1st Annual
Virginia Tech Regional Mathematics Contest
From 9:30 a.m. to 12:00 noon, November 10, 1979

Fill out the individual registration form

1. Show that the right circular cylinder of volume V which has the least surface area is the one whose diameter is equal to its altitude. (The top and bottom are part of the surface.)

2. Let S be a set which is closed under the binary operation $\circ$, with the following properties:

   (i) there is an element $e \in S$ such that $a \circ e = e \circ a = a$, for each $a \in S$,

   (ii) $(a \circ b) \circ (c \circ d) = (a \circ c) \circ (b \circ d)$, for all $a, b, c, d \in S$.

   Prove or disprove:

   (a) $\circ$ is associative on S

   (b) $\circ$ is commutative on S

3. Let A be an $n \times n$ nonsingular matrix with complex elements, and let $\bar{A}$ be its complex conjugate. Let $B = A\bar{A} + I$, where I is the $n \times n$ identity matrix.

   (a) Prove or disprove: $A^{-1}BA = \bar{B}$.

   (b) Prove or disprove: the determinant of $A\bar{A} + I$ is real.

4. Let $f(x)$ be continuously differentiable on $(0, \infty)$ and suppose $\lim_{x \to \infty} f'(x) = 0$. Prove that $\lim_{x \to \infty} f(x)/x = 0$.

5. Show, for all positive integers $n = 1, 2, \ldots$, that 14 divides $3^{4n+2} + 5^{2n+1}$.

6. Suppose $a_n > 0$ and $\sum_{n=1}^{\infty} a_n$ diverges. Determine whether $\sum_{n=1}^{\infty} a_n/S_n^2$ converges, where $S_n = a_1 + a_2 + \cdots + a_n$.

7. Let S be a finite set of non-negative integers such that $|x - y| \in S$ whenever $x, y \in S$.

   (a) Give an example of such a set which contains ten elements.

(b) If A is a subset of S containing more than two-thirds of the elements of S, prove or disprove that *every* element of S is the sum or difference of two elements from A.

8. Let S be a finite set of polynomials in two variables, x and y. For n a positive integer, define $\Omega_n(S)$ to be the collection of all expressions $p_1 p_2 \ldots p_k$, where $p_i \in S$ and $1 \leq k \leq n$. Let $d_n(S)$ indicate the maximum number of linearly independent polynomials in $\Omega_n(S)$. For example, $\Omega_2(\{x^2, y\}) = \{x^2, y, x^2y, x^4, y^2\}$ and $d_2(\{x^2, y\}) = 5$.

   (a) Find $d_2(\{1, x, x+1, y\})$.

   (b) Find a closed formula in n for $d_n(\{1, x, y\})$.

   (c) Calculate the least upper bound over all such sets of $\overline{\lim}_{n\to\infty} \dfrac{\log d_n(S)}{\log n}$.

   ($\overline{\lim}_{n\to\infty} a_n = \lim_{n\to\infty}(\sup\{a_n, a_{n+1}, \ldots\})$, where sup means supremum or least upper bound.)

2nd Annual
Virginia Tech Regional Mathematics Contest
From 9:30 a.m. to 12:00 noon, November 8, 1980

Fill out the individual registration form

1. Let * denote a binary operation on a set S with the property that

$$(w*x)*(y*z) = w*z \quad \text{for all } w,x,y,z \in S.$$

   Show

   (a) If $a*b = c$, then $c*c = c$.

   (b) If $a*b = c$, then $a*x = c*x$ for all $x \in S$.

2. The sum of the first n terms of the sequence

$$1, \quad (1+2), \quad (1+2+2^2), \ldots, (1+2+\cdots+2^{k-1}), \ldots$$

   is of the form $2^{n+R} + Sn^2 + Tn + U$ for all $n > 0$. Find R, S, T and U.

3. Let $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n}$.

   (a) Prove that $\lim_{n\to\infty} a_n$ exists.

   (b) Show that $a_n = \dfrac{(1-(\frac{1}{2})^2)(1-(\frac{1}{4})^2)\ldots(1-(\frac{1}{2n})^2)}{(2n+1)a_n}$.

   (c) Find $\lim_{n\to\infty} a_n$ and justify your answer.

4. Let $P(x)$ be any polynomial of degree at most 3. It can be shown that there are numbers x_1 and x_2 such that $\int_{-1}^{1} P(x)\,dx = P(x_1) + P(x_2)$, where x_1 and x_2 are independent of the polynomial P.

   (a) Show that $x_1 = -x_2$.

   (b) Find x_1 and x_2.

5. For $x > 0$, show that $e^x < (1+x)^{1+x}$.

6. Given the linear fractional transformation of x into $f_1(x) = (2x-1)/(x+1)$, define $f_{n+1}(x) = f_1(f_n(x))$ for $n = 1,2,3,\ldots$. It can be shown that $f_{35} = f_5$. Determine A, B, C, and D so that $f_{28}(x) = (Ax+B)/(Cx+D)$.

7. Let S be the set of all ordered pairs of integers (m,n) satisfying $m > 0$ and $n < 0$. Let $\langle$ be a partial ordering on S defined by the statement: $(m,n) \langle (m',n')$ if and only if $m \leq m'$ and $n \leq n'$. An example is $(5,-10) \langle (8,-2)$. Now let O be a completely ordered subset of S, i.e. if $(a,b) \in O$ and $(c,d) \in O$, then $(a,b) \langle (c,d)$ or $(c,d) \langle (a,b)$. Also let $\mathcal{O}$ denote the collection of all such completely ordered sets.

   (a) Determine whether an arbitrary $O \in \mathcal{O}$ is finite.

   (b) Determine whether the cardinality $\|O\|$ of O is bounded for $O \in \mathcal{O}$.

   (c) Determine whether $\|O\|$ can be countably infinite for any $O \in \mathcal{O}$.

