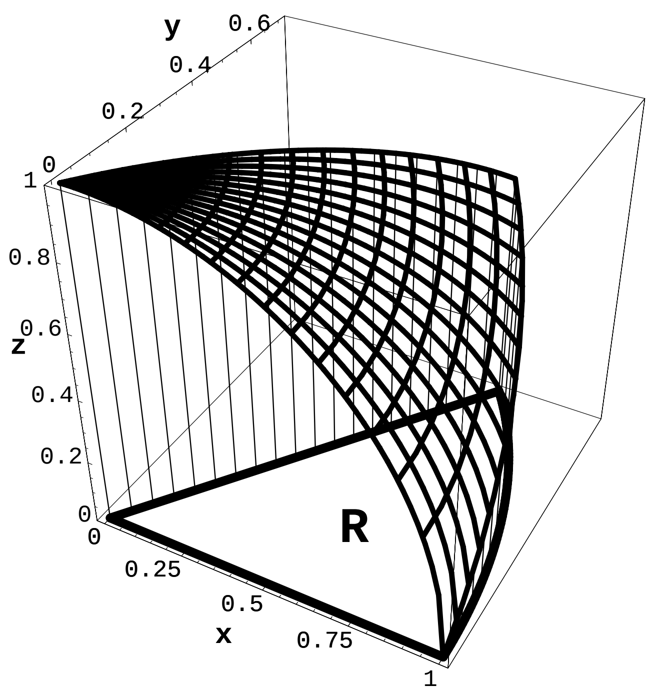


3. We calculate the volume of the region which is in the first octant and above $\{(x,y,0) \mid x \geq y\}$; this is 1/16 of the required volume. The volume is above R, where R is the region in the xy-plane and bounded by $y = 0$, $y = x$ and

$y = \sqrt{1-x^2}$, and below $z = \sqrt{1-x^2}$. This volume is

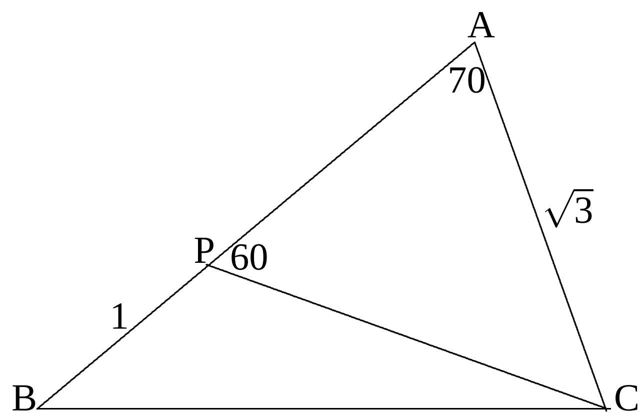
$$\begin{aligned}
&\int_0^{1/\sqrt{2}} \int_0^x \int_0^{\sqrt{1-x^2}} dzdydx + \int_{1/\sqrt{2}}^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dzdydx \\
&= \int_0^{1/\sqrt{2}} \int_0^x \sqrt{1-x^2}\, dydx + \int_{1/\sqrt{2}}^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-x^2}\, dydx \\
&= \int_0^{1/\sqrt{2}} x\sqrt{1-x^2}\, dx + \int_{1/\sqrt{2}}^1 (1-x^2)\, dx \\
&= [-(1-x^2)^{3/2}/3]_0^{1/\sqrt{2}} + [x - x^3/3]_{1/\sqrt{2}}^1 \\
&= \frac{1}{3} - \frac{1}{6\sqrt{2}} + \frac{2}{3} + \frac{1}{6\sqrt{2}} - \frac{1}{\sqrt{2}} \\
&= 1 - 1/\sqrt{2}.
\end{aligned}$$

Therefore the required volume is $16 - 8\sqrt{2}$.



4. We shall prove that $AB = BC$. Using the cosine rule applied to the triangle ABC, we see that $BC^2 = AB^2 + AC^2 - 2(AB)(AC)\cos 70$. Therefore we need to prove $AC = 2AB\cos 70$. By the sine rule applied to the triangle APC, we

find that $AP = 2\sin 50$, so we need to prove $\sqrt{3} = 2(1+2\sin 50)\cos 70$. However $\sin(50+70)+\sin(50-70) = 2\sin 50\cos 70$, $\sin 120 = \sqrt{3}/2$ and $\sin(50-70) = -\cos 70$. The result follows.



5. Since $\sum 1/a_n$ is a convergent series of positive terms, we see that given $M > 0$, there are only finitely many positive integers n such that $a_n < M$. Also rearranging a series with positive terms does not affect its convergence, hence we may assume that $\{a_n\}$ is a monotonic increasing sequence. Then $b_{2n+1} \geq b_{2n} \geq a_n/2$, so the terms of the sequence $\{1/b_n\}$ are at most the corresponding terms of the sequence

$$\frac{2}{a_1}, \frac{2}{a_1}, \frac{2}{a_2}, \frac{2}{a_2}, \frac{2}{a_3}, \frac{2}{a_3}, \ldots$$

Since $\sum 1/a_n$ is convergent, so is the sum of the above sequence and the result now follows from the comparison test for positive term series.

6. We shall assume the theory of writing permutations as a product of disjoint cycles, though this is not necessary. Rule 1 corresponds to the permutation (1 2 3 4 5 6 7 8 9 10) and Rule 2 corresponds to the permutation (2 6)(3 4)(5 9)(7 8). Since

$$(2\ 6)(3\ 4)(5\ 9)(7\ 8)(1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10) = (1\ 6\ 8\ 5\ 2\ 4\ 9\ 10)$$

(where we have written mappings on the left) has order 8, we see the position of the cats repeats once every 16 jumps. Now 10 p.m. occurs after 900 jumps, hence the cats are in the same position then as after 4 jumps and we conclude that the white cat is on post 8 at 10 p.m.

21st VTRMC, 1999, Solutions

1. Since the value of $f(x,y)$ is unchanged when we swap x with y,

$$\int_0^1 \int_0^x f(x+y)\,dydx = \frac{1}{2}\int_0^1 \int_0^1 f(x+y)\,dydx.$$

Also

$$\int_0^1 f(x+y)\,dy = \int_x^{1+x} f(z)\,dz = \int_0^1 f(z)\,dz$$

because $f(z) = f(1+z)$ for all z. Since $\int_0^1 f(z)\,dz = 1999$, we conclude that

$$\int_0^1 \int_0^1 f(x+y)\,dydx = 1999/2.$$

2. For $\alpha = 1$, $\beta = 0$ and $x = 1$, we have $f(1)f(0) = f(1)$. Therefore $f(0) = 1$. By differentiation

$$\alpha f'(\alpha x) f(\beta x) + \beta f(\alpha x) f'(\beta x) = f'(x)$$

holds for all x, and for all α, β satisfying $\alpha^2 + \beta^2 = 1$. Hence $(\alpha+\beta) f'(0) = f'(0)$ holds. By taking $\alpha = \beta = 1/\sqrt{2}$, we wee that $f'(0) = 0$. Set $c = f''(0)$. By Taylor's theorem, $f(y) = 1 + cy^2/2 + \varepsilon(y^2)$, where $\lim_{y\to 0} \varepsilon(y^2)/y^2 = 0$. By taking $\alpha = \beta = 1/\sqrt{2}$ again, we see that $f(x/\sqrt{2})^2 = f(x)$ for all x. By repetition, for every positive integer m,

$$f(x) = \left(f(2^{-m/2}x)\right)^{2^m}.$$

Now fix any x, and $\delta > 0$. There is a positive integer N such that for all $m \ge N$, $2^{-m}(|c| + \delta)x^2 < 1$, and

$$\left(1 + 2^{-m-1}(c-\delta)x^2\right)^{2^m} \le f(x) \le \left(1 + 2^{-m-1}(c+\delta)x^2\right)^{2^m}.$$

Now let $m \to \infty$. We obtain

$$e^{(c-\delta)x^2/2} \le f(x) \le e^{(c+\delta)/x^2}.$$

Since δ was arbitrary, $f(x) = e^{cx^2/2}$. Using the condition $f(1) = 2$, we conclude that $f(x) = 2^{x^2}$.

3. Note that any eigenvalue of A_n has absolute value at most M, because the sum of the absolute values of the entries in any row of A_n is at most M. We may assume that $M > 1$. By considering the characteristic polynomial, we see that the product of nonzero eigenvalues of A_n is a nonzero integer. Write $d = e_n(\delta)$. Then we have $M^n\delta^d > 1$. This can be written as

$$\frac{e_n(\delta)}{n} < \frac{\ln(M)}{\ln(1/\delta)}.$$

The result follows.

4. The points inside the box which are distance at least 1 from all of the sides form a rectangular box with sides 1,2,3, which has volume 6. The volume of the original box is 60. The points outside the box which are distance at most 1 from one of the sides have volume

$$3\times 4+3\times 4+3\times 5+3\times 5+4\times 5+4\times 5 = 94$$

plus the points at the corners, which form eight $\frac{1}{8}$ th spheres of radius 1, plus the points which form 12 $\frac{1}{4}$ th cylinders whose heights are 3,4,5. It follows that the volume required is

$$60-6+94+4\pi/3+12\pi = 148+40\pi/3.$$

5. By differentiating $f(f(x)) = x$, we obtain $f'(f(x))f'(x) = 1$. Since f is continuous, $f'(x)$ can never cross zero. This means that either $f'(x) > 0$ for all x of $f'(x) < 0$ for all x. If $f'(x) > 0$ for all x, then $x > y$ implies $f(x) > f(y)$, and we get a contradiction by considering $f(f(a)) = a$. We deduce that f is monotonically decreasing, and since f is bounded below by 0, we see that $\lim_{x\to\infty} f(x)$ exists and is some nonnegative number, which we shall call L. If $L > 0$, then we obtain a contradiction by considering $f(f(L/2)) = L/2$. The result follows. **Remark** The condition $f(a) \neq a$ is required, otherwise $f(x) = x$ would be a solution.

6. (i) Obviously $n > 4$. Next, $n \neq 5$ because 4 divides $3+5$. Also $n \neq 6$ because 3 divides 6 and $n \neq 7$ because 7 divides $3+4$. Finally $n \neq 8$ since 4 divides 8, and $n \neq 9$ since 3 divides 9. On the other hand $n = 10$ because 3 does not divide 4, 10 and 14. Furthermore 4 does not divide 3, 10 and 13, and 10 does not divide 3, 4 and 7.

(ii) Suppose $\{3,4,10,m\}$ is contained in a set which has property **ND**. Then 3 should not divide m, so m is not of the form $3k$. Also 3 should not divide $10+m$, so m is not of the form $3k+2$. Furthermore 33 should not divide $m+4+10$, so m is not of the form $3k+1$. Here k denotes some integer. If s has property **ND** and contains 3,4 10 and m, then m cannot be of the form $3k$, $3k+1$, $3k+2$. This is impossible and the statement is proven.

22nd VTRMC, 2000, Solutions

1. Let $I = \int_0^\alpha \frac{d\theta}{5-4\cos\theta}$. Using the half angle formula $\cos\theta = 2\cos^2(\theta/2) - 1$, we obtain

$$I = \int_0^\alpha \frac{d\theta}{9-8\cos^2(\theta/2)} = \int_0^\alpha \frac{\sec^2(\theta/2)\,d\theta}{9\sec^2(\theta/2)-8} = \int_0^\alpha \frac{\sec^2(\theta/2)\,d\theta}{9\tan^2(\theta/2)+1}.$$

Now make the substitution $x = 3\tan(\theta/2)$. Then $2\,dx = 3\sec^2(\theta/2)\,d\theta$, consequently

$$3I = \int_0^{3\tan(\alpha/2)} \frac{2\,dx}{1+x^2} = 2\tan^{-1}(3\tan(\alpha/2)).$$

Therefore $I = \frac{2}{3}\tan^{-1}(3\tan(\alpha/2))$. By using the facts that $\tan(\pi/6) = 1/\sqrt{3}$ and $\tan(\pi/3) = \sqrt{3}$, we see that when $\alpha = \pi/3$,

$$I = \frac{2}{3}\tan^{-1}\sqrt{3} = \frac{2\pi}{9}.$$

2. Let J denote the Jordan canonical form of A. Then A and J will have the same trace, and the entries on the main diagonal of J will satisfy $4x^4+1=0$. This equation has roots $\pm 1/2 \pm i/2$, so the trace of A will be a sum of such numbers. But the trace of A is real, hence the imaginary parts must cancel and we see that there must be an even number of terms in the sum. It follows that the trace of A is an integer.

3. Make the substitution $y = x - t$. Then the equation becomes $x' = x^2 - 2xt + 1$. We will show that $\lim_{t\to\infty} x'(t)$ exists and is 0, and then it will follow that $\lim_{t\to\infty} y'(t)$ exists and is -1.

   When $t = 0$, the initial condition tells us that $x = 0$, so $x'(0) = 1$ and we see that $x(t) > 0$ for small t. Suppose for some positive t we have $x(t) \le 0$. Then there is a least positive number T such that $x(T) = 0$. Then $x'(T) = 1$, which leads to a contradiction because $x(t) > 0$ for $t < T$. We deduce that $x(t) > 0$ for all t.

   Now $x' - 1 = x(x-2t)$ and since $x'(0) = 1$, we see that $x - 2t < 0$ for small t. We deduce that for t sufficiently small, $x(t) < t$, consequently $y(t) < 0$ for small t. We now claim that $y(t) < 0$ for all positive t. If this is not the case, then there is a least positive number T such that $y(T) = 0$, and then we must have $y'(S) = 0$ for some S with $0 < S < T$. But from $y' = (y-t)(y+t)$ and $(y+t) > 0$, we would have to have $y(S) = S$, a contradiction because $y(S) < 0$. We deduce that $x(t) < t$ for all positive t.

Now consider $x' = x(x-2t)+1$. Note that we cannot have $x'(t) \geq 0$ for all t, because then $x \to 0$ as $t \to \infty$ which is clearly impossible, consequently x' takes on negative values. Next we have $x'' = 2(xx' - tx' - x)$, so if $x'(t) = 0$, we see that $x''(t) < 0$. We deduce that if $x'(T) < 0$, then $x'(t) < 0$ for all $t > T$. Thus there is a positive number T such that $x'(t) < 0$ for all $t > T$. Now differentiate again to obtain $x''' = 2(xx'' + x'x' - tx'' - 2x')$. Then we see that if $x''(t) = 0$ and $t > T$, then $x'''(t) > 0$, consequently there is a positive number $S > T$ such that either $x''(t) < 0$ for all $t > S$ or $x''(t) > 0$ for all $t > S$. We deduce that $x'(t)$ is monotonic increasing or decreasing for $t > S$ and hence $\lim_{t\to\infty} x'(t)$ exists (possibly infinite).

We now have $x'(t)$ is monotonic and negative for $t > S$, yet $x(t) > 0$ for all $t > S$. We deduce that $\lim_{t\to\infty} x'(t) = 0$ and the result follows.

4. Set $y = \overline{AP}$. Then

$$\begin{aligned} l_2^2 &= (l-x)^2 + y^2 + 2(l-x)y\cos\theta \\ l_1^2 &= x^2 + y^2 - 2xy\cos\theta \end{aligned}$$

Subtracting the second equation from the first we obtain

$$l_2^2 - l_1^2 = l^2 - 2lx + 2ly\cos\theta$$

which yields

$$2y\cos\theta = \frac{l_2^2 - l_1^2 + 2lx - l^2}{l}.$$

From the second equation and the above, we obtain

$$l_1^2 - x^2 + x\frac{l_2^2 - l_1^2 + 2lx - l^2}{l} = y^2.$$

By differentiating the above with respect to x, we now get

$$2x + \frac{l_2^2 - l_1^2 - l^2}{l} = 2y\frac{dy}{dx}$$

and we deduce that $dy/dx = \cos\theta$. Therefore

$$l_2 - l_1 = y(l) - y(0) = \int_0^l \cos\theta\, dx$$

as required.

5. Open out the cylinder so that it is an infinitely long rectangle with width 4. Then the brush paints out two ellipses (one on either side of the cylinder) which have radius $\sqrt{3}$ in the direction of the axis of the cylinder, which we shall call the y-axis, and radius 2 in the perpendicular direction, which we shall call the x-axis. Then the equation of the ellipse is $x^2/4 + y^2/3 = 1$. By considering just one of the ellipses, we see that the area required is

$$4\int_{-1}^{1} y\, dx = 8\sqrt{3}\int_0^1 \sqrt{1 - \frac{x^2}{4}}\, dx.$$

By making the substitution $x = 2\sin\theta$, this evaluates to $6 + \dfrac{4\pi\sqrt{3}}{3}$.

6. Let $\alpha = \sum_{n=1}^{\infty} a_n t^n$. Then

$$\alpha^2 = \sum_{n=1}^{\infty} a_n t^n \sum_{n=1}^{\infty} a_n t^n.$$

Consider the coefficient of t^n on the right hand side of the above; it is

$$a_{n-1}a_1 + a_{n-2}a_2 + \cdots + a_2 a_{n-2} + a_1 a_{n-1}$$

for $n \geq 2$ and 0 for $n = 1$. Using the given hypothesis, we see that this is a_n for all $n \geq 2$. We deduce that $t + \alpha^2 = \alpha$. For the problem under consideration, we need to calculate α when $t = 2/9$, so we want to find α when $\alpha^2 - \alpha + 2/9 = 0$. Since the roots of this equation are 1/3 and 2/3, we are nearly finished. However we ought to check that $\alpha \neq 2/3, \infty$.

Suppose this is not the case. Let $\beta_n = \sum_{m=1}^{n} a_m (2/9)^m$. If $\alpha = 2/3$ or ∞, then there exists a positive integer N such that $b_N < 1/3$ and $b_{N+1} \geq 1/3$. Then the same argument as above gives

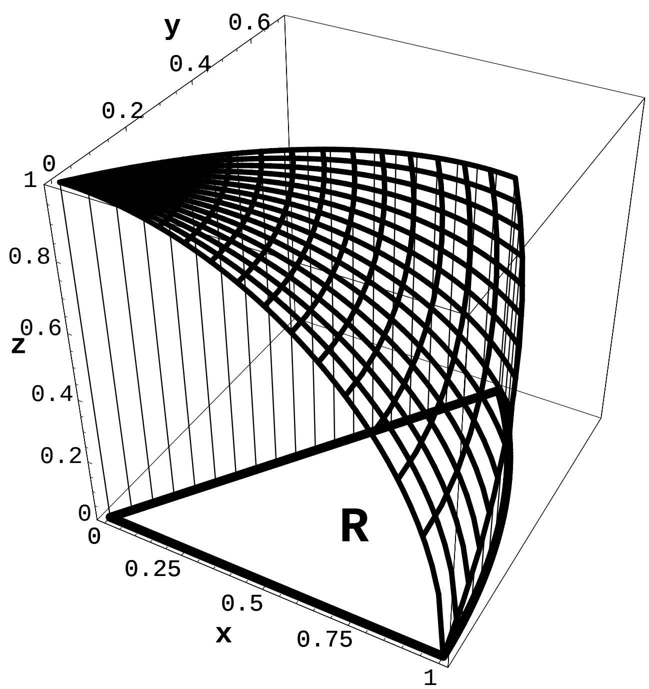
$$\frac{2}{9} + \beta_N^2 > \beta_{N+1},$$

which is not possible, so the result follows.

7. There are three possible positions for the tiles, namely $\left[\begin{smallmatrix}\bullet & \\ & \bullet\end{smallmatrix}\,\middle|\,\bullet\right]$, $\left[\bullet\,\middle|\,\begin{smallmatrix}\bullet & \\ & \bullet\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}\bullet & \\ & \bullet\end{smallmatrix}\,\middle|\,\begin{smallmatrix}\bullet & \\ & \bullet\end{smallmatrix}\right]$, which we shall call A,B and C respectively. Consider A_{n+2}. Then whatever the $n+1$ st tile is in the chain, we can complete it to a chain of length $n+2$. Therefore $A_{n+2} = A_{n+1} + x$ for some nonnegative integer x. However if the $n+1$ st tile is position A, then there is exactly one way to add a tile to get a chain of length $n+2$ (namely add tile B), whereas if the tile is B or C, then there are exactly two ways to add a tile to get a chain of length $n+2$ (namely add tiles A or C). Therefore x is the number of ways a chain ends of length $n+1$ ends in B plus the number of ways a chain of length end in C, which is precisely A_n. Therefore $A_{n+2} = A_{n+1} + A_n$. This recurrence relation is valid for $n \geq 1$. Since $A_1 = 3$ and $A_2 = 5$, we get $A_3 = 8$, $A_4 = 13$ and $A_{10} = 233$.