8. Let $z = x + iy$ be a complex number with x And y rational and with $|z| = 1$.

   (a) Find two such complex numbers.

   (b) Show that $|z^{2n} - 1| = 2|\sin n\theta|$, where $z = e^{i\theta}$.

   (c) Show that $|z^{2n} - 1|$ is rational for every n.

3rd Annual
Virginia Tech Regional Mathematics Contest
From 9:30 a.m. to 12:00 noon, November 7, 1981

Fill out the individual registration form

1. The number $2^{48}-1$ is exactly divisible by what two numbers between 60 and 70?

2. For which real numbers b does the function $f(x)$, defined by the conditions $f(0)=b$ and $f'=2f-x$, satisfy $f(x)>0$ for all $x\geq 0$?

3. Let A be non-zero square matrix with the property that $A^3=0$, where 0 is the zero matrix, but with A being otherwise arbitrary.

   (a) Express $(I-A)^{-1}$ as a polynomial in A, where I is the identity matrix.

   (b) Find a 3×3 matrix satisfying $B^2\neq 0$, $B^3=0$.

4. Define $F(x)$ by $F(x)=\sum_{n=0}^{\infty}F_n x^n$ (wherever the series converges), where F_n is the nth Fibonacci number defined by $F_0=F_1=1$, $F_n=F_{n-1}+F_{n-2}$, $n>1$. Find an explicit closed form for $F(x)$.

5. Two elements A,B in a group G have the property $ABA^{-1}B=1$, where 1 denotes the identity element in G.

   (a) Show that $AB^2=B^{-2}A$.

   (b) Show that $AB^n=B^{-n}A$ for any integer n.

   (c) Find u and v so that $(B^aA^b)(B^cA^d)=B^uA^v$.

6. With k a positive integer, prove that $(1-k^{-2})^k\geq 1-1/k$.

7. Let $A=\{a_0,a_1,\dots\}$ be a sequence of real numbers and define the sequence $A'=\{a'_0,a'_1,\dots\}$ as follows for $n=0,1,\dots$: $a'_{2n}=a_n$, $a'_{2n+1}=a_n+1$. If $a_0=1$ and $A'=A$, find

   (a) a_1,a_2,a_3 and a_4

   (b) a_{1981}

   (c) A simple general algorithm for evaluating a_n, for $n=0,1,\dots$.

8. Let

(i) $0 < a < 1$,

(ii) $0 < M_{k+1} < M_k$, for $k = 0, 1, \ldots$,

(iii) $\lim_{k\to\infty} M_k = 0$.

If $b_n = \sum_{k=0}^{\infty} a^{n-k} M_k$, prove that $\lim_{n\to\infty} b_n = 0$.

4th Annual
Virginia Tech Regional Mathematics Contest

From 9:30 a.m. to 12:00 noon, November 6, 1982

Fill out the individual registration form

1. What is the remainder when $X^{1982}+1$ is divided by $X-1$? Verify your answer.

2. A box contains marbles, each of which is red, white or blue. The number of blue marbles is a least half the number of white marbles and at most on-third the number of red marbles. The number which are white or blue is at least 55. Find the minimum possible number of red marbles.

3. Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be vectors such that $\{\mathbf{a},\mathbf{b},\mathbf{c}\}$ is linearly dependent. Show that

$$\begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix} = 0.$$

4. Prove that $t^{n-1}+t^{1-n}<t^n+t^{-n}$ when $t\neq 1$, $t>0$ and n is a positive integer.

5. When asked to state the Maclaurin Series, a student writes (incorrectly)

$$(*) \quad f(x) = f(x)+xf'(x)+\frac{x^2}{2!}f''(x)+\frac{x^3}{3!}f'''(x)+\cdots.$$

   (a) State Maclaurin's Series for $f(x)$ correctly.

   (b) Replace the left-hand side of (*) by a simple closed form expression in f in such a way that the statement becomes valid (in general).

6. Let S be a set of positive integers and let E be the operation on the set of subsets of S defined by $EA=\{x\in A \mid x \text{ is even}\}$, where $A\subseteq S$. Let CA denote the complement of A in S. $ECEA$ will denote $E(C(EA))$ etc.

   (a) Show that $ECECEA=EA$.

   (b) Find the maximum number of distinct subsets of S that can be generated by applying the operations E and C to a subset A of S an arbitrary number of times in any order.

7. Let $p(x)$ be a polynomial of the form $p(x) = ax^2 + bx + c$, where a, b and c are *integers*, with the property that $1 < p(1) < p(p(1)) < p(p(p(1)))$. Show that $a \geq 0$.

8. For $n \geq 2$, define S_n by $S_n = \sum_{k=n}^{\infty} \frac{1}{k^2}$.

   (a) Prove or disprove that $1/n < S_n < 1/(n-1)$.

   (b) Prove or disprove that $S_n < 1/(n-3/4)$.

5th Annual
Virginia Tech Regional Mathematics Contest

From 9:30 a.m. to 12:00 noon, November 5, 1983

Fill out the individual registration form

1. In the expansion of $(a+b)^n$, where n is a natural number, there are $n+1$ dissimilar terms. Find the number of dissimilar terms in the expansion of $(a+b+c)^n$.

2. A positive integer N (in base 10) is called *special* if the operation C of replacing each digit d of N by its nine's-complement $9-d$, followed by the operation R of reversing the order of the digits, results in the original number. (For example, 3456 is a special number because $R[(C3456)] = 3456$.) Find the sum of all special positive integers less than one million which do not end in zero or nine.

3. Let a triangle have vertices at $O(0,0)$, $A(a,0)$, and $B(b,c)$ in the (x,y)-plane.

   (a) Find the coordinates of a point $P(x,y)$ in the exterior of $\triangle OAB$ satisfying $\text{area}(OAP) = \text{area}(OBP) = \text{area}(ABP)$.

   (b) Find a point $Q(x,y)$ in the interior of $\triangle OAQ$ satisfying $\text{area}(OAQ) = \text{area}(OBQ) = \text{area}(ABQ)$.

4. A finite set of roads connect n towns $T_1, T_2, \ldots, T_n$ where $n \geq 2$. We say that towns T_i and T_j $(i \neq j)$ are *directly connected* if there is a road segment connecting T_i and T_j which does not pass through any other town. Let $f(T_k)$ be the number of other towns directly connected to T_k. Prove that f is not one-to-one.

5. Find the function $f(x)$ such that for all $L \geq 0$, the area under the graph of $y = f(x)$ and above the x-axis from $x = 0$ to $x = L$ equals the arc length of the graph from $x = 0$ to $x = L$. (Hint: recall that $\frac{d}{dx}\cosh^{-1} x = 1/\sqrt{x^2-1}$.)

6. Let $f(x) = 1/x$ and $g(x) = 1-x$ for $x \in (0,1)$. List all distinct functions that can be written in the form $f \circ g \circ f \circ g \circ \cdots \circ f \circ g \circ f$ where $\circ$ represents composition. Write each function in the form $\dfrac{ax+b}{cx+d}$, and prove that your list is exhaustive.

7. If a and b are real, prove that $x^4 + ax + b = 0$ cannot have only real roots.

8. A sequence f_n is generated by the recurrence formula

$$f_{n+1} = \frac{f_n f_{n-1} + 1}{f_{n-2}},$$

for $n = 2, 3, 4, \ldots$, with $f_0 = f_1 = f_2 = 1$. Prove that f_n is integer-valued for all integers $n \geq 0$.

6th Annual
Virginia Tech Regional Mathematics Contest
From 9:30 a.m. to noon November 3, 1984

Fill out the individual registration form

1. Find the units digit (base 10) in the sum $\sum_{k=1}^{99} k!$.

2. Consider any three consecutive positive integers. Prove that the cube of the largest cannot be the sum of the cubes of the other two.

3. A sequence $\{u_n\}$, $n = 0,1,2,\ldots,$ is defined by $u_0 = 5$, $u_{n+1} = u_n + n^2 + 3n+3$, for $n = 0,1,2,\ldots$. If u_n is expressed as a polynomial $u_n = \sum_{k=0}^{d} c_k n^k$, where d is the degree of the polynomial, find the sum $\sum_{k=0}^{d} c_k$.

4. Let the (x,y)-plane be divided into regions by n lines, any two of which may or may not intersect. Describe a procedure whereby these regions may be colored using only two colors so that regions with a common line segment as part of their boundaries have different colors.

5. Let $f(x)$ satisfy the conditions for Rolle's theorem on $[a,b]$ with $f(a) = f(b) = 0$. Prove that for each real number k the function $g(x) = f'(x) + kf(x)$ has at least one zero in (a,b).

6. A matrix is called excellent if it is square and the sum of its elements in each row and column equals the sum of its elements in every other row and column. Let V_n denote the collection of excellent $n \times n$ matrices.

   (a) Show that V_n is a vector space under addition and scalar multiplication (by real numbers).

   (b) Find the dimensions of V_2, V_3, and V_4.

   (c) If $A \in V_n$ and $B \in V_n$, show that $AB \in V_n$.

7. Find the greatest real r such that some normal line to the graph of $y = x^3 + rx$ passes through the origin, where the point of normality is not the origin.

8. Let $f = f(x)$ be an arbitrary differentiable function on $I = [x_0 - h, x_0 + h]$ with $|f'(x)| \leq M$ on I where $M \geq \sqrt{3}$. Let $f(x_0 - h) \leq f(x_0)$ and $f(x_0 + h) \leq$

$f(x_0)$. Find the smallest positive number r such that at least one local maximum of f lies inside or on the circle of radius r centered at $(x_0, f(x_0))$. Express your answer in terms of h, M and $d = \min\{f(x_0) - f(x_0 - h), f(x_0) - f(x_0 + h)\}$.

7th Annual
Virginia Tech Regional Mathematics Contest
From 9:30 a.m. to noon November 2, 1985

Fill out the individual registration form

1. Prove that $\sqrt{ab} \leq (a+b)/2$ where a and b are positive real numbers.

2. Find the remainder r, $1 \leq r \leq 13$, when 2^{1985} is divided by 13.

3. Find real numbers c_1 and c_2 so that
$$I + c_1 M + c_2 M^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$
where $M = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$ and I is the identity matrix.

4. Consider an infinite sequence $\{c_k\}_{k=0}^{\infty}$ of circles. The largest, C_0, is centered at $(1,1)$ and is tangent to both the x and y-axes. Each smaller circle C_n is centered on the line through $(1,1)$ and $(2,0)$ and is tangent to the next larger circle C_{n-1} and to the x-axis. Denote the diameter of C_n by d_n for $n = 0,1,2,\ldots$. Find
   (a) d_1
   (b) $\displaystyle\sum_{n=0}^{\infty} d_n$

5. Find the function $f = f(x)$, defined and continuous on $\mathbb{R}^+ = \{x \mid 0 \leq x < \infty\}$, that satisfies $f(x+1) = f(x) + x$ on $\mathbb{R}^+$ and $f(1) = 0$.

6. (a) Find an expression for 3/5 as a finite sum of distinct reciprocals of positive integers. (For example: $2/7 = 1/7 + 1/8 + 1/56$.)
   (b) Prove that any positive rational number can be so expressed.

7. Let $f = f(x)$ be a real function of a real variable which has continuous third derivative and which satisfies, for a given c and all real x, $x \neq c$,
$$\frac{f(x) - f(c)}{x - c} = (f'(x) + f'(c))/2.$$
Show that $f''(x) = f'(x - f'(c))/(x - c)$.

8. Let $p(x) = a_0 + a_1 x + \cdots + a_n x^n$, where the coefficients a_i are real. Prove that $p(x) = 0$ has at least one root in the interval $0 \leq x \leq 1$ if $a_0 + a_1/2 + \cdots + a_n/(n+1) = 0$.