23rd VTRMC, 2001, Solutions

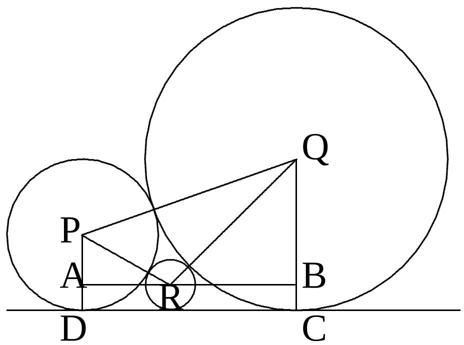
1. We calculate the volume of the region which is in the first octant and above $\{(x,y,0) \mid x \ge y\}$; this is 1/16 of the required volume. The volume is above R, where R is the region in the xy-plane and bounded by $y=0$, $y=x$ and $y=\sqrt{1-x^2}$, and below $z=\sqrt{1-x^2}$. This volume is
$$\begin{aligned}
&\int_0^{1/\sqrt{2}}\int_0^x\int_0^{\sqrt{1-x^2}} dzdydx + \int_{1/\sqrt{2}}^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2}} dzdydx\\
&= \int_0^{1/\sqrt{2}}\int_0^x \sqrt{1-x^2}\,dydx + \int_{1/\sqrt{2}}^1\int_0^{\sqrt{1-x^2}}\sqrt{1-x^2}\,dydx\\
&= \int_0^{1/\sqrt{2}} x\sqrt{1-x^2}\,dx + \int_{1/\sqrt{2}}^1 (1-x^2)\,dx\\
&= [-(1-x^2)^{3/2}/3]_0^{1/\sqrt{2}} + [x-x^3/3]_{1/\sqrt{2}}^1\\
&= \frac{1}{3} - \frac{1}{6\sqrt{2}} + \frac{2}{3} + \frac{1}{6\sqrt{2}} - \frac{1}{\sqrt{2}}\\
&= 1-1/\sqrt{2}.
\end{aligned}$$
Therefore the required volume is $16-8\sqrt{2}$.



2. Let the circle with radius 1 have center P, the circle with radius 2 have center Q, and let R be the center of the third circle, as shown below. Let $\overline{AR} = a$, $\overline{RB} = b$, and let the radius of the third circle be r. By Pythagoras on the triangles PAR and QRB, we obtain

$$(1-r)^2 + a^2 = (1+r)^2, \quad (2-r)^2 + b^2 = (2+r)^2.$$

Therefore $a^2 = 4r$ and $b^2 = 8r$. Also $(a+b)^2 + 1 = 9$, so $2\sqrt{r} + 2\sqrt{2r} = \sqrt{8}$ and we deduce that $r = 6 - 4\sqrt{2}$.



3. Let m, n be a positive integers where $m \le n$. For each $m \times m$ square in an $n \times n$ grid, replace it with the $(m-1) \times (m-1)$ square obtained by deleting the first row and column; this means that the 1×1 squares become nothing. Then these new squares (don't include the squares which are nothing) are in a one-to-one correspondence with the squares of the $(n-1) \times (n-1)$ grid obtained by deleting the first row and column of the $n \times n$ square. Therefore S_{n-1} is the number of squares in an $n \times n$ grid which have size at least 2×2. Since there are n^2 1×1 squares in an $n \times n$ grid, we deduce that $S_n = S_{n-1} + n^2$.

   Thus

$$S_8 = 1^2 + 2^2 + \cdots + 8^2 = 204.$$

4. If $p \le q < (p+1)^2$, then p divides q if and only if $q = p^2$, $p^2 + p$ or $p^2 + 2p$. Therefore if $a_k = p^2$, then $a_{k+3} = (p+1)^2$. Since $a_1 = 1^2$, we see that

$$a_{10000} = (1 + 9999/3)^2 = 3334^2 = 11115556.$$

5. Let $a_n = n^n x^n / n!$. First we use the ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1} x^{n+1} n!}{n^n x^n (n+1)!} = (1 + 1/n)^n x.$$

Since $\lim_{n\to\infty}(1+1/n)^n = e$, we see that the interval of convergence is of the form $\{-1/e, 1/e\}$, where we need to decide whether the interval is open or closed at its two endpoints. By considering $\int_n^{n+1} dx/x$, we see that

$$\frac{1}{n+1} < \ln(1+1/n) < \frac{1}{2}\left(\frac{1}{n} + \frac{1}{n+1}\right) < \frac{3}{3n+1}$$

because $1/n$ is a concave function, consequently $(1+1/n)^{n+1/3} < e < (1+1/n)^{n+1}$. Therefore when $|x| = 1/e$, we see that $|a_n|$ is a decreasing sequence. Furthermore by induction on n and the left hand side of the last inequality, we see that $|a_n| < 1/\sqrt[3]{n}$. Thus when $x = -1/e$, we see that $\lim_{n\to\infty} a_n = 0$ and it follows that the given series is convergent, by the alternating series test. On the other hand when $x = 1/e$, the series is $\sum n^n/(e^n n!)$. By induction on n and the above inequality, we see that $n^n/(e^n/n!) > 1/(en)$ for all $n > 1$. Since $\sum 1/n$ is divergent, we deduce that the given series is divergent when $x = 1/e$, consequently the interval of convergence is $[-1/e, 1/e)$.

6. Let $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$. If we can find a matrix $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $B^2 = A$ or even $4A$ and set $f(x) = \dfrac{ax+b}{cx+d}$, then $f(f(x)) = \dfrac{3x+1}{x+3}$. So we want to find a square root of A. The eigenvalues of A are 2 and 4, and the corresponding eigenvectors are $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ respectively. Thus if $X = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, then $XAX^{-1} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$. Set $C = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 2 \end{pmatrix}$. Then $C^2 = XAX^{-1}$, so if we let $B = (X^{-1}CX)^2$, then $B^2 = A$. Since

$$2B = \begin{pmatrix} 2+\sqrt{2} & 2-\sqrt{2} \\ 2-\sqrt{2} & 2+\sqrt{2} \end{pmatrix}$$

we may define (multiply top and bottom by $1+\sqrt{2}/2$)

$$f(x) = \frac{(3+2\sqrt{2})x+1}{x+3+2\sqrt{2}}.$$

Finally we should remark that f still maps $\mathbb{R}^+$ to $\mathbb{R}^+$. Of course there are many other solutions and answers.

7. Choose $x \in A$ and $y \in B$ so that $f(xy)$ is as large as possible. Suppose we can write xy in another way as ab with $a \in A$ and $b \in B$ (so $a \neq x$). Set $g = ax^{-1}$ and note that $I \neq g \in G$. Therefore either $f(gxy) > f(xy)$ or $f(g^{-1}xy) > f(xy)$. We deduce that either $f(ay) > f(xy)$ or $f(xb) > f(xy)$, a contradiction and the result follows.

24th VTRMC, 2002, Solutions

1. The volume is $\int_0^{\pi/2}\int_0^{1/\sqrt{b^2\cos^2 x+a^2\sin^2 x}} y\,dydx$. We put this in more familiar form by replacing x with θ and y with r to obtain

$$\int_0^{\pi/2}\int_0^{1/\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}} r\,drd\theta.$$

This is simply the area in the first quadrant of $r\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}=1$, equivalently $b^2r^2\cos^2\theta+a^2r^2\sin^2\theta=1$. Putting this in Cartesian coordinates, we obtain $b^2x^2+a^2y^2=1$, so we have to find the area of a quarter ellipse which intersects the positive x and y-axes at $1/b$ and $1/a$ respectively. Therefore the volume required is $\dfrac{\pi}{4ab}$.

2. The only solution is $a=d=e=0$ and $b=c=1$. To check this is a solution, we need to show $\sqrt{7+\sqrt{40}}=\sqrt{2}+\sqrt{5}$. Since both sides are positive, it will be sufficient to show that the square of both sides are equal, that is $7+\sqrt{40}=(\sqrt{2}+\sqrt{5})^2$, which is indeed true because the right hand side is $7+2\sqrt{10}$.

3. Let s be the given integer in S, and for an arbitrary integer y in S, define $f(y)=s-y$ if $y<s$ and $f(y)=99+s-y$ if $y\geq s$ (roughly speaking, $f(y)$ is $s-y$ mod 99). Note that $f\colon S\to S$ is a well defined one-to-one map. Let C denote the integers $f(y)$ for y in B, a subset of S with b elements because f is one-to-one. Since $a+b>99$, we see that A and C must intersect nontrivially, equivalently $f(y)$ must be an integer x in A for some integer y in B. Then $f(y)=x$ which yields $x=s-y$ or $s-y+99$, as required.

4. Since $23=32$, we can rearrange a word consisting of 2's and 3's so that all the 2's appear before the 3's. An even number of 2's gives 1, and an odd number of 2's gives 2, and similarly an even number of 3's gives 1 and an odd number of 3's gives 3. From this, we see that a word consisting of just 2's and 3's can only equal 1 if there are both an even number of 2's and an even number of 3's. Thus we already have that $f(n)=0$ for all positive odd integers n.

Now consider $f(n)$ for n even. By switching the first letter between 2 and 3, we see that the number of words consisting of just 2's and 3's which have an even number of 2's is the same as the number of words with an odd number of 2's. Since the total number of words of length n is 2^n, we deduce that $A(n) = 2^{n-1}$ when n is even. Therefore $A(12) = 2^{11} = 2048$.

5. First we find a recurrence relation for $f(n)$. If the first 0 is in position 1, there are 0 strings; if the first 0 is in position 2, there are $f(n-2)$ strings; if the first 0 is in position 3, there are $f(n-3)$ strings; if the first 0 is in position $(n-2)$ there are $f(2)$ strings; if the first 0 is in position $(n-1)$ there are $f(1)$ strings; and if there are no 0's there is 1 string. We deduce that

$$
\begin{aligned}
f(n) &= f(n-2) + \cdots + f(1) + 1 \\
f(n-1) &= f(n-3) + \cdots + f(1) + 1
\end{aligned}
$$

Therefore $f(n) = f(n-1) + f(n-2)$. We now prove by induction that $f(n) < 1.7^n$ for all n. The result is certainly true for $n = 1, 2$. Suppose it is true for $n-2$, $n-1$, that is $f(n-2) < 1.7^{n-2}$ and $f(n-1) < 1.7^{n-1}$. Then

$$
f(n) = f(n-2) + f(n-1) < 1.7^{n-2} + 1.7^{n-1} = 1.7^n \frac{1.7+1}{1.7^2} < 1.7^n,
$$

which establishes the induction step and the result follows.

6. Let the three matrices in T be X, Y, Z, and let I denote the identity matrix. Let us suppose by way of contradiction that there are no A, B in S such that AB is not in S. If λ is an eigenvalue of X, then λ^r is an eigenvalue of X^r. From this we immediately see that the eigenvalues of X^2 are $\{1, 1\}$. This means that the Jordan Canonical Form of X^2 is $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ where $x = 0$ or 1. If $x = 1$, then the matrices X^{2n} for $n \geq 1$ are all different and members of T, which is not possible because $|T| = 3$. Therefore $X^2 = I$ and in particular I is in T. Thus we may label are members of T as X, Y, I, where I is the identity matrix and $X^2 = Y^2 = I$. Consider XYX. We have $(XYX)(XYX) = XYX^2YX = XY^2X = X^2 = I$, so the eigenvalues of XYX are ± 1. This means that XYX is another member of T, so must be one of X, Y, I. We now show that this is not possible. If $XYX = X$, then $XYXXY = XXY = Y$ which yields $X = Y$. If $XYX = I$, then $XXYXX = XX = I$ which yields $Y = I$. Neither of these is possible because X, Y, I are distinct. Finally if $XYX = Y$, then $(XY)(XY) = I$ which shows that XY is also a member of T, so we must have $XY = X, Y$ or I, and this is easily seen to be not the case.

7. We use the fact that the arithmetic mean is at least the geometric mean, so $(a_1 + \cdots + a_n)/n \geq (a_1 \cdots a_n)^{1/n}$. Since $\sum_1^\infty a_n$ is convergent, it has a sum M say, and then we have $a_1 + \cdots + a_n \leq M$ for all n. We deduce that $b_n \leq M/n$ and hence $b_n^2 \leq M^2/n^2$. But $\sum 1/n^2$ is convergent (p-series with $p = 2 > 1$), hence $\sum M^2/n^2$ is also convergent and the result follows from the comparison test.

25th VTRMC, 2003, Solutions

1. The probability of p gains is the coefficient of $(1/2)^p(1/2)^{n-p}$ in $(1/2+1/2)^n$. Therefore, without the insider trading scenario, on average the investor will have $10000(3/5+9/20)^n$ dollars at the end of n days. With the insider trading, the first term $(3/5)^n$ becomes 0. Therefore on average the investor will have
$$10000\left(\frac{21}{20}\right)^n - 10000\left(\frac{3}{5}\right)^n$$
dollars at the end of n days.

2. We have
$$-\ln(1-x) = x + x^2/2 + x^3/3 + \cdots = \sum_{i=1}^{\infty} \frac{x^n}{n}.$$
Therefore
$$\begin{aligned}(1-x)\ln(1-x) &= -x + x^2/2 + x^3/6 + \ldots \\ &= -x + \sum_{i=1}^{\infty} x^{n+1}\left(\frac{1}{n} - \frac{1}{n+1}\right) = -x + \sum_{i=1}^{\infty} \frac{x^{n+1}}{n(n+1)}.\end{aligned}$$
Dividing by x, we deduce that
$$\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)} = 1 + \frac{1-x}{x}\ln(1-x)$$
for $x \neq 0$, and the sum is 0 for $x = 0$.

3. Let I denote the 2 by 2 identity matrix. Since $A = A^{-1}$, we see that $A^2 = I$ and hence the eigenvalues λ of A must satisfy $\lambda^2 = 1$, so $\lambda = \pm 1$. First consider the case $\det A = 1$. Then A has a repeated eigenvalue ± 1, and A is similar to $\begin{pmatrix} r & s \\ 0 & r \end{pmatrix}$ where $r = \pm 1$ and $s = 0$ or 1. Since $A^2 = I$, we see that $s = 0$ and we conclude that $A = \pm I$.

   Now suppose $\det A = -1$. Then the eigenvalues of A must be $1, -1$, so the trace of A must be 0, which means that A has the form $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ where

a, b are complex numbers satisfying $a^2+b^2=1$. Therefore $b=(1-a^2)^{1/2}$ (where the exponent 1/2 means one of the two complex numbers whose square is $1-a^2$). We conclude that the matrices satisfying $A=A'=A^{-1}$ are $\pm I$ and $\begin{pmatrix} a & (1-a^2)^{1/2} \\ (1-a^2)^{1/2} & -a \end{pmatrix}$ where a is any complex number.

4. Set $R=e^{2\pi i/7}=\cos 2\pi/7+i\sin 2\pi/7$. Since $R\neq 1$ and $R^7=1$, we see that $1+R+\cdots+R^6=0$. Now for n an integer, $R^n=\cos 2n\pi/7+i\sin 2n\pi/7$. Thus by taking the real parts and using $\cos(2\pi-x)=\cos x$, $\cos(\pi-x)=-\cos x$, we obtain

$$1+2\cos\frac{2\pi}{7}-2\cos\frac{\pi}{7}-2\cos\frac{3\pi}{7}=0.$$

Since $\cos\pi/7+\cos 3\pi/7=2\cos(2\pi/7)(\cos\pi/7)$, the above becomes

$$4\cos\frac{2\pi}{7}\cos\frac{\pi}{7}-2\cos\frac{2\pi}{7}=-1.$$

Finally $\cos(2\pi/7)=2\cos^2(\pi/7)-1$, hence $(2\cos^2(\pi/7)-1)(4\cos(\pi/7)-2)=-1$ and we conclude that $8\cos^3(\pi/7)-4\cos^2(\pi/7)-4\cos(\pi/7)=-1$. Therefore the rational number required is $-1/4$.

5. Since $\angle ABC+\angle PQC=90$ and $\angle ACB+\angle PRB=90$, we see that $\angle QPR=\angle ABC+\angle ACB$. Now X,Y,Z being the midpoints of BC, CA, AB respectively tells us that AY is parallel to ZX, AZ is parallel to XY, and BX is parallel to YZ. We deduce that $\angle ZXY=\angle BAC$ and hence $\angle QPR+\angle ZXY=180$. Therefore the points P,Z,X,Y lie on a circle and we deduce that $\angle QPX=\angle ZYX$. Using BZ parallel to XY and BX parallel to ZY from above, we conclude that $\angle ZYX=\angle ABC$. Therefore $\angle QPX+\angle PQX=\angle ABC+\angle PQX=90$ and the result follows.

6. Set $g=f^2$. Note that g is continuous, $g^3(x)=x$ for all x, and $f(x)=x$ for all x if and only if $g(x)=x$ for all x. Suppose $y\in[0,1]$ and $f(y)\neq y$. Then the numbers $y, f(y), f^2(y)$ are distinct. Replacing y with $f(y)$ or $f^2(y)$ and f with g if necessary, we may assume that $y<f(y)<f^2(y)$. Choose $a\in(f(y),f^2(y))$. Since f is continuous, there exists $p\in(y,f(y))$ and $q\in(f(y),f^2(y))$ such that $f(p)=a=f(q)$. Thus $f(p)=f(q)$, hence $f^3(p)=f^3(q)$ and we deduce that $p=q$. This is a contradiction because $p<f(y)<q$, and the result follows.

7. Let the tetrahedron have vertices A,B,C,D and let X denote the midpoint of BC. Then $AX = \sqrt{1-1/4} = \sqrt{3}/2$ and we see that ABC has area $\sqrt{3}/4$. Let R,S,T,U denote the regions vertically above and distance at most 1 from ABC, BCD, ABD, ACD respectively. Then the volumes of R,S,T and U are all $\sqrt{3}/4$. Since these regions are disjoint, they will contribute $\sqrt{3}$ to the volume required.

   Let Y denote the point on AX which is vertically below D. Then Y is the center of ABC (i.e. where the medians meet), in particular $\angle YBX = \pi/6$ and we see that $BY = 1/\sqrt{3}$. Therefore $DY = \sqrt{1-1/3} = \sqrt{2/3}$ and we deduce that $ABCD$ has volume

$$\frac{1}{3} * \frac{\sqrt{3}}{4} * \sqrt{2/3} = \frac{\sqrt{2}}{12}.$$

   Next consider the region which is distance 1 from BC and is between R and S. We need the angle between R and S, and for this we find the angle between DX and DY. Now $DX = AX = \sqrt{3}/2$ and $DY = \sqrt{2/3}$. Therefore $XY = \sqrt{3/4-2/3} = 1/(2\sqrt{3})$. If $\theta = \angle YDX$, then $\sin\theta = XY/DX = 1/3$. We deduce that the angle between R and S is $\pi/2+\theta = \pi/2+\sin^{-1}1/3$. Therefore the region at distance 1 from BC and between R and S has volume $\pi/4+(\sin^{-1}1/3)/2$. There are 6 such regions, which contribute $3\pi/2+3\sin^{-1}1/3$ to the volume required.

   For the remaining volume, we shrink the sides of the tetrahedron to zero. This keeps the remaining volume constant, but the volumes above go to zero. We are left with the volume which is distance 1 from the center of the pyramid, which is $4\pi/3$. Since $3\pi/2+4\pi/3 = 17\pi/6$, we conclude that the volume of the region consisting of points which are distance at most 1 from $ABCD$ is $\sqrt{3}+\sqrt{2}/12+17\pi/6+3\sin^{-1}(1/3) \approx 11.77$. Other expressions for this are $\sqrt{3}+\sqrt{2}/12+13\pi/3-3\cos^{-1}(1/3)$ and $\sqrt{3}+\sqrt{2}/12+13\pi/3-6\sin^{-1}(1/\sqrt{3})$.

26th VTRMC, 2004, Solutions

1. The answer is no. One example is

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

   Then $\det M \neq 0$ (expand by the fourth row), whereas $\det N = 0$ (fourth row consists entirely of 0's). Therefore M is invertible and N is not invertible, as required.

2. For n a non-negative integer, as n increases from n to $n+6$, we add 3 twice, 1 twice and 2 twice to $f(n)$; in other words $f(n+6) = f(n)+12$. We deduce that $f(n) = 2n$ when $n = 0$ (mod 6).

3. Let s_n denote the number of strings of length n with no three consecutive $A's$. Thus $s_1 = 3$, $s_2 = 9$ and $s_3 = 26$. We claim that we have the following recurrence relation:

$$s_n = 2s_{n-1} + 2s_{n-2} + 2s_{n-3} \quad (n \geq 3).$$

   The first term on the right hand side indicates the number of such strings which begin with B or C; the second term indicates the number of such strings which begin with AB of AC, and the third term indicates the number of such strings which begin with AAB or AAC. Using this recurrence relation, we find that $s_4 = 76$, $s_5 = 222$ and $s_6 = 648$. Since the total number of strings is 3^6, we conclude that the probability of a string on 6 symbols not containing 3 successive A's is $648/3^6 = 8/9$.

4. The answer is no. Let us color the chess board in the usual way with alternating black and white squares, say the corners are colored with black squares. Then by determining the number of black squares in each row, working from top to bottom, we see that the number of black squares is

$$4+4+5+4+4+4+5+4+4 = 38.$$

   Since there are 78 squares in all, we see that the number of white squares is 40. Now each domino cover 1 white and 1 black square, so if the board could be covered by dominoes, then there would be an equal number of black and white squares, which is not the case.

5. Expanding the sine, we have

$$f(x) = \cos x \int_0^x \sin(t^2 - t)\,dt + \sin x \int_0^x \cos(t^2 - t)\,dt.$$

Therefore

$$\begin{aligned}
f'(x) &= \cos x \sin(x^2 - x) - \sin x \int_0^x \sin(t^2 - t)\,dt \\
&\quad + \sin x \cos(x^2 - x) + \cos x \int_0^x \cos(t^2 - t)\,dt, \\
f''(x) &= -\sin x \sin(x^2 - x) + (2x - 1)\cos x \cos(x^2 - x) \\
&\quad - \sin x \sin(x^2 - x) - \cos x \int_0^x \sin(t^2 - t)\,dt \\
&\quad + \cos x \cos(x^2 - x) + (1 - 2x)\sin x \sin(x^2 - x) \\
&\quad - \sin x \int_0^x \sin(t^2 - t)\,dt + \cos x \sin(x^2 - x).
\end{aligned}$$

We deduce that

$$\begin{aligned}
f(x) + f''(x) &= (2x+1)(\cos x \cos(x^2 - x) - \sin x \sin(x^2 - x)) \\
&= (2x+1)\cos x^2.
\end{aligned}$$

Set $g(x) = f''(x) + f(x)$. To find $f^{(12)}(0) + f^{(10)}(0)$, we need to compute $g^{(10)}(0)$, which we can find by considering the coefficient of x^{10} in the Maclaurin series expansion for $(2x+1)\cos x^2$. Since $\cos x^2 = 1 - x^4/2! + x^8/4! - x^{12}/6! + \cdots$, we see that this coefficient is 0. Therefore $g^{(10)}(0) = 0$ and the result follows.

6. Suppose first that there is an infinite subset S such that each person only knows a finite number of people in S. Then pick a person A_1 in S. Then there is an infinite subset S_1 of S containing A_1 such that A_1 knows nobody in S_1. Now choose a person A_2 other than A_1 in S_1. Then there is an infinite subset S_2 of S_1 containing $\{A_1, A_2\}$ such that nobody in S_2 knows A_2. Of course nobody in S_2 will know A_1 either. Now choose a person A_3 in S_2 other than A_1 and A_2. Then nobody from $\{A_1, A_2, A_3\}$ knows each other. Clearly we can continue this process indefinitely to obtain an arbitrarily large number of people who don't know each other.

Therefore we may assume in any infinite subset of people there is a person who knows an infinite number of people. So we can pick a person B_1 who knows infinitely many people T_1. Then we can pick a person B_2 in T_1 who knows an infinite number of people T_2 of T_1, because we are assuming in any infinite subset of people, there is somebody who knows infinitely many of them. Of course, B_1 and B_2 know all the people in T_2. Now choose a person B_3 in T_2 who knows an infinite number of people in T_2. Then $\{B_1, B_2, B_3\}$ know each other. Clearly we can continue this process indefinitely to obtain an arbitrarily large number of people who know each other.

Remark A simple application of the axiom of choice shows that we can find an infinite number of people in the party such that either they all know each other or they all don't know each other.

7. Set $b_n = 1 - a_{n+1}/a_n$. Let us suppose to the contrary that $\sum |b_n|$ is convergent. Then $\lim_{n\to\infty} b_n = 0$, so may assume that $|b_n| < 1/2$ for all n. Now

$$a_{n+1} = a_1(1-b_1)(1-b_2)\dots(1-b_n),$$

hence

$$\ln(a_{n+1}) = \ln a_1 + \ln(1-b_1) + \ln(1-b_2) + \cdots + \ln(1-b_n).$$

Since $\lim_{n\to\infty} a_n = 0$, we see that $\lim_{n\to\infty} \ln(a_n) = -\infty$. Now $\ln(1-b) \geq -2|b|$ for $|b| < 1/2$; one way to see this is to observe that $1/(1-x) \leq 2$ for $0 \leq x \leq 1/2$ and then to integrate between 0 and $|b|$. Therefore $\lim_{n\to\infty}(-2|b_1| - \cdots - 2|b_n|) = -\infty$. This proves that $\sum |b_n|$ is divergent and the result follows.

27th VTRMC, 2005, Solutions

1. We note that if $p = 2,3,5,11,19$, then $6(p^3+1) \equiv 4,3,1,2,0 \mod 5$ respectively. Therefore if $n \geq 20$, we may choose p such that $n+6(p^3+1)$ is divisible by 5. Obviously if $n+6(p^3+1)$ is prime, then n is not divisible by 2 or 3. If $12 \leq n \leq 19$, then p may be chosen so that $6(p^3+1) \equiv 1,2,3,4$ mod 5, so 15 is the only possibility for $n+6(p^3+1)$ to be prime; however the above remark tells us this cannot be prime. So we need only check the numbers $1,5,7,11$. For $n = 7,11$, we see that $n+6(p^3+1)$ is divisible by 5 by choosing $p = 3$ and 2 respectively. Finally $5+6(p^3+1)$ is 59 when $p = 2$ and 173 when $p = 3$. Since 59 and 173 are both prime, we conclude that 5 is the largest integer required.

2. First, we must have $p(20) = 12$, $p(19) = 13$, $p(18) = 14$, $p(17) = 15$, $p(16) = 16$, $p(8) = 8$, $p(9) = 7$, $p(10) = 6$, $p(11) = 5$. Then $p(4)$ cannot be 12, hence $p(4) = 4$, $p(3) = 1$, $p(2) = 2$, $p(1) = 3$. Now if $p(n) = 20$, then n must be 12, and we have $p(13) = 19$, $p(14) = 18$ and $p(15) = 17$. Finally we must have $p(5) = 11$, $p(6) = 10$ and $p(7) = 9$. Thus the permutation required is
3,2,1,4,11,10,9,8,7,6,5,20,19,18,17,16,15,14,13,12.

3. If the end of the strip consists of one or two squares, then the number of ways of tiling the strip is $t(n-1)$, which makes a total of $2t(n-1)$. If the end of the strip consists of one or two dominos, then the number of ways of tiling the strip is $t(n-2)$, making for a total of $2t(n-2)$ ways. Finally if the strip ends in one domino and one square, then there may or may not be a square in the penultimate position, and here we get a total of $2t(n-2)$ ways. We conclude that $t(n) = 2t(n-1)+4t(n-2)$. We now have $t(3) = 24$, $t(4) = 80$, $t(5) = 256$ and $t(6) = 832$.

4. The x-coordinate of the beam of light will return to 0 after traveling distance 14. During this period, the y and z coordinates will each have traveled distance 28, and thus will have also returned to their original positions. It follows that the total distance traveled by the beam of light will be $\sqrt{14^2+28^2+28^2} = 14\sqrt{1^2+2^2+2^2} = 42$.

5. Set $z = y\ln(x^2)$. Then $f(x,y) = \dfrac{xz}{(x^2+z^2)\ln(x^2)}$. Since $|xz| \leq x^2+z^2$, we see that $|f(x,y)| \leq 1/\ln(x^2)$. Since $1/\ln(x^2) \to 0$ as $(x,y) \to (0,0)$, it follows that $\lim_{(x,y)\to(0,0)}$ exists and is equal to 0.

6. Divide the rectangle in the xy-plane $0 \le x \le 1$, $0 \le y \le e$ into two regions A, the area above $y = e^{x^2}$, and B, the area below $y = e^{x^2}$. We have area(A) + area(B) = e and area(B) = $\int_0^1 e^{x^2}\,dx$. Also, by interchanging the rôles of x and y, so $y = e^{x^2}$ becomes $x = \sqrt{\ln y}$, we see that area(A) = $\int_1^e \sqrt{\ln y}\,dy$. Now make the substitution $x = (y-1)/(e-1)$, so $y = 1 + ex - x$, we see that this last integral is $\int_0^1 (e-1)\ln(1+ex-x)\,dx$. We conclude that $\int_0^1 ((e-1)\sqrt{\ln(1+ex-x)} + e^{x^2})\,dx = e$.

7. Suppose $AA'\mathbf{x} = \mathbf{0}$, where $\mathbf{x}$ is a column vector with 5 components. Then $\mathbf{x}'AA'\mathbf{x} = \mathbf{0}$, hence $(A'\mathbf{x})'(A'\mathbf{x}) = \mathbf{0}$. Since all the entries of $A'\mathbf{x}$ are real numbers, we deduce that $A'\mathbf{x} = \mathbf{0}$. Thus A' and AA' have the same null space. By hypothesis $\operatorname{rank}(A) = 5$, hence $\operatorname{rank}(A') = 5$ and we deduce that the null space of A' is 0. Therefore AA' has zero null space and we deduce that $\operatorname{rank}(AA') = 5$. This means that every 5×1 matrix can be written in the form $AA'\mathbf{v}$.

28th VTRMC, 2006, Solutions

1. If we write such an integer n in base 3, then it must end in $200\ldots0$, because n contains no 1's. But then n^2 will end in $100\ldots0$ and we conclude that there are *no* positive integers n for which neither n nor n^2 contain a 1 when written out in base 3.

2. The format of such a sequence must either consist entirely of A's and B's, or must be a block of A's, followed by a single B, followed by a block of C's, followed by a string of A's and B's. In the former case, there are 2^n such sequences. In the latter case, the number of such sequences which have k A's and m C's (where $m \geq 1$) is $2^{n-m-k-1}$. Therefore the number of such sequences with k A's is

$$\sum_{m=1}^{n-k-1} 2^{n-m-k-1} = 2^{n-k-1} - 1.$$

We deduce that the total number of such sequences is

$$\sum_{k=0}^{n-2} (2^{n-k-1} - 1) + 2^n = 2^n - 2 - (n-1) + 2^n = 2^{n+1} - (n+1).$$

We conclude that $S(10) = 2^{11} - 11 = 2037$.

3. From the recurrence relation $F(n) = F(n-1) + F(n-2)$, we obtain

$$\begin{aligned} F(n+5) &= F(n+4) + F(n+3) = 2F(n+3) + F(n+2) \\ &= 3F(n+2) + 2F(n+1) = 5F(n+1) + 3F(n). \end{aligned}$$

Thus $F(n+20) = 3^4 F(n) = F(n) \mod 5$ and we deduce that $F(2006) = F(6) \mod 20$. Since $F(6) = 5F(2) + 3F(1) = 8$. it follows that $F(2006)$ has remainder 3 after being divided by 5. Also $F(n+5) = 2F(n+3) + F(n+2)$ tells us that $F(n+5) = F(n+2) \mod 2$ and hence $F(2006) = F(2) = 1 \mod 2$. We conclude that $F(2006)$ is an odd number which has remainder 3 after being divided by 5, consequently the last digit of $F(2006)$ is 3.

4. Set $c_n = (-b_{3n-2})^n - (-b_{3n-1})^n + (-b_{3n})^n$. Then the series $\sum_{n=1}^{\infty} c_n$ can be written as the sum of the three series

$$\sum_{n=1}^{\infty} (-1)^n b_{3n-2}, \quad \sum_{n=1}^{\infty} -(-1)^n b_{3n-1}, \quad \sum_{n=1}^{\infty} (-1)^n b_{3n}.$$

Since each of these three series is alternating in sign with the absolute value of the terms monotonically decreasing with limit 0, the alternating series test tells us that each of the series is convergent. Therefore the sequence $s_k := \sum_{n=1}^{3k}(-1)^n b_n$ is convergent, with limit S say. Since $\lim_{n\to\infty} b_n = 0$, it follows that $\sum_{n=1}^{\infty}(-1)^n b_n$ is also convergent with sum S.

5. We will model the solution on the method reduction of order; let us try a solution of the form $y = u \sin t$ where u is a function of t. Then $y' = u' \sin t + u \cos t$ and $y'' = u'' \sin t + 2u' \cos t - u \sin t$. Plugging into $y'' + py' + qy = 0$, we obtain $u'' \sin t + u'(2\cos t + p \sin t) + u(p \cos t + q \sin t - \sin t) = 0$. We set

$$u'' \sin t + u'(2\cos t + p \sin t) = 0 \quad \text{and} \quad p \cos t + q \sin t - \sin t = 0.$$

There are many possibilities. We want $u = t^2$ to satisfy $u'' + u'(2\cos t + p \sin t)/\sin t = 0$. Since $u = t^2$ satisfies $u'' - u'/t = 0$, we set $2\cot t + p = -1/t$, and then

$$\begin{aligned} p &= -1/t - 2\cot t, \\ q &= 1 - p\cot t = 1 + \frac{\cot t}{t} + 2\cot^2 t, \\ f &= t^2 \sin t. \end{aligned}$$

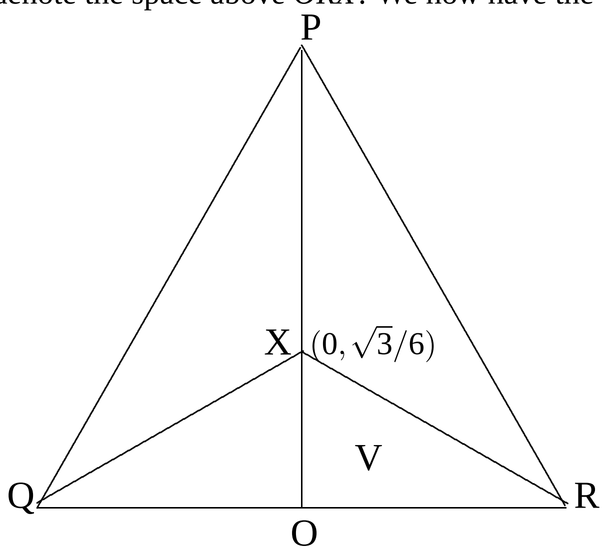
Then p and q are continuous on $(0,\pi)$ (because $1/t$ and $\cot t$ are continuous on $(0,\pi)$), and $y = \sin t$ and $y = f(t)$ satisfy $u'' + pu' + qu = 0$. Also f is infinitely differentiable on the whole real line $(-\infty,\infty)$ and $f(0) = f'(0) = f''(0) = 0$.

6. Let $\beta = \angle QBP$ and $\gamma = \angle QCP$. Then the sine rule for the triangle ABC followed by the double angle formula for sines, and then the addition rules for sines and cosines yields

$$\begin{aligned} \frac{AB + AC}{BC} &= \frac{\sin 2\beta + \sin 2\gamma}{\sin(2\beta + 2\gamma)} = \frac{2\sin(\beta+\gamma)\cos(\beta-\gamma)}{2\sin(\beta+\gamma)\cos(\beta+\gamma)} \\ &= \frac{\cos\beta\cos\gamma + \sin\beta\sin\gamma}{\cos\beta\cos\gamma - \sin\beta\sin\gamma} = \frac{1 + \tan\beta\tan\gamma}{1 - \tan\beta\tan\gamma}. \end{aligned}$$

Since $\tan\beta\tan\gamma = \dfrac{PQ}{BQ}\dfrac{PQ}{QC} = \dfrac{1}{2}$, we see that $\dfrac{AB+AC}{BC} = 3$ and the result is proven.

7. We will call the three spheres A,B,D and let their centers be P,Q,R respectively. Then PQR is an equilateral triangle with sides of length 1. So we will let $O=(0,0,0)$, $P=(0,\sqrt{3}/2,0)$, $Q=(-1/2,0,0)$, $R=(1/2,0,0)$, and $X=(0,1/(2\sqrt{3}),0)$. Then M can be described as the cylinder C with cross-section PQR which is bounded above and below by the spheres A,B,D. Let V denote the space above ORX. We now have the following diagram.



By symmetry, the mass of M is $12\iiint_V z\,dV$. Also above QRX, the mass M is bounded above by the A, which has equation $z=\sqrt{1-x^2-(y-\sqrt{3}/2)^2}$, and the equation of the line XR in the xy-plane is $x+\sqrt{3}y=1/2$. Therefore the mass of M is

$$
\begin{aligned}
&12\int_0^{1/(2\sqrt{3})}\int_0^{1/2-\sqrt{3}y}\int_0^{\sqrt{1-x^2-(y-\sqrt{3}/2)^2}} z\,dz\,dx\,dy\\
&=6\int_0^{1/(2\sqrt{3})}\int_0^{1/2-\sqrt{3}y}(1-x^2-(y-\sqrt{3}/2)^2)\,dx\,dy\\
&=2\int_0^{1/(2\sqrt{3})}[3x-x^3-3x(y-\sqrt{3}/2)^2]_0^{1/2-\sqrt{3}y}dy\\
&=\int_0^{1/(2\sqrt{3})}(1-2\sqrt{3}y)(1/2+4\sqrt{3}y-6y^2)\,dy\\
&=\int_0^{1/(2\sqrt{3})}(12\sqrt{3}y^3-30y^2+3\sqrt{3}y+1/2)\,dy\\
&=[3\sqrt{3}y^4-10y^3+3\sqrt{3}y^2/2+y/2]_0^{\sqrt{3}/6}
\end{aligned}
$$

$$= \sqrt{3}(1/48 - 5/36 + 1/8 + 1/12) = \frac{13}{48\sqrt{3}}.$$

29th VTRMC, 2007, Solutions

1. Let $I = \int \frac{d\theta}{2+\tan\theta}$ We make the substitution $y = \tan\theta$. Then $dy = \sec^2\theta d\theta = (1+y^2)d\theta$ and we find that
$$I = \int \frac{dy}{(1+y^2)(2+y)}.$$
Since $5/((1+y^2)(2+y)) = 1/(2+y) - y/(1+y^2) + 2/(1+y^2)$, we find that
$$5I = \int \frac{dy}{y+2} - \int \frac{ydy}{1+y^2} + \int \frac{2dy}{1+y^2} = \ln(2+y) - (\ln(1+y^2))/2 + 2\tan^{-1} y.$$
Therefore $5I = \ln \frac{2+\tan\theta}{\sec\theta} + 2\theta = \ln(2\cos\theta + \sin\theta) + 2\theta$ and we deduce that
$$I = \frac{2\theta + \ln(2\cos\theta + \sin\theta)}{5},$$
$$\text{hence} \quad \int_0^x \frac{d\theta}{2+\tan\theta} = \frac{2x + \ln(2\cos x + \sin x) - \ln 2}{5}.$$
Plugging in $x = \pi/4$, we conclude that
$$\int_0^{\pi/4} \frac{d\theta}{2+\tan\theta} = \frac{\pi + 2\ln(3/\sqrt{2}) - 2\ln 2}{10} = \frac{\pi + \ln(9/8)}{10}.$$

2. Let $A = 1 + \sum_{n=1}^{\infty}(n+1)/(2n+1)!$ and $B = \sum_{n=1}^{\infty} n/(2n+1)!$, so A and B are the values of the sums in (a) and (b) respectively. Now
$$A + B = \sum_{n=0}^{\infty} 1/(2n)! = (e + e^{-1})/2,$$
$$A - B = \sum_{n=0}^{\infty} 1/(2n+1)! = (e - e^{-1})/2.$$
Therefore $A = e/2$ and $B = 1/(2e)$.

3. We make the substitution $y = e^u$ where u is a function of x to be determined. Then $y' = u'e^u$ and plugging into the given differential equation, we find that $u'e^u = e^u u + e^u e^x$, hence $u' - u = e^x$. This is a first order linear differential equation which can be solved in several ways, for example one method would be to multiply by the integrating factor e^{-x}. We obtain the general solution $u = xe^x + Ce^x$, where C is an arbitrary constant. We are given $y = 1$ when $x = 0$, and then $u = 0$. Therefore $u = xe^x$ and we conclude that $y = e^{xe^x}$.

4. Ceva's theorem applied to the triangle ABC shows that $\frac{AR}{RB}\frac{BP}{PC}\frac{CQ}{QA} = 1$. Since RP bisects $\angle BRC$, we see that $\frac{BP}{PC} = \frac{BR}{RC}$. Therefore $\frac{AR}{RC} = \frac{AQ}{QC}$, consequently $\angle ARQ = \angle QRC$ and the result follows.