8th Annual
Virginia Tech Regional Mathematics Contest

From 9:30 a.m. to 12:00 noon, November 1, 1986

Fill out the individual registration form

1. Let $x_1 = 1$, $x_2 = 3$, and
$$x_{n+1} = \frac{1}{n+1}\sum_{i=1}^{n} x_i \quad \text{for } n = 2, 3, \ldots.$$
Find $\lim_{n\to\infty}$ and give a proof of your answer.

2. Given that $a > 0$ and $c > 0$, find a necessary and sufficient condition on b so that $ax^2 + bx + c > 0$ for all $x > 0$.

3. Express $\sinh 3x$ as a polynomial in $\sinh x$. As an example, the identity $\cos 2x = 2\cos^2 x - 1$ shows that $\cos 2x$ can be expressed as a polynomial in $\cos x$. (Recall that sinh denotes the hyperbolic sine defined by $\sinh x = (e^x - e^{-x})/2$.)

4. Find the quadratic polynomial $p(t) = a_0 + a_1 t + a_2 t^2$ such that $\int_0^1 t^n p(t)\,dt = n$ for $n = 0, 1, 2$.

5. Verify that, for $f(x) = x + 1$,
$$\lim_{r\to 0^+}\left(\int_0^1 (f(x))^r\,dx\right)^{1/r} = e^{\int_0^1 \ln f(x)\,dx}.$$

6. Sets A and B are defined by $A = \{1, 2, \ldots, n\}$ and $B = \{1, 2, 3\}$. Determine the number of distinct functions from A onto B. (A function $f\colon A \to B$ is "onto" if for each $b \in B$ there exists $a \in A$ such that $f(a) = b$.)

7. A function f from the positive integers to the positive integers has the properties:
   - $f(1) = 1$,
   - $f(n) = 2$ if $n \geq 100$,
   - $f(n) = f(n/2)$ if n is even and $n < 100$,

- $f(n) = f(n^2+7)$ if n is odd and $n > 1$.

(a) Find all positive integers n for which the stated properties require that $f(n) = 1$.

(b) Find all positive integers n for which the stated properties do not determine $f(n)$.

8. Find all pairs N, M of positive integers, $N < M$, such that

$$\sum_{j=N}^{M} \frac{1}{j(j+1)} = \frac{1}{10}.$$

9th Annual
Virginia Tech Regional Mathematics Contest
From 9:30 a.m. to 12:00 noon, October 31, 1987

Fill out the individual registration form

1. A path zig-zags from $(1,0)$ to $(0,0)$ along line segments $\overline{P_nP_{n+1}}$, where P_0 is $(1,0)$ and P_n is $(2^{-n},(-2)^{-n})$, for $n>0$. Find the length of the path.

2. A triangle with sides of lengths a, b, and c is partitioned into two smaller triangles by the line which is perpendicular to the side of length c and passes through the vertex opposite that side. Find *integers* $a<b<c$ such that each of the two smaller triangles is similar to the original triangle and has sides of integer lengths.

3. Let $a_1,a_2,\ldots,a_n$ be an arbitrary rearrangement of $1,2,\ldots,n$. Prove that if n is odd, then $(a_1-1)(a_2-2)\ldots(a_n-n)$ is even.

4. Let $p(x)$ be given by $p(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n$ and let $|p(x)|\le|x|$ on $[-1,1]$.

   (a) Evaluate a_0. (b) Prove that $|a_1|\le 1$.

5. A sequence of integers $\{n_1,n_2,\ldots\}$ is defined as follows: n_1 is assigned arbitrarily and, for $k>1$,

$$n_k=\sum_{j=1}^{j=k-1} z(n_j),$$

   where $z(n)$ is the number of 0's in the binary representation of n (each representation should have a leading digit of 1 except for zero which has the representation 0). An example, with $n_1=9$, is $\{9,2,3,3,3,\ldots\}$, or in binary, $\{1001,10,11,11,11,\ldots\}$.

   (a) Find n_1 so hat $\lim_{k\to\infty} n_k=31$, and calculate $n_2,n_3,\ldots,n_{10}$.

   (b) Prove that, for every choice of n_1, the sequence $\{n_k\}$ converges.

6. A sequence of polynomials is given by $p_n(x)=a_{n+2}x^2+a_{n+1}x-a_n$, for $n\ge 0$, where $a_0=a_1=1$ and, for $n\ge 0$, $a_{n+2}=a_{n+1}+a_n$. Denote by r_n and s_n the roots of $p_n(x)=0$, with $r_n\le s_n$. Find $\lim_{n\to\infty} r_n$ and $\lim_{n\to\infty} s_n$.

7. Let $A = \{a_{ij}\}$ and $B = \{b_{ij}\}$ be $n \times n$ matrices such that A^{-1} exists. Define $A(t) = \{a_{ij}(t)\}$ and $B(t) = \{b_{ij}(t)\}$ by $a_{ij}(t) = a_{ij}$ for $i < n$, $a_{nj}(t) = ta_{nj}$, $b_{ij}(t) = b_{ij}$ for $i < n$, and $b_{nj}(t) = tb_{nj}$. For example, if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then $A(t) = \begin{bmatrix} 1 & 2 \\ 3t & 4t \end{bmatrix}$. Prove that $A(t)^{-1}B(t) = A^{-1}B$ for $t > 0$ and any n. (Partial credit will be given for verifying the result for $n = 3$.)

8. On Halloween, a black cat and a witch encounter each other near a large mirror positioned along the y-axis. The witch is *invisible except by reflection* in the mirror. At $t = 0$, the cat is at $(10, 10)$ and the witch is at $(10, 0)$. For $t \geq 0$, the witch moves toward the cat at a speed numerically equal to their distance of separation and the cat moves toward the apparent position of the witch, as seen by reflection, at a speed numerically equal to their reflected distance of separation. Denote by $(u(t), v(t))$ the position of the cat and by $(x(t), y(t))$ the position of the witch.