5. Let

$$\begin{aligned}
A &= (2+\sqrt{5})^{100}((1+\sqrt{2})^{100} + (1+\sqrt{2})^{-100}) \\
B &= (\sqrt{5}-2)^{100}((1+\sqrt{2})^{100} + (1+\sqrt{2})^{-100}) \\
C &= (\sqrt{5}+2)^{100} + (\sqrt{5}-2)^{100} \\
D &= (\sqrt{2}+1)^{100} + (\sqrt{2}-1)^{100}
\end{aligned}$$

First note that C and D are integers; one way to see this is to use the binomial theorem. Also $\sqrt{2}-1 = (\sqrt{2}+1)^{-1}$. Thus $A+B = CD$ is an integer. Now $\sqrt{5}-2 < 1/4$, $\sqrt{2}+1 < 2.5$ and $\sqrt{2}-1 < 1$. Therefore $0 < B < (5/8)^{100} + (1/4)^{100} < 10^{-4}$. We conclude that there is a positive number $\varepsilon < 10^{-4}$ such that $A+\varepsilon$ is an integer, hence the third digit after the decimal point of the given expression A is 9.

6. Suppose $\det(A^2+B^2) = 0$. Then A^2+B^2 is not invertible and hence there exists a nonzero $n \times 1$ matrix (column vector) u with real entries such that $(A^2+B^2)u = 0$. Then $u'A^2u + u'B^2u = 0$, where u' denotes the transpose of u, a $1 \times n$ matrix. Therefore $(Au)'(Au) + (Bu)'(Bu) = 0$ and we deduce that $u'A = u'B = 0$, consequently $u'(AX+BY) = 0$. This shows that $\det(AX+BY) = 0$, a contradiction and the result follows.

7. We claim that $x^{1/(\ln(\ln x))^2} > (\ln x)^2$ for large x. Indeed by taking logs, we need $(\ln x)/(\ln(\ln x))^2 > 2\ln(\ln(x))$, that is $\ln x > 2(\ln(\ln x))^3$. So by making the substitution $y = \ln x$, we want $y > 2(\ln y)^3$, which is true for y large. It now follows that for large n,

$$n^{-(1+\frac{1}{(\ln(\ln n))^2})} = \frac{1}{n}\frac{1}{n^{1/(\ln(\ln n))^2}} < \frac{1}{n(\ln n)^2}.$$

However $\sum 1/(n(\ln n)^2)$ is well known to be convergent, by using the integral test, and it now follows from the basic comparison test that the given series is also convergent.

30th VTRMC, 2008, Solutions

1. Write $f(x) = xy^3 + yz^3 + zx^3 - x^3y - y^3z - z^3x$. First we look for local maxima, so we need to solve $\partial f/\partial x = \partial f/\partial y = \partial f/\partial z = 0$. Now $\partial f/\partial x = y^3 + 3x^2z - z^3 - 3x^2y$. If $y = z$, then $f(x,y,z) = 0$ and this is not a maximum. Thus we may divide by $y - z$ and then $\partial f/\partial x = 0$ yields $y^2 + yz + z^2 = 3x^2$. Similarly $x^2 + xz + z^2 = 3y^2$ and $x^2 + xy + y^2 = 3z^2$. Adding these three equations, we obtain $(x-y)^2 + (y-z)^2 + (z-x)^2 = 0$, which yields $x = y = z$. This does not give a maximum, because $f = 0$ in this case, and we conclude that the maximum of f must occur on the boundary of the region, so at least one of x,y,z is 0 or 1.

   Let's look at f on the side $x = 0$. Here $f = yz^3 - y^3z$ and $0 \le y \le 1$, $0 \le z \le 1$. To find local maxima, we solve $\partial f/\partial y = \partial f/\partial z = 0$. This yields $y = z = 0$ and $f = 0$, which is not a maximum, so the maximum occurs on the edges of the region considered. If y or $z = 0$, we get $f = 0$ which is not a maximum. If $y = 1$, then $f = z^3 - z \le 0$, which won't give a maximum. Finally if $z = 1$, then $f = y - y^3$. Since $df/dy = 1 - 3y^2$, we see that f has a maximum at $y = 1/\sqrt{3}$. This gives that the maximum value of f on $x = 0$ is $1/\sqrt{3} - 1/\sqrt{3}^3 = 2\sqrt{3}/9$.

   Similarly if y or $z = 0$, the maximum value of f is $2\sqrt{3}/9$. Now let's look at f on the side $x = 1$. Here $f = y^3 + yz^3 + z - y - y^3z - z^3$. Again we first look for local maxima: $\partial f/\partial y = 3y^2 + z^3 - 1 - 3y^2z$. Then $\partial f/\partial y = 0$ yields either $z = 1$ or $3y^2 = z^2 + z + 1$. If $z = 1$, then $f = 0$ which is not a maximum, so $3y^2 = z^2 + z + 1$. Similarly $3z^2 = y^2 + y + 1$. Adding these two equations, we find that $y^2 - y/2 + z^2 - z/2 = 1$. Thus $(y - 1/2)^2 + (z - 1/2)^2 = 3/2$. This has no solution in the region considered $0 \le y \le 1$, $0 \le z \le 1$. Thus f must have a maximum on one of the edges. If y or z is 0, then we are back in the previous case. On the other hand if y or z is 1, then $f = 0$, which is not a maximum.

   We conclude that the maximum value of f on $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$ is $2\sqrt{3}/9$.

2. For each positive integer n, let $f(n)$ denote the number of sequences of 1's and 3's that sum to n. Then $f(n+3) = f(n+2) + f(n)$, and we have $f(1) = 1$, $f(2) = 1$, and $f(3) = 2$. Thus $f(4) = f(3) + f(1) = 3$, $f(5) = f(4) + f(2) = 4$, $f(6) = 6, \ldots, f(15) = 189$, $f(16) = 277$. Thus the number of sequences required is 277.

3. Let R denote the specified region, i.e. $\{(x,y) \mid x^4+y^4 \le x^2-x^2y^2+y^2\}$. Then R can be described as the region inside the curve $x^4+x^2y^2+y^4 = x^2+y^2$ ($(x,y) \neq (0,0)$). This can be rewritten as

$$(x^2+y^2-xy)(x^2+y^2+xy) = x^2+y^2.$$

Now change to polar coordinates: write $x = r\cos\theta$, $y = r\sin\theta$; then the equation becomes $(r^2 - r^2\cos\theta\sin\theta)(r^2+r^2\cos\theta\sin\theta) = r^2$. Since $r \neq 0$ and $2\cos\theta\sin\theta = \sin 2\theta$, we now have $r^2(1-\frac{1}{4}\sin^2 2\theta) = 1$. Therefore the area A of R is

$$\begin{aligned}\iint_R r\,dr\,d\theta &= \int_0^{2\pi}\int_0^{(1-\frac{1}{4}\sin^2 2\theta)^{-1/2}} r\,dr\,d\theta = \int_0^{2\pi} \frac{d\theta}{2(1-\frac{1}{4}\sin^2 2\theta)} \\ &= \int_0^{\pi/4} \frac{16\,d\theta}{3+\cos^2 2\theta} = \int_0^{\pi/4} \frac{16\sec^2 2\theta\,d\theta}{4+3\tan^2 2\theta}.\end{aligned}$$

Now make the substitution $2z = \sqrt{3}\tan 2\theta$, so $dz = \sqrt{3}\sec^2 2\theta\,d\theta$ and we obtain

$$A = \frac{4}{\sqrt{3}}\int_0^\infty \frac{dz}{1+z^2} = 2\pi/\sqrt{3}.$$

4. Ceva's theorem applied to the triangle ABC shows that $\dfrac{AP}{PB}\dfrac{BM}{MC}\dfrac{CN}{NA} = 1$. Since $BM = MC$, we see that $\dfrac{AP}{PB} = \dfrac{AN}{NC}$ and we deduce that PN is parallel to BC. Therefore $\angle NPX = \angle PCB = \angle NAX$ and we conclude that $APXN$ is a cyclic quadrilateral. Since that opposite angles of a cyclic quadrilateral sum to 180°, we see that $\angle APX + \angle XNA = 180^\circ$, and the result follows.

5. Let $\mathscr{T} = \{a_n \mid n \in \mathbb{N}\}$ and for t a positive number, let $A_t = \{n \in \mathbb{N} \mid a_n \ge t\}$. Since $\sum a_n = 1$ and $a_n \ge 0$ for all n, we that if $\delta > 0$, then there are only finitely many numbers in $\mathscr{T}$ greater than δ, and also A_t is finite. Thus we may label the nonzero elements of $\mathscr{T}$ as $t_1, t_2, t_3, \ldots$, where $t_1 > t_2 > t_3 > \cdots > 0$. We shall use the notation $X \triangle Y$ to indicate the symmetric difference $\{X \setminus Y \cup Y \setminus X\}$ of two subsets X, Y.

   Consider the sum
$$\sum_{i \ge 1} (t_i - t_{i+1})|A_{t_i} \triangle \pi^{-1}A_{t_i}|.$$

   Note that $n \in A_t \setminus \pi^{-1}A_t$ if and only if $a_n \ge t > a_{\pi n}$, and $n \in \pi^{-1}A_t \setminus A_t$ if and only if $a_n < t \le a_{\pi n}$. Write $a_n = t_p$ and $a_{\pi n} = t_q$. We have three cases to examine:

(a) $a_n = a_{\pi n}$. Then n does not appear in the above sum.

(b) $a_n > a_{\pi n}$. Then $p < q$ and n is in $A_{t_r} \setminus \pi^{-1}A_{t_r}$ whenever $t_p \geq t_r > t_q$, that is $q > r \geq p$ and we get a contribution $(t_p - t_{p+1}) + (t_{p+1} - t_{p+2}) + \cdots + (t_{q-1} - t_q) = t_p - t_q = a_n - a_{\pi n} = |a_n - a_{\pi n}|$.

(c) $a_n < a_{\pi n}$. Then $p > q$ and n is in $\pi^{-1}A_{t_r} \setminus A_{t_r}$ whenever $t_q \geq t_r > t_p$, that is $p > r \geq q$ and we get a contribution $(t_q - t_{q+1}) + (t_{q+1} - t_{q+2}) + \cdots + (t_{p-1} - t_p) = t_q - t_p = a_{\pi n} - a_n = |a_n - a_{\pi n}|$.

We conclude that

$$\sum_{n=1}^{\infty} |a_n - a_{\pi n}| = \sum_{i \geq 1} (t_i - t_{i+1}) |A_{t_i} \triangle \pi A_{t_i}|,$$

because $|A_{t_i} \triangle \pi^{-1} A_{t_i}| = |A_{t_i} \triangle \pi A_{t_i}|$. Similarly

$$\sum_{n=1}^{\infty} |a_n - a_{\rho n}| = \sum_{i \geq 1} (t_i - t_{i+1}) |A_{t_i} \triangle \rho A_{t_i}|,$$

and we deduce that

$$\sum_{n=1}^{\infty} (|a_n - a_{\pi n}| + \sum_{n=1}^{\infty} |a_n - a_{\rho n}|) = \sum_{i \geq 1} (t_i - t_{i+1})(|A_{t_i} \triangle \pi A_{t_i}| + |A_{t_i} \triangle \rho A_{t_i}|).$$

Therefore $\sum_{i \geq 1} (t_i - t_{i+1})(|A_{t_i} \triangle \pi A_{t_i}| + |A_{t_i} \triangle \rho A_{t_i}|) < \varepsilon$. We also have $\sum_{i \geq 1} (t_i - t_{i+1}) |A_{t_i}| = 1$. Therefore for some i, we must have $|A_{t_i} \triangle \pi A_{t_i}| + |A_{t_i} \triangle \rho A_{t_i}| < \varepsilon |A_{t_i}|$ and the result follows.

6. Multiply $a^4 - 3a^2 + 1$ by b and subtract $(a^3 - 3a)(ab - 1)$ to obtain $a^3 - 3a + b$. Now multiply by b and subtract $a^2(ab - 1)$ to obtain $a^2 - 3ab + b^2$. Thus we want to know when $ab - 1$ divides $(a - b)^2 - 1$, where a, b are positive integers. We cannot have $a = b$, because $a^2 - 1$ does not divide -1. We now assume that $a > b$.

   Suppose $ab - 1$ does divide $(a - b)^2 - 1$ where a, b are positive integers. Write $(a - b)^2 - 1 = k(ab - 1)$, where k is an integer. Since $(a - b)^2 - 1 \geq 0$, we see that k is nonnegative. If $k = 0$, then we have $(a - b)^2 = 1$, so $a - b = \pm 1$. In this case, $ab - 1$ does divide $a^4 - 3a^2 + 1$, because $a^4 - 3a^2 + 1 = (a^2 + a - 1)(a^2 - a - 1)$. We now assume that $k \geq 1$.

   Now fix k and choose a, b with b as small as possible. Then we have $a^2 + a(-2b - kb) + b^2 + k - 1 = 0$. Consider the quadratic equation $x^2 + x(-2b -$

$kb) + b^2 + k - 1 = 0$. This has an integer root $x = a$. Let v be its other root. Since the sum of the roots is $2b + kb$, we see that v is also an integer. Also $av = b^2 + k - 1$. Since $b, k \geq 1$, we see that v is also positive. We want to show that $v < b$; if this was not the case, then we would have $b^2 + k - 1 \geq ab$, that is $k \geq ab - b^2 + 1$. We now obtain

$$(a-b)^2 - 1 \geq (ab - b^2 + 1)(ab - 1).$$

This simplifies to $a^2 - ab \geq (ab - b^2 + 1)ab$, that is $a - b \geq (ab - b^2 + 1)b$ and we obtain $(a-b)(b^2-1) + b \leq 0$, which is not the case. Thus $v < b$ and we have $v^2 + v(-2b - k) + b^2 + k - 1 = 0$. Set $u = b$. Then we have $(u-v)^2 - 1 = k(uv - 1)$, where u, v are positive and $v < b$. By minimality of b, we conclude that there are no a, b such that $(a-b)^2 - 1 = k(ab - 1)$.

Putting this altogether, the positive integers required are all a, b such that $b = a \pm 1$.

7. Note that for fixed $x > 1$, the sequence $1/f_n(x)$ is decreasing with respect to n and positive, so the given limit exists which means that g is well-defined. Next we show that $g(e^{1/e}) \geq 1/e$, equivalently $\lim_{n\to\infty} f_n(e^{1/e}) \leq e$. To do this, we show by induction that $f_n(e^{1/e}) \leq e$ for all positive integers n. Certainly $f_1(e^{1/e}) = e^{1/e} \leq e$. Now if $f_n(e^{1/e}) \leq e$, then

$$f_{n+1}(e^{1/e}) = (e^{1/e})^{f_n(e^{1/e})} \leq (e^{1/e})^e = e,$$

so the induction step passes and we have proven that $g(e^{1/e}) \geq 1/e$.

We now prove that $g(x) = 0$ for all $x > e^{1/e}$; this will show that g is discontinuous at $x = e^{1/e}$. We need to prove that $\lim_{n\to\infty} f_n(x) = \infty$. If this is not the case, then we may write $\lim_{n\to\infty} f_n(x) = y$ where y is a positive number > 1. We now have

$$y = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} f_{n+1}(x) = x^{\lim_{n\to\infty} f_n(x)} = x^y.$$

Therefore $\ln y = y \ln x$ and $x = y^{1/y}$. Since $(dx/dy)/x = (1 - \ln y)/y^2$, we see by considering the graph of $y^{1/y}$ that it reaches its maximum when $y = e$, and we deduce that $x \leq e^{1/e}$. This is a contradiction and we conclude that $\lim_{n\to\infty} f_n(x) = 0$. Thus we have shown that $g(x)$ is discontinuous at $x = e^{1/e}$.

31st VTRMC, 2009, Solutions

1. Let x and y meters denote the distance between the walker and the jogger when the dog returns to the walker for the n th and $n+1$ st times respectively. Then starting at the walker for the n th time, it will take the dog x seconds to reach the jogger, whence the jogger will be distance $2x$ from the walker, and then it will take the dog a further $2x/4$ seconds to return to the walker. Thus $y = x + x + 2x/4 = 5x/2$, and also the time taken for the dog to return to the walker will be $x + 2x/4 = 3x/2$ seconds. It follows that the total distance travelled by the dog to return to the walker for the n th time is

$$3 * \frac{3}{2}(1 + 5/2 + (5/2)^2 + \cdots + (5/2)^{n-1}) = \frac{9((5/2)^n - 1)}{2(5/2 - 1)}$$

meters. Therefore $f(n) = 3((5/2)^n - 1)$.

2. By examining the numbers 5,10,15,20,25,30,35,40 (at most 40 and divisible by 5), we see that the power of 5 dividing 40! is 9. Also it is easy to see that the power of 2 dividing 40! is at least 9. Therefore 10^9 divides 40! and we deduce that $p = q = r = s = t = u = v = w = x = 0$.

   Next note that 999999 divides 40!. This is because $999999 = 9 * 111111 = 9 * 111 * 1001 = 9 * 3 * 37 * 11 * 91 = 27 * 7 * 11 * 13 * 37$. Since $999999 = 10^6 - 1$, it follows that if we group the digits of 40! in sets of six starting from the units digit and working right to left, then the sum of these digits will be divisible by 999999. Therefore $abcdef + 283247 + 897734 + 345611 + 269596 + 115894 + 272000$ is divisible by 999999, that is

$$abcdef + 2184082 = 999999y$$

   for some integer y. Clearly $y = 3$ and we conclude that $abcdef = 815915$.

3. By symmetry, $f(x)$ is twice the integral over the triangle bounded by $u = v$, $v = 0$ and $u = x$, so $f(x) = 2\int_0^x \int_0^u e^{u^2v^2}\,dv\,du$ and it follows that $f'(x) = 2\int_0^x e^{x^2v^2}\,dv$. Set $t = xv$, so $dv = dt/x$ and $f'(x) = (2/x)\int_0^{x^2} e^{t^2}\,dt$. Therefore

$$f''(x) = \frac{-2}{x^2}\int_0^{x^2} e^{t^2}\,dt + 2x\frac{2}{x}e^{x^4}.$$

   We conclude that $xf''(x) + f'(x) = 4xe^{x^4}$ and hence $2f''(2) + f'(2) = 8e^{16}$.

4. Let the common tangent at X meet AQ and PB at Y and Z respectively. Then we have $\angle APX = \angle AXY$ (because YZ is tangent to α at X) $= \angle ZXB$ (because opposite angles are equal) $= \angle PQB$ (because YZ is tangent to β at X). It follows that AP is parallel to QB, as required.

5. Let $a_3X^3 + a_2X^2 + a_1X + a_0I$ denote the minimum polynomial of A, where $a_i \in \mathbb{C}$ and at least one of the a_i is 1. It is the unique monic polynomial $f(X)$ of minimal degree such that $f(A) = 0$. Since $AB = 0$, we see that A is not invertible, hence 0 is an eigenvalue of A, consequently $a_0 = 0$. Set $D = a_3A^2 + a_2A + a_1I$. Then $D \neq 0$, because $a_3X^3 + a_2X^2 + a_1X$ is the minimum polynomial of A. Also $AD = DA = a_3A^3 + a_2A^2 + a_1A + a_0I = 0$ and the result is proven.

6. Suppose $n^4 - 7n^2 + 1$ is a perfect square; we may assume that n is positive. We have $n^4 - 7n^2 + 1 = (n^2 - 3n + 1)(n^2 + 3n + 1)$. Suppose p is a prime that divides both $n^2 - 3n + 1$ and $n^2 + 3n + 1$. Then $p \big| 6n$ and we see that p divides 2,3 or n. By inspection, none of these are possible. Thus no prime can divide both $n^2 - 3n + 1$ and $n^2 + 3n + 1$, consequently both these numbers are perfect squares, in particular $n^2 + 3n + 1$ is a perfect square. By considering this mod 3, we see that $n \equiv 0 \mod 3$, so we may write $n = 3m$ where m is a positive integer. Thus $9m^2 + 9m + 1$ is a perfect square. However $(3m+1)^2 < 9m^2 + 9m + 1 < (3m+2)^2$, a contradiction and the result follows.

7. It is easy to check that $f(x) = ax + b$ satisfies the differential equation for arbitrary constants a, b; however it cannot satisfy the numerical condition. We need to find another solution to the differential equation. Since the equation is linear, it's worth trying a solution in the form $f(x) = e^{rx}$, where r is a complex constant. Plugging into the differential equation, we obtain $re^{rx} = e^{r(x+1)} - e^{rx}$, which simplifies to $e^r = 1 + r$. We need a solution with $r \neq 0$. Write $r = a + ib$, where $a, b \in \mathbb{R}$. We want to solve $e^{a+ib} = 1 + a + ib$. By equating the real and imaginary parts, we obtain

$$
\begin{aligned}
e^a \cos b &= 1 + a \\
e^a \sin b &= b.
\end{aligned}
$$

Eliminating a, we find that $g(b) := -b\cos b + (1 + \log b - \log(\sin b))\sin b = 0$. Since $\lim_{x \to 0+} x \log x = 0$, we see that $\lim_{b \to 2\pi+} (\log(\sin b)) \sin b = 0 =$

$\lim_{b\to 3\pi-}(\log(\sin b))\sin b$. Therefore

$$\lim_{b\to 2\pi+} g(b) = -2\pi < 0 \text{ and } \lim_{b\to 3\pi-} g(b) = 3\pi > 0.$$

Since g is continuous on $(2\pi, 3\pi)$, we see that there exists $q \in (2\pi, 3\pi)$ such that $g(q) = 0$. Set $p = \log q - \log(\sin q)$. Then $e^{p+iq} + p + iq = 1$. Since the differential equation is linear, the real and imaginary parts of f will also satisfy the differential equation, and in particular $f(x) := e^{px}\sin qx$ will satisfy the differential equation. A routine calculation shows that $f''(0) = 2pq \neq 0$, and so the answer to the question is “yes”.

Remark It can be shown that $p \approx 2.09$ and $q \approx 7.46$.

32nd VTRMC, 2010, Solutions

1. It is easily checked that 101 is a prime number (divide 101 by the primes whose square is less than 101, i.e. the primes ≤ 7). Therefore for $1 \leq r \leq 100$, we may choose a positive integer q such that $rq \equiv 1 \mod 101$. Since $(I+A+\cdots+A^{100})(I-A) = I-A^{101}$, we see that $A^{101} = I$, in particular A is invertible with inverse A^{100}. Suppose $1 \leq n \leq 100$ and set $r = 101-n$. Then $1 \leq r \leq 100$ and $A^n + \cdots + A^{100}$ is invertible if and only if $I + \cdots + A^{r-1}$ is invertible. We can think of $I + \cdots + A^{r-1}$ as $(I-A^r)/(I-A)$, which should have inverse $(I-A)/(I-A^r)$. However $A = (A^r)^q$ and so $(I-A)/(I-A^r) = I + A^r + \cdots + (A^r)^{q-1}$. It is easily checked that

$$(I + \cdots + A^{r-1})(I + A^r + \cdots + (A^r)^{q-1}) = I.$$

   It follows that $A^n + \cdots + A^{100}$ is invertible for all positive integers $n \leq 100$. We conclude that $A^n + \cdots + A^{100}$ has determinant ± 1 for all positive integers $n \leq 100$.

2. First we will calculate $f_n(75) \mod 16$. Note that if a, b are odd positive integers and $a \equiv b \mod 16$, then $a^a \equiv b^b \mod 16$. Also $3^3 \equiv 11 \mod 16$ and $11^{11} \equiv 3 \mod 16$. We now prove by induction on n that $f_{2n-1}(75) \equiv 11 \mod 16$ for all $n \in \mathbb{N}$. This is clear for $n = 1$ so suppose $f_{2n-1}(75) \equiv 11 \mod 16$ and set $k = f_{2n-1}(75)$ and $m = f_{2n}(75)$. Then

$$f_{2n}(75) \equiv k^k \equiv 11^{11} \equiv 3 \mod 16$$
$$f_{2n+1}(75) \equiv m^m \equiv 3^3 \equiv 11 \mod 16$$

   and the induction step is complete. We now prove that $f_n(a) \equiv f_{n+2}(a) \mod 17$ for all $a, n \in \mathbb{N}$ with a prime to 17 and n even. In fact we have

$$f_{n+1}(a) \equiv a^3 \mod 17, \quad f_{n+2}(a) \equiv (a^3)^{11} \equiv a \mod 17.$$

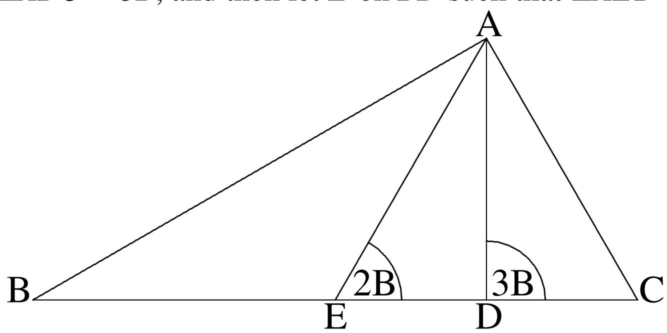
   Thus $f_{100}(75) \equiv f_2(75) \mod 17$. Therefore $f_{100}(75) \equiv 7^{11} \equiv 14 \mod 17$.

3. First note the $e^{2\pi i/7}$ satisfies $1 + x + \cdots + x^6 = 0$, so by taking the real part, we obtain $\sum_{n=0}^{n=6} \cos 2n\pi/7 = 0$. Since $\cos 2\pi/7 = \cos 12\pi/7$, $\cos 4\pi/7 = \cos 10\pi/7 = -\cos 3\pi/7$ and $\cos 6\pi/7 = \cos 8\pi/7 = -\cos \pi/7$, we see that $1 - 2\cos \pi/7 + 2\cos 2\pi/7 - 2\cos 3\pi/7 = 0$.

   Observe that if $1 - 2\cos\theta + 2\cos 2\theta - 2\cos 3\theta = 0$, then by using $\cos 2\theta = 2\cos^2\theta - 1$ and $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, we find that $\cos\theta$ satisfies

$8x^3-4x^2-4x+1=0$. Thus in particular $\cos\pi/7$ satisfies this equation. Next note that $1-2\cos 3\pi/7+2\cos 6\pi/7-2\cos 9\pi/7 = 1-2\cos 3\pi/7-2\cos\pi/7+2\cos 2\pi/7$, so $\cos 3\pi/7$ is also a root of $8x^3-4x^2-4x+1$. Finally since the sum of the roots of this equation is 1/2, we find that $-\cos 2\theta$ is also a root. Thus the roots of $8x^3-4x^2-4x+1$ are $\cos\pi/7$, $-\cos 2\pi/7$, $\cos 3\pi/7$.

4. The equation $4A+3C=540^\circ$ tells us that $A=3B$. Let D on BC such that $\angle ADC = 3B$, and then let E on BD such that $\angle AED = 2B$.



Since triangles ABD and AED are similar, we see that

$$\frac{BD}{AD}=\frac{AD}{ED}=\frac{AB}{AE}.$$

Also $BE=AE$ because $B=\angle BAE$, and $BE=BD-BE$. We deduce that $BD^2=AD^2+AB\cdot AD$. Since $BD=BC-CD$, we conclude that $(BC-CD)^2=AD^2+AB\cdot AD$.

Next triangles ABC and ADC are similar, consequently

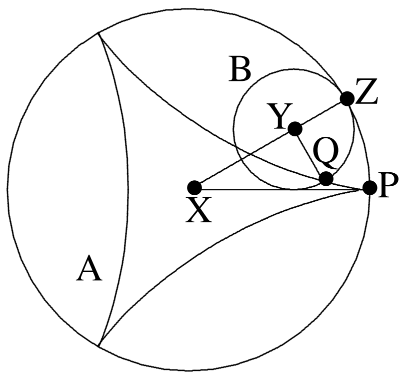
$$\frac{BC}{AC}=\frac{AB}{AD}=\frac{AC}{CD}.$$

Thus $AD=AB\cdot AC/BC$ and $CD=AC^2/BC$. We deduce that

$$(a-b^2/a)^2=c^2b^2/a^2+c^2b/a.$$

Therefore $(a^2-b^2)^2=bc^2(a+b)$ and the result follows.

5. Let X denote the center of A, let Y denote the center of B, let Z be where A and B touch (so X,Y,Z are collinear), and let $\theta=\angle PXY$. Note that YQ makes an angle 2θ downwards with respect to the horizontal, because $\angle QYZ=3\theta$.



Choose (x,y)-coordinates such that X is the origin and XP is on the line $y=0$. Let (x,y) denote the coordinates of Q. Then we have

$$x = 2\cos\theta + \cos 2\theta$$
$$y = 2\sin\theta - \sin 2\theta.$$

By symmetry the area above the x-axis equals the area below the x-axis (we don't really need this observation, but it may make things easier to follow). Also θ goes from 2π to 0 as circle B goes round circle A. Therefore the area enclosed by the locus of Q is

$$\begin{aligned}
2\int_\pi^0 y\frac{dx}{d\theta}\,d\theta &= 2\int_\pi^0 (2\sin\theta - \sin 2\theta)(-2\sin\theta - 2\sin 2\theta)\,d\theta \\
&= 2\int_0^\pi (4\sin^2\theta + 2\sin\theta\sin 2\theta - 2\sin^2 2\theta)\,d\theta \\
&= \int_0^\pi (4 - 4\cos 2\theta + 2\cos\theta - 2\cos 3\theta - 2 + 2\cos 4\theta)\,d\theta = 2\pi.
\end{aligned}$$

6. Note that if $0 < x, y < 1$, then $0 < 1 - y/2 < 1$ and $0 < x(1-y/2) < 1$, and it follows that (a_n) is a positive monotone decreasing sequence consisting of numbers strictly less that 1. This sequence must have a limit z where $0 \le z \le 1$. In particular $a_{n+2} - a_{n+1} = a_n a_{n+1}/2$ has limit 0, so $\lim_{n\to\infty} a_n a_{n+1} = 0$. It follows that $z = 0$.

   Set $b_n = 1/a_n$. Then $b_{n+2} = b_{n+1}/(1 - a_n/2) = b_{n+1}(1 + a_n/2 + O(a_n^2))$. Therefore $b_{n+2} - b_{n+1} = b_{n+1}(a_n/2 + O(a_n^2))$. Also

$$a_n/a_{n+1} = (1 - a_{n-1}/2)^{-1} = 1 + O(a_n)$$

and we deduce that $b_{n+2}-b_{n+1}=b_n(1+O(a_n))(a_n/2+O(a_n^2))$. Therefore $b_{n+2}-b_{n+1}=1/2+O(a_n)$. Thus given $\varepsilon>0$, there exists $N\in\mathbb{N}$ such that $|b_{n+1}-b_n-1/2|<\varepsilon$ for all $n>N$. We deduce that if k is a positive integer, then $|b_{n+k}/k-b_n/k-1/2|<\varepsilon$. Thus for k sufficiently large, $|b_{n+k}/(n+k)-1/2|<2\varepsilon$. We conclude that $\lim_{n\to\infty}b_n/n=1/2$ and hence $\lim_{n\to\infty}na_n=2$.

7. It will be sufficient to prove that $\sum_{n=1}^{\infty}\frac{n^2}{1/a_1^2+\cdots+1/a_n^2}$ is convergent. Note we may assume that (a_n) is monotonic decreasing, because rearranging the terms in series $\sum a_n$ does not affect its convergence, whereas the terms of the above series become largest when (a_n) is monotonic decreasing. Next observe that if $\sum_{n=1}^{\infty}a_n=S$, then $a_n\le S/n$ for all positive integers n. Now consider $\frac{(2n)^2}{1/a_1^2+\cdots+1/a_{2n}^2}$. This is $\le\frac{(2n)^2S}{1/a_1+2/a_2+\cdots+2n/a_{2n}}\le\frac{4n^2S}{n^2/a_n}=4Sa_n$. The result follows.

33rd VTRMC, 2011, Solutions

1. Write $I = \int_1^4 \frac{x-2}{(x^2+4)\sqrt{x}}\,dx$ and make the substitution $y = \sqrt{x}$. Then $dx = 2y\,dy$ and $I = \int_1^2 \frac{2y^2-4}{y^4+4}\,dy$. Now $y^4+4 = (y^2-2y+2)(y^2+2y+2)$ and using partial fractions, we find

$$\frac{2y^2-4}{y^4+4} = \frac{y-1}{y^2-2y+2} - \frac{y+1}{y^2+2y+2}.$$

It follows that $2I = [\ln(y^2-2y+2) - \ln(y^2+2y+2)]_1^2 = \ln 2 - \ln 1 - (\ln 10 - \ln 5) = 0$, so the answer is 0.

2. The first few terms (starting with a_{-1}) are $-1, 0, 3, 8$, so it seems reasonable that $a_n = n^2-1$; let us prove this by induction on n. The result is certainly true if $n = 0$ or 1. So suppose $a_r = r^2-1$ for $r \le n$. Then

$$\begin{aligned} a_{n+1} &= (n^2-1)^2 - (n+1)^2\big((n-1)^2-1\big) - 1 \\ &= n^4 - 2n^2 + 1 - (n^4 - 2n^2 + 1) + n^2 + 2n + 1 - 1 \\ &= n^2 + 2n = (n+1)^2 - 1. \end{aligned}$$

and the induction is complete. Thus $a_n = n^2-1$ for all $n \in \mathbb{N}$ and we deduce that $a_{100} = 9999$.

3. Let $S = \sum_{k=1}^n \frac{k^2-2}{(k+2)!}$ where $n \in \mathbb{N}$. We have

$$\frac{k^2-2}{(k+2)!} = \frac{1}{k!} - \frac{1}{(k+2)!} - 3\Big(\frac{1}{(k+1)!} - \frac{1}{(k+2)!}\Big).$$

By telescoping series, we find that $S = 1 + 1/2 - 3/2 - 1/(n+1)! - 1/(n+2)! + 3/(n+2)! = -1/(n+1)! + 2/(n+2)!$. Since $\lim_{n\to\infty} 1/n! = 0$, it follows that the required sum is 0.

4. We repeatedly use the Chinese remainder theorem without further comment. Let $b \in \mathbb{Z}$. We claim that $|\{[br] \mid r \in \mathbb{Z}\}| = mn/(mn,b)$. Set $k = (mn,b)$. Then $\{[kr] \mid r \in \mathbb{Z}\} = \{[kr] \mid r = 1,2,\dots,mn/k\}$ so $|\{[kr] \mid r \in \mathbb{Z}\}| = mn/k$. It follows that $|\{[br] \mid r \in \mathbb{Z}\}| \le mn/k$. On the other hand there exists $c \in \mathbb{Z}$ such that and $bc \equiv k \mod mn$. We conclude that $|\{[br] \mid r \in \mathbb{Z}\}| \ge m$ and the claim is established.

Therefore $mn/(a,mn) = m$. Thus $(a,mn) = n$, hence $n \mid a$ and we may write $a = sn$ where $s \in \mathbb{Z}$; clearly $(s,m) = 1$. Now if $t \in \mathbb{Z}$, then $(s+tm)n \equiv a \mod mn$, i.e. $[n(s+tm)] = [a]$. Since $(s,m) = 1$, we may choose t so that $(s+tm,mn) = 1$. The result follows by setting $q = s+tm$.

5. Set $f(x) = x^{1+1/x} - x - \ln x$. We first show that $\lim_{x\to\infty} f(x) = 0$. Note that $\lim_{x\to\infty} x^{1/x} = 1$ because $\lim_{x\to\infty} \ln\bigl(x^{1/x}\bigr) = \lim_{x\to\infty}(\ln x)/x = 0$. Thus $\lim_{x\to\infty} f(x)/x = 0$. Now we apply l'hôpital's rule. We obtain

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{f(x)/x}{1/x} \\
&= \lim_{x\to\infty} \frac{(f(x)/x)'}{-1/x^2} \\
&= \lim_{x\to\infty} -x^2(x^{1/x} - 1 - (\ln x)/x)' \\
&= \lim_{x\to\infty} -x^2\bigl(x^{1/x}/x^2 - (x^{1/x}\ln x)/x^2 - 1/x^2 + (\ln x)/x^2\bigr) \\
&= \lim_{x\to\infty} (x^{1/x} - 1)(\ln x - 1).
\end{aligned}$$

Thus we need to prove $\lim_{x\to\infty}(x^{1/x} - 1)\ln x = 0$. Again we apply l'hôpital's rule.

$$\begin{aligned}
\lim_{x\to\infty} (x^{1/x} - 1)\ln x &= \lim_{x\to\infty} \frac{x^{1/x} - 1}{1/\ln x} \\
&= \lim_{x\to\infty} \frac{x^{1/x}(1-\ln x)/x^2}{-1/(x(\ln x)^2)} \\
&= \lim_{x\to\infty} \frac{(\ln x)^3}{x} = 0.
\end{aligned}$$

Thus $\lim_{x\to\infty} f(x) = 0$ and we deduce that $\lim_{x\to\infty} f(2x) - f(x) = 0$. But

$$\begin{aligned}
f(2x) - f(x) &= (2x)^{1+1/(2x)} - 2x - \ln 2x - x^{1+1/x} + x + \ln x \\
&= (2x)^{1+1/(2x)} - x^{1+1/x} - x - \ln 2
\end{aligned}$$

and we deduce that $\lim_{x\to\infty}(2x)^{1+1/(2x)} - x^{1+1/x} - x = \ln 2$.

6. Let $<$ indicate the usual order on $\mathbb{Q}$; thus $<$ is an asymmetric relation on $\mathbb{Q}$. Define $T = S \times \mathbb{Q}$ and let $\prec$ denote the lexicographic asymmetric relation on T, that is $(a,p) \prec (b,q)$ if and only if $a < b$ or $a = b$ and $p < q$. It is

easily checked that $\prec$ is asymmetric. Also we may identify S with $S \times 0$ via $s \mapsto (s,0)$, and then the restriction of $\prec$ to S is $<$. Now suppose $(a,p) \prec (b,q)$. If $a < b$, then $(a,p) \prec (a,p+1) \prec (b,q)$. On the other hand if $a = b$, then $p < q$ and $(a,p) \prec (a,(p+q)/2) \prec (a,q)$ and the result is proven.

7. Assume that the roots $x_1,\ldots,x_{100}$ are all real. By the Viete relations we have $\sum_{i=1}^{100} x_i = -20$ and $\sum_{i<j} x_i x_j = 198$. Therefore $\sum_{i=1}^{100} x_i^2 = (-20)^2 - 2 * 198 = 4$ and we deduce that $(\sum_{i=1}^{100} x^i)^2 = 100 \sum_{i=1}^{100} x_i^2$. Now apply the Cauchy-Schwartz triangle inequality, that is $(\sum_i x_i y_i)^2 \leq \sum_i x_i^2 \sum_i y_i^2$ with equality if and only if one of $(x_i),(y_i)$ is a scalar multiple of the other, in particular taking $y_i = 1$ for all i, we obtain $(\sum_{i=1}^{100} x_i)^2 \leq 100 \sum_{i=1}^{100} x_i^2$ with equality if and only if (x_i) is a scalar multiple of $(1,\ldots,1)$, i.e. all the x_i are equal. We deduce that all the x_i are equal. But the product of the roots is 1, consequently all the x_i are 1 or all the x_i are -1, which contradicts the fact that $\sum_{i=1}^{100} x_i = -20$.

34th VTRMC, 2012, Solutions

1. Let I denote the value of the integral. We make the substitution $y = \pi/2 - x$. Then $dx = -dy$, and as x goes from 0 to $\pi/2$, y goes from $\pi/2$ to 0. Also $\sin(\pi/2 - x) = \cos x$ and $\cos(\pi/2 - x) = \sin x$. Thus

$$I = \int_0^{\pi/2} \frac{\sin^4 x + \sin x \cos^3 x + \sin^2 x \cos^2 x + \sin^3 x \cos x}{\sin^4 x + \cos^4 x + 2\sin x \cos^3 x + 2\sin^2 x \cos^2 x + 2\sin^3 x \cos x}\, dx.$$

Adding the above to the given integral, we obtain $2I = \int_0^{\pi/2} dx$. Therefore $I = \pi/4$.

2. We necessarily have $x \geq -2$. Also the left hand side becomes negative for $x \geq 2$. Therefore we may assume that $x = 2\cos t$ for $0 \leq t \leq \pi$. After making this substitution, the equation becomes $\cos 3t + \cos(t/2) = 0$. Using a standard trig formula ($2\cos A \cos B = \cos(A+B) + \cos(A-B)$), this becomes $\cos(7t/4)\cos(5t/4) = 0$. This results in the solutions $t = 2\pi/5$, $2\pi/7$, $6\pi/7$. Therefore $x = 2\cos(2\pi/5)$, $2\cos(2\pi/7)$, $2\cos(6\pi/7)$.

3. We make a,b,c,d,e be the roots of the quintic equation $x^5 + px^4 + qx^3 + rx^2 + sx + t = 0$. Using the first and last equations, we get $p = t = 1$. Let $Z = \{a,b,c,d,e\}$. Then

$$2q = \sum_{u,v \in Z,\ u \neq v} uv = (a + \cdots + e)^2 - (a^2 + \cdots + e^2) = -14,$$

so $q = -7$. Next $s = abcde(1/a + \cdots + 1/e) = -1/-1 = 1$. Finally

$$\begin{aligned} r = abcde\Big(\sum_{u,v \in Z,\ u \neq v} uv\Big) &= abcde\big((1/a + \cdots + 1/e)^2 - (1/a^2 + \cdots + 1/e^2)\big) \\ &= -14, \end{aligned}$$

so $r = -7$.