   (a) Set up the equations of motion of the cat and the witch for $t \geq 0$.

   (b) Solve for $x(t)$ and $u(t)$ and find the time when the cat strikes the mirror. (Recall that the mirror is a perpendicular bisector of the line joining an object with its apparent position as seen by reflection.)

10th Annual
Virginia Tech Regional Mathematics Contest

From 9:30 a.m. to 12:00 noon, October 22, 1988

Fill out the individual registration form

1. A circle C of radius r is circumscribed by a parallelogram S. Let θ denote one of the interior angles of S, with $0 < \theta \leq \pi/2$. Calculate the area of S as a function of r and θ.

2. A man goes into a bank to cash a check. The teller mistakenly reverses the amounts and gives the man cents for dollars and dollars for cents. (Example: if the check was for \$5.10, the man was given \$10.05.). After spending five cents, the man finds that he still has twice as much as the original check amount. What was the original check amount? Find all possible solutions.

3. Find the general solution of $y(x) + \int_1^x y(t)\,dt = x^2$.

4. Let a be a positive integer. Find all positive integers n such that $b = a^n$ satisfies the condition that $a^2 + b^2$ is divisible by $ab + 1$.

5. Let f be differentiable on $[0,1]$ and let $f(\alpha) = 0$ and $f(x_0) = -.0001$ for some α and $x_0 \in (0,1)$. Also let $|f'(x)| \geq 2$ on $[0,1]$. Find the smallest upper bound on $|\alpha - x_0|$ for all such functions.

6. Find positive real numbers a and b such that $f(x) = ax - bx^3$ has four extrema on $[-1,1]$, at each of which $|f(x)| = 1$.

7. For any set S of real numbers define a new set $f(S)$ by $f(S) = \{x/3 \mid x \in S\} \cup \{(x+2)/3 \mid x \in S\}$.

   (a) Sketch, carefully, the set $f(f(f(I)))$, where I is the interval $[0,1]$.

   (b) If T is a bounded set such that $f(T) = T$, determine, *with proof*, whether T can contain 1/2.

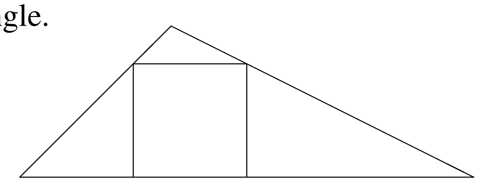
8. Let $T(n)$ be the number of incongruent triangles with integral sides and perimeter $n \geq 6$. Prove that $T(n) = T(n-3)$ if n is even, or disprove by a counterexample. (*Note*: two triangles are *congruent* if there is a one-to-one correspondence between the sides of the two triangles such that corresponding sides have the same length.)

11th Annual
Virginia Tech Regional Mathematics Contest

From 9:30 a.m. to noon October 21, 1989

Fill out the individual registration form

1. A square of side a is inscribed in a triangle of base b and height h as shown. Prove that the area of the square cannot exceed one-half the area of the triangle.



2. Let A be a 3×3 matrix in which each element is either 0 or 1 but is otherwise arbitrary.

   (a) Prove that $\det(A)$ cannot be 3 or -3.

   (b) Find all possible values of $\det(A)$ and prove your result.

3. The system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

   has the solution $x_1 = -1$, $x_2 = 3$, $x_3 = 2$ when $b_1 = 1$, $b_2 = 0$, $b_3 = 1$ and it has the solution $x_1 = 2$, $x = -2$, $x_3 = 1$ when $b_1 = 0$, $b_2 = -1$, $b_3 = 1$. Find a solution of the system when $b_1 = 2$, $b_2 = -1$, $b_3 = 3$.

4. Let a, b, c, d be distinct integers such that the equation

$$(x-a)(x-b)(x-c)(x-d) - 9 = 0$$

   has an integer root r. Show that $4r = a+b+c+d$. (This is essentially a problem from the 1947 Putnam examination.)

5. (i) Prove that $f_0(x) = 1 + x + x^2 + x^3 + x^4$ has no real zero.

(ii) Prove that, for every integer $n \geq 0$, $f_n(x) = 1 + 2^{-n}x + 3^{-n}x^2 + 4^{-n}x^3 + 5^{-n}x^4$ has no real zero. (Hint: consider $(d/dx)(xf_n(x))$.)

6. Let g be defined on $(1, \infty)$ by $g(x) = x/(x-1)$, and let $f^k(x)$ be defined by $f^0(x) = x$ and for $k > 0$, $f^k(x) = g(f^{k-1}(x))$. Evaluate $\sum_{k=0}^{\infty} 2^{-k} f^k(x)$ in the form $\dfrac{ax^2 + bx + c}{dx + e}$.

7. Three farmers sell chickens at a market. One has 10 chickens, another has 16, and the third has 26. Each farmer sells at least one, but not all, of his chickens before noon, all farmers selling at the same price per chicken. Later in the day each sells his remaining chickens, all again selling at the same *reduced* price. If each farmer received a total of \$35 from the sale of his chickens, what was the selling price before noon and the selling price after noon? (From “Math Can Be Fun” by Ya Perelman.)

8. The integer sequence $\{a_0, a_1, \ldots, a_{n-1}\}$ is such that, for each i $(0 \leq i \leq n-1)$, a_i is the number of i’s in the sequence. (Thus for $n = 4$ we might have the sequence $\{1, 2, 1, 0\}$.)

   (a) Prove that, if $n \geq 7$, such a sequence is a unique.

   (b) Find such a sequence for $n = 7$.

   Hint: show that the sum of all the terms is n, and that there are $n - a_0 - 1$ nonzero terms other than a_0 which sum to $n - a_0$. (This problem is slightly modified from one on the Cambridge Men’s Colleges Joint Awards and Entrance Examination, 24 November 1970.)