Similarly $s = 1$ and $r = -7$. Therefore a,b,c,d,e are the roots of $x^5 + x^4 - 7x^3 - 7x^2 + x + 1 = 0$. By inspection, -1 is a root and the equation factors as

$$(x+1)(x^4 - 7x^2 + 1) = (x+1)(x^2 - 3x + 1)(x^2 + 3x + 1).$$

Using the quadratic formula, it follows that a,b,c,d,e are (in whatever order you like)

$$-1, \frac{\pm 3 \pm \sqrt{5}}{2}.$$

4. We repeatedly use the fact that if n is a positive integer and $a \in \mathbb{Z}$ is prime to n, then $a^{\phi(n)} \equiv 1 \bmod n$ where ϕ is Euler's totient function.

   We first show that $f(n) \equiv 3 \bmod 4$ for all $n \geq 1$. We certainly have $f(1) \equiv 3 \bmod 4$. Since $f(n)$ is always odd, we see that $f(n+1) \equiv 3^{f(n)} \equiv 3^{f(n-1)} \equiv f(n) \bmod 4$ and we deduce that $f(n) \equiv 3 \bmod 4$ for all $n \geq 1$.

   Now we show that $f(n) \equiv f(3) \bmod 25$ for all $n \geq 3$. First observe that $f(n+1) \equiv 3^{f(n)} \equiv 3^{f(n-1)} \equiv f(n) \bmod 5$ for $n \geq 2$, provided $f(n) = f(n-1) \bmod 4$, which is true by the previous paragraph. It follows that $f(n+1) \equiv f(n) \bmod 20$ for all $n \geq 2$. Therefore $f(n+1) \equiv 3^{f(n)} \equiv 3^{f(n-1)} \equiv f(n) \bmod 25$, provided $n \geq 3$, and our assertion is proven. Since the last two digits of $f(3)$ are 87, the last two digits of $f(2012)$ are also 87.

5. Let $f(n) = 1/(\ln n) - (1/\ln n)^{(n+1)/n}$. Then $f(n)\ln n = 1 - (\ln n)^{-1/n}$. Assume that $n > 27$. Since $\ln n > e$, we see that $f(n)\ln n > 1 - e^{-1/n}$. Therefore $nf(n)\ln n > n(1 - e^{-1/n})$. By L'hôpital's rule, $\lim_{n\to\infty} n(1 - e^{-1/n}) = 1$. Therefore $nf(n)\ln n > 1/2$ for sufficiently large n, so $f(n) > 1/(2n\ln n)$. Since $\sum_{n=2}^{\infty} 1/(n\ln n)$ is divergent, it follows that $\sum_{n=2}^{\infty} f(n)$ is also divergent.

6. We shall prove by induction that $a_n = p$ if p is a prime and $n = p^m$ for some positive integer m, and 1 otherwise. This is clear in the case $n = p^m$, because then there are exactly $m-1$ nontrivial divisors of p^m, and each contributes p to the denominator of the displayed fraction. The case $n = pq$, where p, q are distinct primes, is also clear, because then p and q are the only nontrivial divisors of n, and they contribute p and q respectively to the denominator.

   Now assume that n is neither a prime power, nor a product of two distinct primes, and assume the result is true for all smaller values of n. Then we may write $n = pm$, where p is a prime and m is not a prime power. Write $m = p^a k$, where a is a nonnegative integer and k is prime to p. If $d \mid n$, then either $d \mid m$ or $d = p^{a+1} r$, where $r \mid k$. Note that in the latter case, d is a prime power only when $r = 1$. Therefore

$$a_n = a_m \frac{p}{a_{p^{a+1}}} = 1\frac{p}{p} = 1.$$

   by induction, which proves the claim. It follows that $a_{999000} = 1$.

7. Let 0 and I denote the zero and identity 2×2 matrices respectively. Let A denote one of the three matrices. The result is clear if $A = 0$ or I, because

every matrix commutes with 0 and I. Next note that $A^i = A^j$, where $0 < i < j < 5$, and we see that the minimum polynomial of A divides $x^j - x^i$.

Suppose 0 is not an eigenvalue of A. Then A is invertible and it follows that $A^{j-i} = I$, in particular I is one of the matrices. Since I commutes with all matrices, the result follows in this case.

Thus we may assume that 0 is an eigenvalue of A. Next suppose both the eigenvalues of A are 0 (i.e. A has a repeated eigenvalue 0). Then the Jordan canonical form of A is either 0 or $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. In both case, $A^2 = 0$, and since 0 commutes with all matrices, the result follows in this case.

Therefore we may assume that A has one eigenvalue 0 and another eigenvalue $\lambda \neq 0$. Since the minimum polynomial of A divides $x^j - x^i$ where $0 < i < j < 5$, we see that the possibilities for λ are 1, -1, or ω where ω is a primitive cube root of 1. Since the eigenvalues of A are distinct, it is diagonalizable and in particular, its Jordan canonical form will be $\begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}$. If $\lambda = \omega$, then $\{A, A^2, A^3\}$ are three distinct commuting matrices, and the result is proven in this case. Thus we may assume that the Jordan canonical form for A is $\begin{pmatrix} \pm 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Now not all the A_i can have Jordan canonical form $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, because then $\operatorname{tr}(A_1 + A_2 + A_3) = 3$, so at least one of the matrices, say A_1, has trace -1. It should be pointed out that we may assume that the A_i are distinct, if not, then the three matrices come from $\{A_1, A_1^2\}$, and since A_1 commutes with A_1^2, the result follows in this case.

Suppose $\operatorname{tr}(A_2) = -1$ and $\operatorname{tr}(A_3) = 1$. Then $A_1^2 = A_2^2 = A_3$ and A_3 commutes with A_1 and A_2, and the result is proven in this case.

Finally suppose $\operatorname{tr}(A_2) = \operatorname{tr}(A_3) = 1$. Then without loss of generality, we may assume that $A_1^2 = A_2$, and so $A_1 = -A_2$. Thus $-A_3 \neq A_1$ or A_2. Since $A_2A_3 = -A_1A_3$, we see that $A_2A_3 = A_1$ or A_2. Similarly $A_3A_2 = A_1$ or A_2. Since $\operatorname{tr}(A_2A_3) = \operatorname{tr}(A_3A_2)$, we deduce that $A_2A_3 = A_3A_2$. Also $A_1A_2 = A_2A_1$, and the result follows.

35th VTRMC, 2013, Solutions

1. Make the substitution $t = 2y$, so $dt = 2dy$. Then $I = \int_0^{x/2} 6\sqrt{2}\frac{\sqrt{(1+1\cos 2y)/2}}{17-8\cos 2y}dy = \int_0^{x/2} 3\sqrt{2}\frac{2\sqrt{2}\cos y}{9+16\sin^2 y}dy$. Now make the substitution $z = \sin y$. Then $dz = dy\cos y$ and $I = 12\int_0^{\sin x/2} \frac{dz}{3^2+4^2z^2} = \tan^{-1}\frac{4}{3}\sin x/2$. If $\tan I = 2/\sqrt{3}$, then $2\sqrt{3} = \frac{4}{3}\sin x/2$ and we deduce that $x = 2\pi/3$.

2. Without loss of generality we may assume that $BC = 1$, and then we set $x := BD$, so $AD = 2x$. Write $\theta = \angle CAD$, $y = AC$ and $z = DC$. The area of ADC is both x and $(yz\sin\theta)/2$. Also $y^2 = 1+9x^2$ and $z^2 = 1+x^2$. Therefore $4x^2 = (1+9x^2)(1+x^2)\sin^2\theta$. We need to maximize θ, equivalently $\sin^2\theta$, which in turn is equivalent to minimizing $(1+9x^2)(1+x^2)/(4x^2)$. Therefore we need to find x such that $x^{-2}+9x^2$ is minimal. Differentiating, we find $-2x^{-3}+18x = 0$, so $x^2 = 1/3$. It follows that $\sin^2\theta = 1/4$ and we deduce that the maximum value of $\angle CAD = \theta$ is 30°.

3. We need to show that a_n is bounded, equivalently $\ln a_n$ is bounded, i.e. $\ln 2\sum_{n=1}^\infty \ln(1+n^{-3/2})$ is bounded. But $\ln(1+n^{-3/2}) < n^{-3/2}$ and $\sum_{n=1}^\infty n^{-3/2}$ is convergent. It follows that (a_n) is convergent.

4. (a) $25 = 50/2 = \dfrac{5^2+5^2}{1^2+1^2}$.

   (b) Assume that 2013 is special. Then we have

$$x^2+y^2 = 2013(u^2+v^2) \tag{1}$$

for some positive integers x, y, u, v. Also, we assume that x^2+y^2 is minimal with this property. The prime factorization of 2013 is $3\cdot 11\cdot 61$. From (1) it follows $3|x^2+y^2$. It is easy to check by looking to the residues mod 3 that $3|x$ and $3|y$, hence we have $x = 3x_1$ and $y = 3y_1$. Replacing in (1) we get

$$3(x_1^2+y_1^2) = 11\cdot 61(u^2+v^2), \tag{2}$$

i.e. $3|u^2+v^2$. It follows $u = 3u_1$ and $v = 3v_1$, and replacing in (2) we get

$$x_1^2+y_1^2 = 2013(u_1^2+v_1^2).$$

Clearly, $x_1^2+y_1^2 < x^2+y^2$, contradicting the minimality of x^2+y^2.

(c) Observe that $2014 = 2 \cdot 19 \cdot 53$ and 19 is a prime of the form $4k+3$. If 2014 is special, then we have,

$$x^2 + y^2 = 2014(u^2 + v^2),$$

for some positive integers x, y, u, v. As in part (b), we may assume that $x^2 + y^2$ is minimal with this property. Now, we will use the fact that if a prime p of the form $4k+3$ divides x^2+y^2, then it divides both x and y. Indeed, if p does not divide x, then it does not divide y too. We have $x^2 \equiv -y^2 \mod p$ implies $(x^2)^{\frac{p-1}{2}} \equiv (-y^2)^{\frac{p-1}{2}} \mod p$. Because $\frac{p-1}{2} = 2k+1$, the last relation is equivalent to $(x^2)^{\frac{p-1}{2}} \equiv -(y^2)^{\frac{p-1}{2}} \mod p$, hence $x^{p-1} \equiv -y^{p-1} \mod p$. According to the Fermat's little theorem, we obtain $1 \equiv -1 \mod p$, that is p divides 2, which is not possible.

Now continue exactly as in part (b) using the prime 19, and contradict the minimality of x^2+y^2.

5. Write $x = \tan A$, $y = \tan B$, $z = \tan C$, where $0 < A, B, C < \pi/2$. Using the formula $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$ twice, we see that

$$\tan(A+B+C) = \frac{x+y+z-xyz}{1-yz-zx-xy} = 0$$

and therefore $A+B+C = \pi$. Now $\sin A = \frac{x}{1+x^2}$, so we need to prove that $\sin A + \sin B + \sin C \leq 3\sqrt{3}/2$. However $\sin t$ is a concave function, so we may apply Jensen's inequality (or consider the tangent at $t = (A+B+C)/3$) to deduce that

$$\frac{\sin A + \sin B + \sin C}{3} \leq \frac{\sin(A+B+C)}{3} = \sin(\pi/3) = \sqrt{3}/2,$$

and the result follows.

6. Let $C = X^{-1} + (Y^{-1} - X)^{-1}$. Observe that $(Y^{-1} - X) = (X - XYX)X^{-1}Y^{-1}$, consequently $(Y^{-1} - X)^{-1} = YX(X - XYX)^{-1}$. Therefore $C(X - XYX)^{-1} = I - YX + YX = I$ and we deduce that $XY - BY = (X - X + XYXD)Y = XYXY$. Therefore we can take $M = XY = \begin{pmatrix} 190 & 81 & 65 \\ -49 & 64 & -191 \\ -56 & 74 & 86 \end{pmatrix}$.

7. For $|q|<1$, we have $\sum_{k=1}^{\infty} q^k = q/(q-1)$. Therefore for $|q|>1$,

$$\begin{aligned}
-\sum_{n=1}^{\infty}\frac{(-1)^n}{q^n-1} &= -\sum_{n=1}^{\infty}\frac{(-1)^n q^{-n}}{1-q^{-n}} \\
&= -\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(q^{-n})^k(-1)^{n+1} \\
&= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{-n}}{1-q^{-n}}, \\
\sum_{n=1}^{\infty}\frac{1}{q^n+1} &= \sum_{n=1}^{\infty}\frac{q^{-n}}{1+q^{-n}} \\
&= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(-1)^{k+1}(q^{-n})^k \\
&= \sum_{k=1}^{\infty}\frac{(-1)^{k+1}q^{-k}}{1-q^{-k}}.
\end{aligned}$$

It follows that $-\sum_{n=1}^{\infty}\frac{(-1)^n}{q^n-1} = \sum_{n=1}^{\infty}\frac{1}{q^n+1}$. Now

$$\begin{aligned}
\frac{d}{dx}\frac{1}{x^n-1} &= \frac{-n}{x(x^{n/2}-x^{-n/2})^2} \\
\frac{d}{dx}\frac{1}{x^n+1} &= \frac{-n}{x(x^{n/2}+x^{-n/2})}.
\end{aligned}$$

We deduce that

$$-\sum_{n=1}^{\infty}\frac{(-1)^n n}{q(q^{n/2}-q^{-n/2})^2} = \sum_{n=1}^{\infty}\frac{n}{q(q^{n/2}+q^{-n/2})^2}$$

Now set $q=4$. We conclude that $\sum_{n=1}^{\infty}\frac{n}{(2^n+2^{-n})^2}+\frac{(-1)^n n}{(2^n-2^{-n})^2}=0.$

36th VTRMC, 2014, Solutions

1. Let S denote the sum of the given series. By partial fractions,

$$2\frac{n^2-2n-4}{n^4+4n^2+16} = \frac{n-2}{n^2-2n+4} - \frac{n}{n^2+2n+4}.$$

If $f(n) = \frac{n-2}{n^2-2n+4}$, then $2S = \sum_{n=2}^{n=\infty} f(n) - f(n+2)$. Since $\lim_{n\to\infty} f(n) = 0$, it follows by telescoping series that the series is convergent and $2S = f(2) - f(4) + f(3) - f(5) + f(4) - f(6) + \cdots$, so $2S = f(2) + f(3)$ and we deduce that $S = 1/14$.

2. Let I denote the given integral. First we make the substitution $y = x^2$, so $dy = 2xdx$. Then

$$2I = \int_0^4 \frac{16-y}{16-y+\sqrt{(16-y)(12+y)}}\,dy = \int_0^4 \frac{\sqrt{16-y}}{\sqrt{16-y}+\sqrt{12+y}}\,dy.$$

Now make the substitution $z = 4-y$, so $dz = -dy$. Then

$$2I = \int_0^4 \frac{\sqrt{12+z}}{\sqrt{12+z}+\sqrt{16-z}}\,dz.$$

Adding the last two equations, we obtain $4I = \int_0^4 dz = 4$ and hence $I = 1$.

3. Let $m = \phi(2^{2014}) = 2^{2013}$ (here $\phi(x)$ is Euler's totient function, the number of positive integers $< x$ which are prime to x). Then $19^m \equiv 1 \mod 2^{2014}$ by Euler's theorem. Therefore n divides 2^{2013}, so $n = 2^k$ for some positive integer k. Now

$$19^{2^k} - 1 = (19-1)(19+1)(19^2+1)(19^4+1)\ldots(19^{2^{k-1}}+1);$$

we calculate the power of 2 in the above expression. This is $1+2+1+1+\cdots+1 = k+2$. Therefore $k+2 = 2014$ and it follows that $n = 2^{2012}$.

4. Put i^{a+2b} in the square in the (a,b) position. Note that the sum of all the entries in a 4×1 or 1×4 rectangle is zero, because $\sum_{k=0}^{3} i^{a+k+2b} = (1+i+i^2+i^3)i^{a+2b} = 0$ and $\sum_{k=0}^{3} i^{a+2(b+k)} = (1+i^2+i^4+i^6)i^{a+2b} = 0$. Therefore if we have a tiling with 4×1 and 1×4 rectangles, the sum of the entries in

all 361 squares is the value on the central square, namely $i^{10+20} = -1$. On the other hand this sum is also

$$\begin{aligned}(i+i^2+\cdots+i^{19})(i^2+i^4+\cdots+i^{38}) &= i\frac{i^{19}-1}{i-1}\cdot(-1+1-\cdots-1)\\ &= i\frac{-i-1}{i-1}\cdot -1 = 1.\end{aligned}$$

This is a contradiction and therefore we have no such tiling.

5. Suppose by way of contradiction we can write $n(n+1)(n+2) = m^r$, where $n \in \mathbb{N}$ and $r \geq 2$. If a prime p divides $n(n+2)$ and $n+1$, then it would have to divide $n+1$, and n or $n+2$, which is not possible. Therefore we may write $n(n+2) = x^r$ and $n+1 = y^r$ for some $x, y \in \mathbb{N}$. But then $n(n+2)+1 = (n+1)^2 = z^r$ where $z = y^2$. Since $(n+1)^2 > n(n+2)$, we see that $z > x$ and hence $z \geq x+1$, because $x, z \in \mathbb{N}$. We deduce that $z^r \geq (x+1)^r > x^r + 1$, a contradiction and the result follows.

6. (a) Since A and B are finite subsets of T, we may choose $a \in A$ and $b \in B$ so that $f(ab)$ is as large as possible. Suppose we can write $g := ab = cd$ with $c \in A$ and $d \in B$. Let $h = d^{-1}b$ and $d \neq b$. Note that $g, h \in T$. Then $h \neq I$ and we see that either $f(gh^{-1}) > f(g)$ or $f(gh) > f(g)$. This contradicts the maximality of $f(ab)$. Therefore $d = b$ and because b is an invertible matrix, we deduce that $a = c$ and the result is proven.

   (b) Set $M = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$. Then $M \in S$ and $M^3 = I$. Suppose $f(M) > f(I)$. Then ($X = M$ and $Y = M$) we obtain either $f(M^2) > f(M)$ or $f(I) > f(M)$, hence $f(M^2) > f(M)$. Now do the same with $X = M^2$ and $Y = M$: we obtain $f(M^3) > f(M^2)$. Since $M^3 = I$, we now have $f(I) > f(M^2) > f(M) > f(I)$, a contradiction. The argument is similar if we start out with $f(M) < f(I)$. This shows that there is no such f.

7. (a) Let $A = (x_A, y_A)$ and $B = (x_B, y_B)$. Then $d(A,B) = \begin{pmatrix} x_B - x_A + y_A - y_B \\ x_B - x_A \end{pmatrix}$.

   (b) By definition $\det M = d(A_1,B_1)d(A_2,B_2) - d(A_1,B_2)d(A_2,B_1)$. Note that the first term counts all pairs of paths (π_1, π_2) where $\pi_i : A_i \to B_i$, and the second term is the negative of the number of pairs (π_1, π_2) where $\pi_1 \colon A_1 \to B_2$ and $\pi_2 \colon A_2 \to B_1$. The configuration of the points implies that every pair of paths (π_1, π_2) where $\pi_1 : A_1 \to B_2$ and $\pi_2 : A_2 \to B_1$

must intersect. Let $\mathscr{I} := \{(\pi_1, \pi_2) : \pi_1 \cap \pi_2 \neq \emptyset\}$ (this is the set of all intersecting paths, regardless of their endpoints). Define $\Phi\colon \mathscr{I} \to \mathscr{I}$ as follows. If $(\pi_1, \pi_2) \in \mathscr{I}$ then $\Phi((\pi_1, \pi_2)) = (\pi_1', \pi_2')$ and the new pair of paths is obtained from the old one by switching the tails of π_1, π_2 after their *last* intersection point. In particular, the pairs (π_1, π_2) and (π_1', π_2') must appear in different terms of $\det M$. But it is clear that $\Phi \circ \Phi = id_{\mathscr{I}}$, therefore Φ is an involution. This implies that all intersecting pairs of paths must cancel each other, and that the only pairs which contribute to the determinant are those from the set $\{(\pi_1, \pi_2)\colon \pi_1 \cap \pi_2 = \emptyset\}$. Since all the latter pairs can appear only with positive sign (in the first term of $\det M$), this finishes the solution. (In fact, we proved that $\det M = \#\{(\pi_1, \pi_2) : \pi_1 \cap \pi_2 = \emptyset\}$.)