12th Annual
Virginia Tech Regional Mathematics Contest
From 9:30 a.m. to 12:00 noon, October 20, 1990

Fill out the individual registration form

1. Three pasture fields have areas of 10/3, 10 and 24 acres, respectively. The fields initially are covered with grass of the same thickness and new grass grows on each at the same rate per acre. If 12 cows eat the first field bare in 4 weeks and 21 cows eat the second field bare in 9 weeks, how many cows will eat the third field bare in 18 weeks? Assume that all cows eat at the same rate. (From *Math Can be Fun* by Ya Perelman.)

2. A person is engaged in working a jigsaw puzzle that contains 1000 pieces. It is found that it takes 3 minutes to put the first two pieces together and that when x pieces have been connected it takes $\dfrac{3(1000-x)}{1000+x}$ minutes to connect the next piece. Determine an accurate estimate of the time it takes to complete the puzzle. Give both a formula and an approximate numerical value in hours. (You may find useful the approximate value $\ln 2 = .69$.)

3. Let f be defined on the natural numbers as follows: $f(1) = 1$ and for $n > 1$, $f(n) = f(f(n-1)) + f(n - f(n-1))$. Find, with proof, a simple explicit expression for $f(n)$ which is valid for all $n = 1, 2, \ldots$.

4. Suppose that $P(x)$ is a polynomial of degree 3 with integer coefficients and that $P(1) = 0$, $P(2) = 0$. Prove that at least one of its four coefficients is equal to or less than -2.

5. Determine all real values of p for which the following series converge.

$$\text{(a)} \sum_{n=1}^{\infty} (\sin\frac{1}{n})^p \quad \text{(b)} \sum_{n=1}^{\infty} |\sin n|^p$$

6. The number of individuals in a certain population (in arbitrary real units) obeys, at discrete time intervals, the equation

$$y_{n+1} = y_n(2 - y_n) \quad \text{for } n = 0, 1, 2, \ldots,$$

where y_0 is the initial population.

(a) Find all "steady-state" solutions y^* such that, if $y_0 = y^*$, then $y_n = y^*$ for $n = 1, 2, \ldots$.

(b) Prove that if y_0 is any number in $(0,1)$, then the sequence $\{y_n\}$ converges monotonically to one of the steady-state solutions found in (a).

7. Let the following conditions be satisfied:

   (i) $f = f(x)$ and $g = g(x)$ are continuous functions on $[0,1]$,

   (ii) there exists a number a such that $0 < f(x) \leq a < 1$ on $[0,1]$,

   (iii) there exists a number u such that $\max\limits_{0 \leq x \leq 1} (g(x) + uf(x)) = u$.

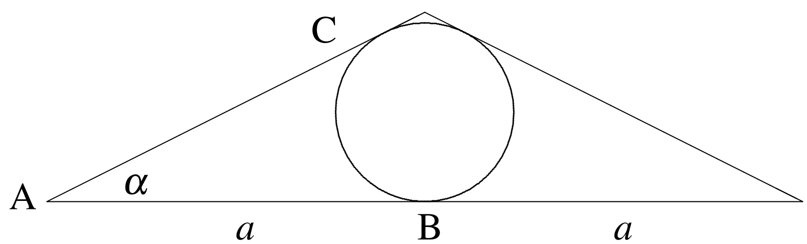
   Find constants A and B such that $F(x) = \dfrac{Ag(x)}{f(x)+B}$ is a continuous function on $[0,1]$ satisfying $\max\limits_{0 \leq x \leq 1} F(x) = u$, and prove that your function has the required properties.

8. Ten points in space, no three of which are collinear, are connected, each one to all the others, by a total of 45 line segments. The resulting framework F will be "disconnected" into two disjoint nonempty parts by the removal of one point from the interior of each of the 9 segments emanating from any one vertex of f. Prove that F cannot be similarly disconnected by the removal of only 8 points from the interiors of the 45 segments.

13th Annual
Virginia Tech Regional Mathematics Contest
From 9:30 a.m. to 12:00 noon, October 19, 1991

Fill out the individual registration form

1. An isosceles triangle with an inscribed circle is labeled as shown in the figure. Find an expression, in terms of the angle α and the length a, for the area of the curvilinear triangle bounded by sides AB and AC and the arc BC.



2. Find all differentiable functions f which satisfy $f(x)^3 = \int_0^x f(t)^2\,dt$ for all real x.

3. Prove that if α is a real root of $(1-x^2)(1+x+x^2+\cdots+x^n)-x=0$ which lies in $(0,1)$, with $n=1,2,\ldots$, then α is also a root of $(1-x^2)(1+x+x^2+\cdots+x^{n+1})-1=0$.

4. Prove that if $x>0$ and $n>0$, where x is real and n is an integer, then
$$\frac{x^n}{(x+1)^{n+1}} \le \frac{n^n}{(n+1)^{n+1}}.$$

5. Let $f(x)=x^5-5x^3+4x$. In each part (i)–(iv), prove or disprove that there exists a real number c for which $f(x)-c=0$ has a root of multiplicity

   (i) one, (ii) two, (iii) three, (iv) four.

6. Let $a_0=1$ and for $n>0$, let a_n be defined by
$$a_n = -\sum_{k=1}^{n} \frac{a_{n-k}}{k!}.$$

   Prove that $a_n=(-1)^n/n!$, for $n=0,1,2,\ldots$.

7. A and B play the following money game, where a_n and b_n denote the amount of holdings of A and B, respectively, after the n th round. At each round a player pays one-half his holdings to the bank, then receives one dollar from the bank if the *other* player had less than c dollars at the end of the *previous* round. If $a_0 = .5$ and $b_0 = 0$, describe the behavior of a_n and b_n when n is large, for

   (i) $c = 1.24$ and (ii) $c = 1.26$.

8. Mathematical National Park has a collection of trails. There are designated campsites along the trails, including a campsite at each intersection of trails. The rangers call each stretch of trail between adjacent campsites a "segment". The trails have been laid out so that it is possible to take a hike that starts at any campsite, covers each segment exactly once, and ends at the beginning campsite. Prove that it is possible to plan a collection $\mathscr{C}$ of hikes with all of the following properties:

   (i) Each segment is covered exactly once in one hike $h \in \mathscr{C}$ and never in any of the other hikes of $\mathscr{C}$.

   (ii) Each $h \in \mathscr{C}$ has a base campsite that is its beginning and end, but which is never passed in the middle of the hike. (Different hikes of $\mathscr{C}$ may have different base campsites.)

   (iii) Except for its base campsite at beginning and end, no hike in $\mathscr{C}$ passes any campsite more than once.

14th Annual
Virginia Tech Regional Mathematics Contest
From 9:30 a.m. to 12:00 noon, October 31, 1992

Fill out the individual registration form

1. Find the inflection point of the graph of $F(x) = \int_0^{x^3} e^{t^2}\,dt$, for $x \in \mathbb{R}$.

2. Assume that $x_1 > y_1 > 0$ and $y_2 > x_2 > 0$. Find a formula for the shortest length l of a planar path that goes from (x_1, y_1) to (x_2, y_2) and that touches both the x-axis and the y-axis. Justify your answer.

3. Let $f_n(x)$ be defined recursively by

$$f_0(x) = x, \quad f_1(x) = f(x), \quad f_{n+1}(x) = f(f_n(x)), \quad \text{for} \quad n \geq 0,$$

   where $f(x) = 1 + \sin(x-1)$.

   (i) Show that there is a unique point x_0 such that $f_2(x_0) = x_0$.

   (ii) Find $\sum_{n=0}^{\infty} \frac{f_n(x_0)}{3^n}$ with the above x_0.

4. Let $\{t_n\}_{n=1}^{\infty}$ be a sequence of positive numbers such that $t_1 = 1$ and $t_{n+1}^2 = 1 + t_n$, for $n \geq 1$. Show that t_n is increasing in n and find $\lim_{n\to\infty} t_n$.

5. Let $A = \begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix}$. Find A^{100}. You have to find all four entries.

6. Let $p(x)$ be the polynomial $p(x) = x^3 + ax^2 + bx + c$. Show that if $p(r) = 0$, then

$$\frac{p(x)}{x-r} - 2\frac{p(x+1)}{x+1-r} + \frac{p(x+2)}{x+2-r} = 2$$

   for all x except $x = r$, $r-1$ and $r-2$.

7. Find $\lim_{n\to\infty} \frac{2\log 2 + 3\log 3 + \cdots + n\log n}{n^2 \log n}$.

8. Some goblins, N in number, are standing in a row while "trick-or-treat"ing. Each goblin is at all times either $2'$ tall or $3'$ tall, but can change spontaneously from one of these two heights to the other at will. While lined up in such a row, a goblin is called a **Local Giant Goblin** (LGG) if he/she/it is not standing beside a taller goblin. Let $G(N)$ be the total of all occurrences of LGG's as the row of N goblins transmogrifies through all possible distinct configurations, where height is the only distinguishing characteristic. As an example, with $N = 2$, the distinct configurations are $\hat{2}\hat{2}$, $2\hat{3}$, $\hat{3}2$, $\hat{3}\hat{3}$, where a cap indicates an LGG. Thus $G(2) = 6$.

   (i) Find $G(3)$ and $G(4)$.

   (ii) Find, with proof, the general formula for $G(N)$, $N = 1, 2, 3, \ldots$.

15th Annual
Virginia Tech Regional Mathematics Contest

From 9:00 a.m. to 11:30 a.m., October 30, 1993

Fill out the individual registration form

1. Prove that $\displaystyle\int_0^1 \int_{x^2}^1 e^{y^{3/2}}\,dy\,dx = \frac{2e-2}{3}$.

2. Prove that if $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and $f(x) = \int_0^x f(t)\,dt$, then $f(x)$ is identically zero.

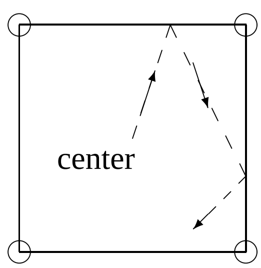
3. Let $f_1(x) = x$ and $f_{n+1}(x) = x^{f_n(x)}$, for $n = 1, 2 \ldots$. Prove that $f_n'(1) = 1$ and $f_n''(1) = 2$, for all $n \geq 2$.

4. Prove that a triangle in the plane whose vertices have integer coordinates cannot be equilateral.

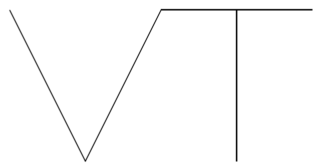
5. Find $\displaystyle\sum_{n=1}^{\infty} \frac{3^{-n}}{n}$.

6. Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a surjective map with the property that if the points A, B and C are collinear, then so are $f(A)$, $f(B)$ and $f(C)$. Prove that f is bijective.

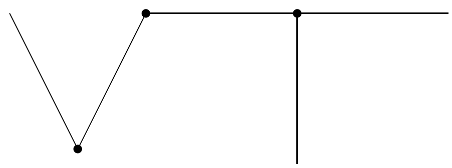
7. On a small square billiard table with sides of length 2 ft., a ball is played from the center and after rebounding off the sides several times, goes into a cup at one of the corners. Prove that the total distance travelled by the ball is **not** an integer number of feet.