37th VTRMC, 2015, Solutions

1. We have $f(n) := n^4 + 6n^3 + 11n^2 + 3n + 31 = (n^2 + 3n + 1)^2 - 3(n - 10)$. Therefore $f(10)$ is a perfect square, and we now show there is no other integer n such that $f(n)$ is a perfect square. We have $(n^2+3n+2)^2 - (n^2 + 3n+1)^2 = 2n^2+6n+3$ and $(n^2+3n+1)^2 - (n^2+3n)^2 = 2n^2+6n+1$. We have four cases to consider.

   (a) $n > 10$. Then we have $3(n-10) \geq 2n^2 + 6n + 1$, which is not possible.

   (b) $2 < n < 10$. Then we have $3(10-n) \geq 2n^2 + 6n + 3$, which is not possible.

   (c) $n < -6$. Then we have $3(10-n) \geq 2n^2 + 6n + 3$, which is not possible.

   (d) $-6 \leq n \leq 2$. Then we can check individually that the 9 values of n do not make $f(n)$ a perfect square.

   We conclude that $f(n)$ is a perfect square only when $n = 10$.

2. The folded 3-dimensional region can be described as a regular tetrahedron with four regular tetrahedrons at each vertex cut off. The four smaller tetrahedrons have side length 2cm., while the big tetrahedron has sides of length 6cm. Recall that the volume of a regular tetrahedron of side of length 1 is $\sqrt{2}/12$ (or easy calculation). Therefore the volume required in cm^3 is

$$6^3\sqrt{2}/12 - 4 \cdot 2^3\sqrt{2}/12 = 46\sqrt{2}/3.$$

3. Let $n = 2015$. If one regards $a_1, \ldots, a_n$ as variables, the determinant is skew symmetric (i.e. if we interchange a_i and a_j where $i \neq j$, we obtain $-\det A$). We deduce that $a_i - a_j$ divides $\det A$ for all $i \neq j$, hence

$$\det A \text{ is divisible by } \prod_{1 \leq i < j \leq n} (a_i - a_j).$$

   For $k \in \mathbb{N}$, we prove by induction on k that if a number is divisible $a_1 \cdots a_k$ and $\prod_{1 \leq i < j \leq k}(a_i - a_j)$, then it is divisible by $k!$; the case $k = 1$ is immediate. So assume the result for $i \leq k$. If one of the a_i is divisible by $k+1$, then the result is true for $k+1$ by induction. On the other hand if none of the a_i is divisible by $k+1$, then at least one of the numbers $a_i - a_j$ is divisible by $k+1$ and the induction step is complete. The result follows.

4. We first show the result is true if $0 < p \leq 1$ for $p \in \mathbb{Q}$ (positive number excludes 0, however the result is even true here by taking the sum of a zero number of terms). Write $p = a/b$ where $a, b \in \mathbb{N}$. The result is obviously true if $a = 1$. We now prove the result by induction on a; we may assume that $a < b$. Let $n \geq 2$ be the unique positive integer such that $1/n \leq p < 1/(n-1)$. Then we have $b \leq an$ and $0 \neq an - a < b$. Set $q = (an-b)/bn$. Since $an - b < a$, we may write q as a partial sum S of the $1/m$, and then we have $p = S + 1/n$. Also the integers m which appear in S must have $m > n$, because $p < n-1$. This completes the induction step, and we have proven the result for $p \leq 1$.

   Let $p \in \mathbb{Q}$ and let $s_n = \sum_{k=1}^{n} 1/k$. Since the harmonic series is divergent, there exists a unique $m \in \mathbb{N}$ such that $s_m < p \leq s_{m+1}$. Then $p - s_m < 1$, so by the previous paragraph is a partial sum S of the $1/n$, and then we have $p = S + 1/m$. Also $S \leq 1/(m+1)$ so none of the $1/n$ appearing in S can be equal to $1/m$, and the proof is complete.

5. Let n be a positive integer. Then

$$\int_0^n \int_1^\pi \frac{1}{1+(xy)^2}\, dxdy = \int_1^\pi \int_0^n \frac{1}{1+(xy)^2}\, dydx.$$

   Therefore

$$\int_0^n \frac{\arctan(\pi x) - \arctan(x)}{x}\, dx = \int_1^\pi \frac{\arctan(ny)}{y}\, dy.$$

   Set $u = \arctan(ny)$ and $dv = 1/y$ and use integration by parts to obtain

$$\int_1^\pi \frac{\arctan(ny)}{y}\, dy = \arctan(n\pi)\ln\pi - \int_1^\pi \frac{n \ln y}{1+n^2y^2}\, dy.$$

   On the other hand, $0 \leq \dfrac{n \ln y}{1+n^2} \leq \dfrac{n \ln \pi}{1+n^2}$ for all $y \in [1, \pi]$. Therefore

$$\lim_{n\to\infty} \int_1^\pi \frac{\arctan(ny)}{y}\, dy = \frac{\pi \ln \pi}{2}$$

   and we deduce that

$$\int_0^\infty \frac{\arctan(\pi x) - \arctan(x)}{x}\, dx = \lim_{n\to\infty} \int_0^n \frac{\arctan(\pi x) - \arctan(x)}{x}\, dx = \frac{\pi \ln \pi}{2}.$$

6. If $(x,y) \in S := \sum_{i=1}^{n} \mathbb{Z}(a_i,b_i)$, then there exist $k_i \in \mathbb{Z}$ such that $(x,y) = \sum_{i=1}^{n} k_i(a_i,b_i)$. We choose the k_i such that $d := \sum_{i=1}^{n} |k_i|$ is minimal and then define $d(x,y) = d$. On the other hand if $(x,y) \notin S$, then define $d(x,y) = +\infty$ (thus $d(x,y) = \infty$ if and only if $(x,y) \notin S$). Now choose a positive integer m such that $m \geq n/\varepsilon$ and define

$$f(x,y) = \begin{cases} 1 - d(x,y)/m & \text{if } d(x,y) \leq m; \\ 0 & \text{if } d(x,y) > m. \end{cases}$$

If $(x,y) \notin S$, then $(x+a_i, y+b_i) \notin S$ for all i and therefore $d(x,y) = d(x+a_i, y+b_i) = 0$ and hence $f(x,y) = 0$ if $d(x,y) = +\infty$. On the other hand if $(x,y) \in S$, then $|d(x,y) - d(x+a_i,y+b_i)| \leq 1$ for all i. It then follows that $f(x,y) = 0$ if $d(x,y) > m$, hence $f(x,y) \neq 0$ for only finitely many (x,y), and furthermore $|f(x,y) - f(x+a_i,y+b_i)| = 0$ or $1/m$ for all i. Thus $f(x,y)$ satisfies the required condition, so the answer is "yes".

7. Note that the hypotheses show that there exists a positive integer a such that $a\langle u,v\rangle \in \mathbb{Z}$ for all $u,v \in S$. Therefore there exists a positive integer b such that $b\|u\|^2 = b\langle u,u\rangle$ is a positive integer for all $0 \neq u \in S$, so we may choose $0 \neq s \in S$ such that $\|s\|$ is minimal.

   First suppose that the x_i are all contained in $\mathbb{R}s$ (i.e. the points of S are collinear). Then the same is true of S and we claim that $S = \mathbb{Z}s$. If $u \in S$, then $u = cs$ for some $c \in \mathbb{R}$. Also $a\|s\| \leq \|u\| < (a+1)\|s\|$ for some nonnegative integer a, hence $\|as\| \leq \|cs\| < \|(a+1)s\|$. We deduce that $\|(a-c)s\| < \|s\|$ and since $(a-c)s \in S$, we conclude that $\|(a-c)s\| = 0$. Therefore $u = as$ and the claim is established. Now we place disks of radius $R := 3\|s\|/4$ and center $(2n+1/2)s$ for all $n \in \mathbb{Z}$ and the result is proven in this case.

   Now suppose that not all the x_i are not contained $\mathbb{R}s$. Then we may choose $t \in S \setminus \mathbb{R}s$ with $\|t\|$ minimal. We claim that $S = T := \{ms + nt \mid m,n \in \mathbb{Z}\}$. If this is not the case, we may choose $u \in S \setminus T$. Note that $\mathbb{R}s + \mathbb{R}t = \mathbb{R}^2$, so we may write $u = ps + qt$ for some $p,q \in \mathbb{R}$ and then there exist $a,b \in \mathbb{Z}$ such that $a \leq p < a+1$ and $b \leq q < b+1$, so u is inside the parallelogram with vertices (as,bt), $(as+s,bt)$, $(as,bt+t)$, $(as+s,bt+t)$. Since $\|s\| \leq \|t\|$ we see that u is distance at most $\|t\|$ from one of these vertices. Furthermore $\|u-v\| \geq \|t\|$ for all $u \neq v \in S$, so we must have $u \in S$.

   Now we can place disks with radius $R := \|s\|/2$ and centers at $((2m+1/2)s, nt)$ for $m,n \in \mathbb{Z}$. Clearly every disk contains at least two points of S, namely $(2ms, nt)$ and $(2ms+1, nt)$ for the disk centered at $((2m+1/2)s, nt)$, and these disks accounts for all the points in S. We only need to show that any two distinct disks intersect in at most one point, and thus we need to show that two different centers are distance at least $\|s\|$ apart. So consider two different centers, say at $((2m+1/2)s, nt)$ and $((2m'+1/2)s, nt')$. Then the distance between theses two centers is the same at the distance between $(2ms, nt)$ and $(2m's, n't)$, which is at least $\|s\|$ by minimality of $\|s\|$. This completes the proof. (This actually proves the stronger statement that every point of S lies in exactly one disk, which is how the problem was meant to be stated; the argument can be significantly shortened for the actual problem.)

38th VTRMC, 2016, Solutions

1. Write $I = \int_1^2 \frac{\ln x}{2-2x+x^2}\,dx$. We make the substitution $y = 2/x$. Then $dx = -2y^{-2}dy$ and we have
$$I = \int_2^1 \frac{-2y^{-2}\ln(2/y)}{2-4/y-4/y^2}\,dy = \int_1^2 \frac{\ln 2 - \ln y}{y^2-2y+2}\,dy.$$
Therefore
$$2I = \int_1^2 \frac{\ln 2}{y^2-2y+2}\,dy = \int_0^1 \frac{\ln 2}{x^2+1}\,dx$$
by making the substitution $x = y-1$. We conclude that $I = \dfrac{\pi \ln 2}{8}$.

2. Set $a_n = \frac{(2n)!}{4^n n! n!}$. Then $a_n/a_{n-1} = (2n-1)/(2n) = 1-1/(2n)$. Therefore
$$(n-1)/n \le (a_n/a_{n-1})^2 \le n/(n+1)$$
for all $n \in \mathbb{N}$. Now if $b_n = 1/n$, then
$$b_n/b_{n-1} \le (a_n/a_{n-1})^2 \le b_{n+1}/b_n.$$
Therefore $1/4n \le a_n^2 \le 1/(n+1)$ and hence
$$\frac{1}{(4n)^{k/2}} \le a_n \le \frac{1}{(n+1)^{k/2}}.$$
Since $\sum 1/n^{k/2}$ is convergent if and only if $k > 2$, we deduce that the series is convergent for $k > 2$ and divergent for $k \le 2$.

3. Let I denote the identity matrix in $\mathrm{M}_n(\mathbb{Z}_2)$. If $A \in \mathrm{M}_n(\mathbb{Z}_2)$ and $A^2 = 0$, then $(I+A)^2 = I + 2A + A^2 = I$ because we are working mod 2, and we see that $I + A \in \mathrm{GL}_n(\mathbb{Z}_2)$, the invertible matrices in $\mathrm{M}_n(\mathbb{Z}_2)$. Conversely if $X \in \mathrm{GL}_n(\mathbb{Z}_2)$, and $X^2 = I$, then $(I+X)^2 = 0$. We deduce that the number of matrices A satisfying $A^2 = 0$ is precisely the number of matrices satisfying $X^2 = I$. Since $n \ge 2$, the number of matrices in $\mathrm{GL}_n(\mathbb{Z}_2)$ is even (if $Y \in \mathrm{GL}_n(\mathbb{Z}_2)$, then we can pair it with the matrix Y' obtained from Y by interchanging the first two rows of Y, and note that $Y \ne Y'$ otherwise Y would have two rows equal and therefore would not be invertible). Now if $Z \in \mathrm{GL}_n(\mathbb{Z}_2)$ and $Z^2 \ne I$, then we can pair it with Z^{-1} and we see that the number of matrices satisfying $Z^2 \ne I$ in $\mathrm{GL}_n(\mathbb{Z}_2)$ is even. Therefore the number of matrices satisfying $X^2 = I$ is even and the result follows.

4. First observe that if $p > 2$ is a prime and $a < p$ is such that a^2+1 is divisible by p, then $a \neq p-a$ and $P(a) = P(p-a) = p$. Indeed a^2+1 and $(p-a)^2 + 1 = (a^2+1) + p(p-2a)$ are divisible by p and are smaller than p^2, so they cannot be divisible by any prime greater than p.

   We will prove the stronger statement that there are infinitely many primes p for which $P(x) = p$ has at least three positive integer solutions, so assume by way of contradiction that there are finitely many such primes and let s be the maximal prime among these; if there are no solutions, set $s = 2$. Let S be the product of all primes not exceeding s. If $p = P(S)$, then p is coprime to S and thus $p > s$. Let a be the least positive integer such that $a \equiv S \bmod p$. Then a^2+1 is divisible by p, hence $P(a) = P(p-a) = p$ because $p > a$. Let $b = a$ if a is even, otherwise let $b = p-a$. Then $(b+p)^2+1$ is divisible by $2p$, so $P(b+p) \geq p$. If $P(b+p) = p$, we arrive at a contradiction. Therefore $P(b+p) =: q > p$ and $(b+p)^2+1$ is divisible by $2pq$ and thus $(b+p)^2 + 1 \geq 2pq$. This means $q < b+p$, otherwise $(b+p)^2+1 \leq (2p-1)q+1$ (because $b < p$) $< 2pq$. Now let c be the least positive integer such that $c = b+p \bmod q$. We have $P(c) = P(q-c) = P(b+p) = q > p > s$, another contradiction and the proof is finished.

5. The equality yields $1+m-n\sqrt{3} = (2-\sqrt{3})^{2r-1}$ and hence $(1+m)^2 - 3n^2 = 1^{2r-1} = -1$. Therefore $m(m+2) = 3n^2$. If $p \neq 2,3$ is a prime and p^a is the largest power of p dividing n, then p^{2a} is the largest power of p dividing $3n^2$. Since p cannot divide both m and $m+2$, we see that either $p \nmid m$ or $p^{2a} \mid m$, in either case the power of p that divides m is an even. It remains to prove that the largest power of 2 and 3 that divides m is also even. Now if 2 divides m, then the largest power of 2 that divides $m(m+2)$, and hence also $3n^2$, is odd which is not possible. All that remains to be proven is that 3 does not divide m. However we have $1+m = 2^{2r-1} \bmod 3$, which shows that 3 does not divide m as required.

6. Write $M = \begin{pmatrix} I+A & -X \\ -Y & I+P \end{pmatrix}, N = \begin{pmatrix} I+B & X \\ Y & I+Q \end{pmatrix}.$

   Then

$$MN = \begin{pmatrix} I+A+B+AB-XY & AX-XQ \\ PY-YB & I+P+Q+PQ-YX \end{pmatrix} = I.$$

   Therefore $NM = I$ and in particular $I+A+B+BA-XY = I$. The result follows.

7. Proceed by induction on k. Let c_k denote the constant term of f_k. For the base case $k = 1$, we need only observe that $f_1(X) = (1-X)(1-qX^{-1}) = 1+q-X-qX^{-1}$ and $c_1 = (1-q^2)/(1-q) = 1+q$. For any k, we have

$$c_{k+1} = \frac{(1-q^{2k+1})(1-q^{2k+2})}{(1-q^{k+1})^2} c_k = \frac{(1-q^{2k+1})(1+q^{k+1})}{1-q^{k+1}} c_k.$$

We will prove that the constant term of $f_k(X)$ satisfies the same recurrence relation, which gives the induction step. Let $a_k^{(i)}$ denote the coefficient of X^i in f_k. From

$$\begin{aligned} f_{k+1}(X) &= (1-q^kX)(1-q^{k+1}X^{-1})f_k(X) \\ &= (1-q^kX-q^{k+1}X^{-1}+q^{2k+1})f_k(X) \end{aligned}$$

we deduce that

$$a_{k+1}^{(0)} = (1+q^{2k+1})a_k^{(0)} - q^k a_k^{(-1)} - q^{k+1}a_k^{(1)}.$$

We want a recurrence relation for $a_k^{(0)}$. To relate $a_k^{(\pm 1)}$ to $a_k^{(0)}$, we consider

$$\begin{aligned} f_k(qX) &= \prod_{i=0}^{k-1}\Big((1-q^{i+1}X)(1-q^iX^{-1})\Big) \\ &= \frac{(1-q^kX)(1-X^{-1})}{(1-X)(1-q^kX^{-1})} f_k(X) \\ &= \frac{1-q^kX}{q^k-X} f_k(X). \end{aligned}$$

Hence $(q^k-X)f_k(qX) = (1-q^kX)f_k(X)$. Equating coefficients of X^0 and X^1 on both sides, we obtain

$$a_k^{(-1)} = q\frac{q^k-1}{1-q^{k+1}}a_k^{(0)}, \qquad a_k^{(1)} = \frac{q^k-1}{1-q^{k+1}}a_k^{(0)}.$$

Therefore

$$a_{k+1}^{(0)} = \left(1+q^{2k+1}-2q^{k+1}\frac{q^k-1}{1-q^{k+1}}\right)a_k^{(0)} = \frac{(1-q^{2k+1})(1+q^{k+1})}{1-q^{k+1}}a_k^{(0)}$$

and this completes the proof.

39th VTRMC, 2017, Solutions

1. Set $f(x) = 2x^6 - 6x^4 - 6x^3 + 12x^2 + 1 = 0$ and $g(x) = 2x^6 - 6x^4 - 4\sqrt{2}x^3 + 12x^2$. By raising to the sixth power, we see that a solution to the given equation also satisfies f. Furthermore to have a real solution, we need $x \le \sqrt{2}$. Therefore if we can show that $f(x)$ has no solutions with $x \le \sqrt{2}$, then it will follow that the original equation has no solutions. Now $g(x) = 2x^2(x-\sqrt{2})^2(x^2+2\sqrt{2}x+3)$. Thus g has zeros at 0 and $\sqrt{2}$ (of multiplicity 2), and is positive otherwise, because $x^2+2\sqrt{2}x+3 > 0$ for all $x \in \mathbb{R}$. Now $f(x) - g(x) = (4\sqrt{2}-6)x^3 + 1$ which is positive for $x \le \sqrt{2}$, because the function is decreasing and $(4\sqrt{2}-6)\sqrt{2}^3 + 1 > 0$. To see this, we need to show that $17 - 12\sqrt{2} > 0$. However multiplying by $17 + 12\sqrt{2}$, we see that we need to show $17^2 - 144 \cdot 2 > 0$, which is true. It follows that the given equation has no real solutions.

2. Write $t = \tan(x/2)$. Then $\cos^2(x/2) = 1/(1+t^2)$, so
$$\cos x = \cos^2(x/2) - \sin^2(x/2) = \frac{1-t^2}{1+t^2}$$
and since $\tan x = 2t/(1-t^2)$,
$$\sin x = \cos x \tan x = \frac{2t}{1+t^2}.$$
Write $I = \int_0^a \frac{dx}{1+\cos x+\sin x}$. Since $dt/dx = \frac{\sec^2(x/2)}{2} = (1+t^2)/2$, we see that
$$I = \int_0^{\tan(a/2)} \frac{2dt}{(1+t^2)+(1-t^2)+2t} = \int_0^{\tan(a/2)} \frac{dt}{1+t}.$$
Therefore $I = \ln(1+\tan(a/2))$. (An alternative answer is $\frac{1}{2}\ln\frac{1+\sin a}{1+\cos a} + \frac{1}{2}\ln 2$.) When $a = \pi/2$, we have $\tan(a/2) = 1$ and we deduce that $I = \ln 2$ as required.

3. We may assume that $AB = 1$. Since $\angle APB = 150$, the sine rule yields, $\sin 150/AB = \sin 20/AP = \sin 10/BP$ and $\sin 30/AP = \sin 40/CP$. Therefore $PC = 4\sin 20 \sin 40 = 2\cos 20 - 1$. Write $\angle PBC = \theta$. Since $\angle BPC =$

100, we see that $\angle PCB = 80 - \theta$, and then the sine rule for triangle BPC yields

$$\frac{2\cos 20 - 1}{\sin\theta} = \frac{2\sin 10}{\sin(80-\theta)} = \frac{2\sin 10}{\cos(\theta+10)}.$$

Therefore

$$2\cos 20\cos(\theta+10) = 2\sin 10\sin\theta + \cos(\theta+10) = \cos(\theta-10).$$

We deduce that $\cos(30+\theta)+\cos(10-\theta) = \cos(\theta-10)$ and hence $\cos(30+\theta) = 0$. We conclude that $\theta = 60$.

4. Denote the vertices of the triangle by A, B and C (counterclockwise). Let P be an interior point of the triangle and draw lines parallel to the three sides, partitioning the triangle into three triangles and three parallelograms. Let EH be the segment parallel to AC, let FI be the segment parallel to BC, and let JG be the segment parallel AB. Here the points E, F lie on the edge AB; the points G, H lie on the edge BC, and the points I, J lie on the edge AC. Suppose that the area of the triangle EFP is a, the area of the triangle PGH is b, and the area of the triangle JPI is c. Note that the triangles EFP, PGH, JPI and ABC are similar. Therefore $EF/PG = \sqrt{a}/\sqrt{b}$ and $JP/PG = \sqrt{c}/\sqrt{b}$. Thus $(EF+JP)/PG = (\sqrt{a}+\sqrt{c})/\sqrt{b}$ and hence $1+(EF+JP)/PG = 1+(\sqrt{a}+\sqrt{c})/\sqrt{b}$, i.e.

$$\frac{PG+EF+JP}{PG} = \frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{\sqrt{b}}.$$

Since $PG = FB$ and $JP = AE$, because $FBGP$ and $AEJP$ are parallelograms, $AB/PG = (\sqrt{a}+\sqrt{b}+\sqrt{c})/\sqrt{b}$. Because ABC is similar to PGH, we have $AB/PG = \sqrt{T}/\sqrt{b}$. Therefore $\sqrt{T} = \sqrt{a}+\sqrt{b}+\sqrt{c}$.

5. Let $(a,b) \in S$ and let $d = \gcd(a,b)$. Then $a = dm$ and $b = dn$ with $\gcd(m,n) = 1$. Since $g(a,b) \in \mathbb{N}$, we see that $ab = d^2mn$ is a perfect square and hence mn is a perfect square. Therefore m and n are both perfect squares, because $\gcd(m,n) = 1$. Thus we may write $a = ds^2$ and $b = dt^2$ with $\gcd(s,t) = 1$.

By assumption, $h(a,b) = 2ds^2t^2/(s^2+t^2) \in \mathbb{N}$. Since $\gcd(s^2+t^2,s^2) = \gcd(s^2+t^2,t^2) = \gcd(s^2,t^2) = 1$, it follows that s^2+t^2 divides $2d$. Thus $a = k(s^2+t^2)s^2/2$ and $b = k(s^2+t^2)t^2/2$ for some $k \in \mathbb{N}$.

Now $a \neq b$ because $s \neq \pm 1$. Also $f(a,b) = k(s^2+t^2)^2/4 \in \mathbb{N}$. We have two cases to consider.

- If s^2+t^2 is odd, then $4|k$ and hence $f(a,b) \geq 4(1^2+2^2)^2/4 = 25$.
- If s^2+t^2 is even, then s and t are odd because $\gcd(s,t) = 1$ and hence $f(a,b) \geq (1^2+3^2)/4 = 25$.

We conclude that $f(a,b) \geq 25$. However $f(5,45) = f(10,40) = 25$, so the minimum of f over S is 25.

6. Set $g(x) = f(x) - x^2 + 4x - 2$. Then $g(1) = g(4) = g(8) = 0$. Therefore we may write $g(x) = (x-1)(x-4)(x-8)q(x)$ where $q(x) \in \mathbb{Z}[x]$. Since $f(n) = n^2 - 4n - 18$, we see that $g(n) = -20$ and hence $(n-1)(n-4)(n-8)q(n) = -20$. By inspection, $n = 3$ or 6. We note that both of these values of n can be obtained, by taking (for example) $q(x) = -2$ and 1 respectively, and then $f(x) = -2(x-1)(x-4)(x-8)+x^2-4x+2$ and $(x-1)(x-4)(x-8)+x^2-4x+2$ respectively.

7. First we look at small values of n: the given equation is a quadratic in m. If $n \in \{0,1,2,4\}$, there are no solutions. If $n = 3$, then $m = 6$ or 9. If $n = 5$, then $m = 9$ or 54. We now proceed by contradiction to show that there is no solution if $n \geq 6$. So suppose (m,n) is a solution with $n \geq 6$. Then m divides $2 \cdot 3^n$ and so either $m = 3^a$ for some $0 \leq a \leq n$, or $m = 2 \cdot 3^b$ for some $0 \leq b \leq n$. If $m = 3^a$, then
$$2^{n+1} - 1 = m + 2 \cdot 3^n/3^a = 3^a + 2 \cdot 3^{n-a}.$$
On the other hand if $m = 2 \cdot 3^b$, then
$$2^{n+1} - 1 = m + 2 \cdot 3^n/m = 2 \cdot 3^b + 3^{n-b}.$$
Therefore there must be nonnegative integers a, b such that
$$2^{n+1} - 1 = 3^a + 2 \cdot 3^b, \quad a + b = n.$$
Note that $3^a < 2^{n+1} < 3^{2(n+1)/3}$ and $2 \cdot 3^b < 2^{n+1} < 2 \cdot 3^{2(n+1)/3}$, because $3^{2/3} > 2$. Thus $a, b < 2(n+1)/3$. Since $a + b = n$, we deduce that
$$(n-2)/3 < a < 2(n+1)/3 \quad \text{and} \quad (n-2)/3 < b < 2(n+1)/3.$$
Now let $t = \min(a,b)$. Then $t > (n-2)/3$ and since $n \geq 6$, it follows that $t > 1$. Because 3^t divides 3^a and $2 \cdot 3^b$, we see that 3^t divides $2^{n+1} - 1$. Since

$t \geq 2$, we deduce that $2^{n+1} \equiv 1 \mod 9$. Now $2^{n+1} \equiv 1 \mod 9$ if and only if 6 divides $n+1$, so $n+1 = 6r$ for some $r \in \mathbb{N}$. Therefore

$$2^{n+1} - 1 = 4^{3r} - 1 = (4^{2r} + 4^r + 1)(4^r - 1) = (4^{2r} + 4^r + 1)(2^r - 1)(2^r + 1).$$

Since 3^t divides $2^{n+1} - 1$, we see that 3^t divides $(4^{2r} + 4^r + 1)(2^r - 1)(2^r + 1)$. Note that 9 does not divide $4^{2r} + 4^r + 1$, hence 3^{t-1} divides $(2^r - 1)(2^r + 1)$. Since $\gcd(2^r - 1,\ 2^r + 1) = 1$, either $3^{t-1} \mid 2^r - 1$ or $2^r + 1$. In any case, $3^{t-1} \leq 2^r + 1$. Then $3^{t-1} \leq 2^r + 1 \leq 3^r = 3^{(n+1)/6}$. Therefore $(n-2)/3 - 1 < t - 1 \leq (n+1)/6$. This yields $n < 11$, which is a contradiction, because $n \geq 6$ and we proved that $6 \mid n+1$.

40th VTRMC, 2018, Solutions

1. Let $I = \int_1^2 \frac{\arctan(1+x)}{x}\,dx$. First we integrate by parts to obtain

$$\begin{aligned} I &= [\ln(x)\arctan(1+x)]_1^2 - \int_1^2 \frac{\ln x}{1+(1+x)^2}\,dx \\ &= \ln(2)\arctan(3) - \int_1^2 \frac{\ln x}{2+2x+x^2}\,dx. \end{aligned}$$

Now let $J = \int_1^2 \frac{\ln x}{2+2x+x^2}\,dx$ and make the substitution $x = 2/y$. We obtain

$$J = \int_2^1 \frac{\ln 2 - \ln y}{2+4/y+4/y^2}(-2/y^2)\,dy = \int_1^2 \frac{\ln 2}{1+(1+y)^2}\,dy - J.$$

Therefore $2J = \int_1^2 \frac{\ln 2}{1+(1+y)^2}\,dy = [\ln(2)\arctan(1+y)]_1^2 = \ln(2)(\arctan(3) - \arctan(2))$ and we deduce that $I = \ln(2)(\arctan(3)+\arctan(2))/2$. Now $\tan(\arctan(3)+\arctan(2)) = (3+2)(1-6) = -1$, which shows that $\arctan(3) - \arctan(2) = 3\pi/4$. Therefore $I = 3\pi\ln(2)/8$, and the answer is $q = 3/8$.

2. First we'll show that if $X, Y \in \mathrm{M}_6(\mathbb{Z})$, $X \equiv I \equiv Y \bmod 3$, and $XYX = Y$, then $X = I$. Suppose $X \neq I$ and write $X = I + pC$ where p is a positive power of 3 and $C \not\equiv 0 \bmod 3$. Note that $XY^rX = Y^r$ for all odd integers r. Write $Y = I + 3D$ where $D \in \mathrm{M}_6(\mathbb{Z})$. Then $Y^p \equiv I \bmod 3p$, so $X^2 \equiv I \bmod 3p$. Therefore $I + 2pC + p^2C \equiv I \bmod 3p$ which is not the case. Thus $X = I$ and we conclude that $A^3 = I$. Now write $A = I + qD$ where q is a positive power of 3 and $D \not\equiv 0 \bmod 3$. Then $(I+qD)^3 \equiv I \bmod 9q$, which shows that $3qD \equiv 0 \bmod 9q$ which is not the case.

3. Let $\mathbb{M} = \{2, 3, \dots\} = \mathbb{N} \setminus \{1\}$. Then $f^2(\mathbb{N}) = \mathbb{M}$ and therefore $f(\mathbb{N}) = \mathbb{N}$ or $\mathbb{M}$. The former yields $f^2(\mathbb{N}) = \mathbb{N}$, which is not the case, so we must have the latter which yields $f(\mathbb{M}) = \mathbb{M}$. It follows that $f^2(\mathbb{M}) = \mathbb{M}$ and we have a contradiction, so there is no such f, as required.

4. Let $d = \gcd(m,n)$. Then $d = an + bm$ for some integers a and b. Now $\binom{n}{m} = \frac{n}{m}\binom{n-1}{m-1}$, therefore

$$\frac{d}{n}\binom{n}{m} = (a + bm/n)\binom{n}{m} = a\binom{n}{m} + b\binom{n-1}{m-1}.$$

Since $\binom{n}{m}$ and $\binom{n-1}{m-1}$ are integers, the result follows.

5. We'll show that (a_n) is unbounded. We have $a_{n-1} = \int_0^{1/\sqrt{n-1}} \frac{|1-e^{nit}|}{|1-e^{it}|}\,dt$. Note that $|1-e^{it}| \le t$ for $t \ge 0$. To see this, by squaring both sides, this is equivalent to $2-2\cos t \le t^2$, i.e. $t^2+2\cos t-2 \ge 0$, which is true because we have equality when $t=0$, and the derivative of the left hand side is non-negative for $t \ge 0$ by using the inequality $\sin t \le t$ for $t \ge 0$. Therefore it will be sufficient to show that $b_n := \int_0^{1/\sqrt{n-1}} |1-e^{nit}|/t\,dt$ is unbounded (because $\pi/4 < 1$). However for $n \in \mathbb{Z}$,

$$\int_{\pi r/n}^{\pi(r+1)/n} |1-e^{nit}|\,dt = \int_{\pi r/n}^{\pi(r+1)/n} \sqrt{2-2\cos nt} = 4/n.$$

Let $k = [\sqrt{n-1}/\pi]$, so k is the greatest positive integer such that $k\pi < \sqrt{n-1}$. Note that $k \to \infty$ as $n \to \infty$. Then $b_n \ge \frac{4}{\pi}(1+1/2+\cdots+1/k)$, which is unbounded because the harmonic series is divergent.

6. First we show that $a_n - b_n \ge 0$ for all $n \ge 1$. This is equivalent to proving

$$(1+\frac{1}{n})(1/2+1/4+\cdots+\frac{1}{2n}) \le 1+1/3+\cdots+\frac{1}{2n-1},$$

that is

$$1+1/2+1/3+\cdots+1/n \le n\big((2-1)+(2/3-2/4)+\cdots+(\frac{2}{2n-1}-\frac{2}{2n})\big).$$

Since $1+1/2+\cdots+1/n \le n$, the assertion follows. Since $a_1 - b_1 = 0$, we see that the minimum of $a_n - b_n$ is zero.

Next we show that $a_n - b_n$ is decreasing for n sufficiently large. We have

$$\begin{aligned}(a_n-b_n)-(a_{n+1}-b_{n+1}) &= a_n - a_{n+1} - (b_n - b_{n+1})\\ &= \frac{1}{(n+1)(n+2)}(1+1/3+\cdots+\frac{1}{2n-1}) - \frac{1}{(n+2)(2n+1)}\\ &\quad - \frac{1}{n(n+1)}(1/2+1/4+\cdots+\frac{1}{2n}) + \frac{1}{(n+1)(2n+2)}.\end{aligned}$$

Now $\frac{1}{(n+1)(2n+2)} - \frac{1}{(n+2)(2n+1)} > 0$ for all $n \ge 1$, so we need to prove

$$\frac{1}{(n+1)(n+2)}(1+1/3+\cdots+\frac{1}{2n-1}) > \frac{1}{n(n+1)}(1/2+1/4+\cdots+\frac{1}{2n})$$

for n sufficiently large. Multiplying by $n(n+1)(n+2)$ and then subtracting $n(1/2+1/4+\cdots+\frac{1}{2n})$ from both sides, means we want to prove

$$n(1/2+1/12+\cdots+\frac{1}{(2n-1)2n}) > 1+1/2+\cdots+1/n$$

for sufficiently large n. However this is clear for $n \geq 4$. Therefore $a_n - b_n$ takes its maximum value for some $n \leq 4$. By inspection, the maximum value occurs when $n = 3$, which is 7/90.

7. Note that if g and h are continuous piecewise-monotone functions on $[a,b]$, then $\ell(gh) \leq \ell(g)\ell(h)$. Thus $\ell(f^n) \leq (\ell(f))^n$ for all $n \in \mathbb{N}$. Now fix a positive integer k. Given $n \in \mathbb{N}$, there are integers q and r such that $n = qk+r$ with $0 \leq r < k$. Then $\ell(f^n) \leq (\ell(f^k))^q(\ell(f))^r$, consequently

$$\sqrt[n]{\ell(f^n)} \leq (\ell(f^k))^{q/n}(\ell(f))^{r/n}.$$

Since k is fixed, $r/n \to 0$ and $q/n \to 1/k$ as $n \to \infty$. Therefore $\limsup \sqrt[n]{\ell(f^n)} \leq \sqrt[k]{\ell(f^k)}$ and we deduce that

$$\limsup \sqrt[n]{\ell(f^n)} \leq \inf \sqrt[k]{\ell(f^k)} \leq \liminf \sqrt[k]{\ell(f^k)}$$

and the result follows.

41st VTRMC, 2019, Solutions

1. Let M denote the minimal value of $f(n)$. Clearly $M \leq 2+7+7+1=17$. We will show that $M=17$, so assume by way of contradiction that $M<17$. Choose $n \in \mathbb{N}$ with $f(n)=M$, and write n in reverse order as $1a_1 \ldots a_d$ where $a_d \neq 0$ (so n is a $(d+1)$-digit number). We have $f(n) \equiv 2771^n \equiv (-1)^n \mod 9$. First assume that n is odd, so $f(n) \equiv -1 \mod 9$, so we must have $a_1+\cdots+a_d=7$. We also have $1-a_1+a_2-\cdots \equiv 2771^n \equiv -1 \mod 11$, so $-a_1+a_2-a_3+\cdots=-2$. Adding these two equations, we obtain $2a_2+2a_4+\cdots=5$, a contradiction because the left hand side is an even integer and the right hand side is an odd integer. Now assume that n is an even integer. Then we have $f(n) \equiv 1 \mod 9$ and therefore $a_1+\cdots+a_d=9$. Also $1-a_1+a_2-\cdots \equiv 1 \mod 11$ and therefore $-a_1+a_2-a_3+\cdots=0$. Adding the last two equations, we obtain $2a_2+2a_4+\cdots=9$, again a contradiction and the result follows.

2. Since $BX/XA=9$, we see that $AX=AB/10$ and we deduce that the area of AXC is 1/10 of the area of ABC, because they have the same height. Using the fact that the area of XYC is 9/100 of the area of ABC, we find that the area XYB is 81/100 of the area of ABC. Therefore the area of XBY is 9/10 of the area of XBC. Let H be the point on AB such that $\angle AHC=90^\circ$. Since XBY and XBC have the same base, we see that $MY=(9/10)CH$. Now MBY and HBC are similar, consequently

$$HB=(10/9)MB=(10/9)\cdot(1/2)\cdot XB=(10/9)\cdot(1/2)\cdot(9/10)AB=(1/2)AB.$$

Therefore $AC=BC$ and hence $BC=20$.

3. Define $g(x)=\int_0^x(1-t)f(t)\,dt$. Then $g(0)=0$ and

$$\begin{aligned} g(1) &= \int_0^1 f(x)\,dx-\int_0^1 xf(x)\,dx \\ &= \sum_{d=0}^{n}\frac{a_d}{d+1}-\sum_{d=0}^{n}\frac{a_d}{d+2} \\ &= \sum_{d=0}^{n}\frac{a_d}{(d+2)(d+1)}=0. \end{aligned}$$

By Rolle's theorem, there exists $q \in (0,1)$ such that $g'(q)=0$, that is $(1-q)f(q)=0$. Since $q \neq 1$, we deduce that $f(q)=0$ as required.

4. Let $I = \int_0^1 \frac{x^2}{x+\sqrt{1-x^2}}\,dx$. We make the substitution $x = \sin t$. Then $dx = dt\cos t$ and we see that $I = \int_0^{\pi/2} \frac{\sin^2 t\cos t}{\sin t+\cos t}\,dt$. Also by making the substitution $x = \cos t$, we see that $I = \int_0^{\pi/2} \frac{\cos^2 t\sin t}{\sin t+\cos t}\,dt$ and we deduce that
$$2I = \int_0^{\pi/2} \frac{\sin^2 t\cos t + \cos^2 t\sin t}{\sin t + \cos t}\,dt.$$
Since $\sin^2 t\cos t + \cos^2 t\sin t = \sin t\cos t(\sin t + \cos t)$, we find that $2I = \int_0^{\pi/2} \sin t\cos t\,dt$. Therefore $4I = \int_0^{\pi/2} \sin 2t\,dt$ and we conclude that $I = 1/4$.

5. We make the substitution $t = 1/x$. Let y' and y'' denote the first and second derivatives of y with respect to t, respectively.Then $dy/dx = -t^2y$ and $d^2y/dx^2 = 2t^3y' + t^4y''$ and by substituting back into the original equation, we obtain $y'' + (2t^{-1} - 2)y' + (1 - 2t^{-1})y = 0$. It is easy to see that $y = e^t$ is a solution to this equation. We now use reduction of order to obtain a second solution, so let $y = f(t)e^t$ be another solution, where f is to be determined. Then $f'' + 2t^{-1}f' = 0$, which has the solution $f = t^{-1}$. We deduce that $e^{1/x}$ and $xe^{1/x}$ are solutions to the original equation. Since these solutions are clearly linearly independent, it follows that the general solution to the original equation is $y = C_1e^{1/x} + C_2xe^{1/x}$, where C_1 and C_2 are arbitrary constants.

6. For each $s \in S$, there exist $m,n,p,q \in \mathbb{N}$ and $a,b \in \{\pm 1\}$ such that $s \in (am/n, bp/q)$ and $S \cap (am/n, bp/q) = \{s\}$. Then we may define
$f(s) = 2^{a+1}3^{b+1}5^m7^n11^p13^q$.

7. For $d \in \mathbb{N}$, the number of d-digit integers in S is 9^d, because we have 9 choices for each digit, and all these integers are $\geq 10^{d-1}$. Therefore the series is bounded by
$$\sum_{d\in\mathbb{N}} 9^d \left(10^{d-1}\right)^{-99/100} = \sum_{d\in\mathbb{N}} 9^d 10^{-99(d-1)/100}.$$
This is a geometric series with ratio between successive terms $9 \cdot 10^{-99/100}$; we show that this ratio is < 1. Rearranging, we find that we need to prove $10^{99}/9^{99} > 9$, equivalently $(1 + 1/9)^{99} > 9$, which is true by the binomial series. It follows that the geometric series is convergent, and we conclude by the comparison test that the original series is convergent.