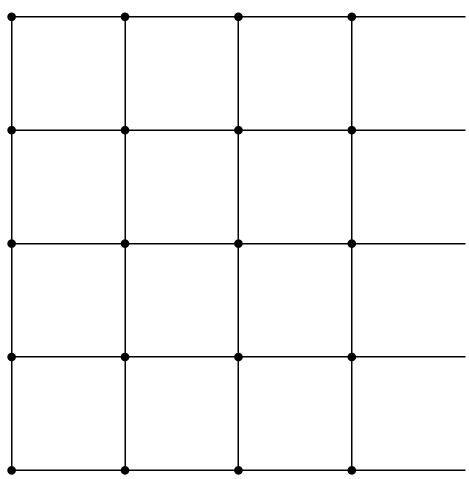
8. A popular Virginia Tech logo looks something like



Suppose that wire-frame copies of this logo are constructed of 5 equal pieces of wire welded at three places as shown:



If bending is allowed, but no re-welding, show clearly how to cut the maximum possible number of ready-made copies of such a logo from the piece of welded wire mesh shown. Also, prove that no larger number is possible.



16th Annual
Virginia Tech Regional Mathematics Contest
From 9:00 a.m. to 11:30 a.m., October 29, 1994

Fill out the individual registration form

1. Evaluate $\int_0^1 \int_0^x \int_0^{1-x^2} e^{(1-z)^2}\, dz dy dx.$

2. Let f be continuous real function, strictly increasing in an interval $[0,a]$ such that $f(0) = 0$. Let g be the inverse of f, i.e., $g(f(x)) = x$ for all x in $[0,a]$. Show that for $0 \le x \le a$, $0 \le y \le f(a)$, we have
$$xy \le \int_0^x f(t)\, dt + \int_0^y g(t)\, dt.$$

3. Find all continuously differentiable solutions $f(x)$ for
$$f(x)^2 = \int_0^x \left(f(t)^2 - f(t)^4 + (f'(t))^2\right) dt + 100$$
where $f(0)^2 = 100$.

4. Consider the polynomial equation $ax^4 + bx^3 + x^2 + bx + a = 0$, where a and b are real numbers, and $a > 1/2$. Find the maximum possible value of $a+b$ for which there is at least one positive real root of the above equation.

5. Let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ be a function which satisfies $f(0,0) = 1$ and
$$f(m,n) + f(m+1,n) + f(m,n+1) + f(m+1,n+1) = 0$$
for all $m,n \in \mathbb{Z}$ (where $\mathbb{Z}$ and $\mathbb{R}$ denote the set of all integers and all real numbers, respectively). Prove that $|f(m,n)| \ge 1/3$, for infinitely many pairs of integers (m,n).

6. Let A be an $n \times n$ matrix and let α be an n-dimensional vector such that $A\alpha = \alpha$. Suppose that all the entries of A and α are positive real numbers. Prove that α is the only linearly independent eigenvector of A corresponding to the eigenvalue 1. Hint: if β is another eigenvector, consider the minimum of $\alpha_i/|\beta_i|$, $i = 1, \ldots, n$, where the α_i's and β_i's are the components of α and β, respectively.

7. Define $f(1) = 1$ and $f(n+1) = 2\sqrt{f(n)^2+n}$ for $n \geq 1$. If $N \geq 1$ is an integer, find $\sum_{n=1}^{N} f(n)^2$.

8. Let a sequence $\{x_n\}_{n=0}^{\infty}$ of rational numbers be defined by $x_0 = 10$, $x_1 = 29$ and $x_{n+2} = \dfrac{19x_{n+1}}{94x_n}$ for $n \geq 0$. Find $\displaystyle\sum_{n=0}^{\infty} \frac{x_{6n}}{2^n}$.

17th Annual
Virginia Tech Regional Mathematics Contest
From 9:00 a.m. to 11:30 a.m., October 28, 1995

Fill out the individual registration form

1. Evaluate $\displaystyle\int_0^3 \int_0^2 \frac{1}{1+(\max(3x,2y))^2\,dxdy}$.

2. Let $\mathbb{R}^2$ denote the xy-plane, and define $\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $\theta(x,y) = (4x-3y+1, 2x-y+1)$. Determine $\theta^{100}(1,0)$, where θ^{100} indicates applying θ 100 times.

3. Let $n \geq 2$ be a positive integer and let $f(x)$ be the polynomial
$$1-(x+x^2+\cdots+x^n)+(x+x^2+\cdots+x^n)^2-\cdots+(-1)^n(x+x^2+\cdots+x^n)^n.$$
If r is an integer such that $2 \leq r \leq n$, show that the coefficient of x^r in $f(x)$ is zero.

4. Let $\tau = (1+\sqrt{5})/2$. Show that $[\tau^2 n] = [\tau[\tau n]+1]$ for every positive integer n. Here $[r]$ denotes the largest integer that is not larger than r.

5. Let $\mathbb{R}$ denote the real numbers, and let $\theta\colon \mathbb{R} \to \mathbb{R}$ be a map with the property that $x > y$ implies $(\theta(x))^3 > \theta(y)$. Prove that $\theta(x) > -1$ for all x, and that $0 \leq \theta(x) \leq 1$ for at most one value of x.

6. A straight rod of length 4 inches has ends which are allowed to slide along the perimeter of a square whose sides each have length 12 inches. A paint brush is attached to the rod so that it can slide between the two ends of the rod. Determine the total possible area of the square which can be painted by the brush.

7. If n is a positive integer larger than 1, let $n = \prod p_i^{k_i}$ be the unique prime factorization of n, where the p_i's are distinct primes, 2, 3, 5, 7, 11, ..., and define $f(n)$ by $f(n) = \sum k_i p_i$ and $g(n)$ by $g(n) = \lim_{m\to\infty} f^m(n)$, where f^m is meant the m-fold application of f. Then n is said to have *property H* if $n/2 < g(n) < n$.

   (i) Evaluate $g(100)$ and $g(10^{10})$.

   (ii) Find all positive odd integers larger than 1 that have property H.

18th Annual

Virginia Tech Regional Mathematics Contest

From 9:00 a.m. to 11:30 a.m., October 26, 1996

Fill out the individual registration form

1. Evaluate $\int_0^1 \int_{\sqrt{y-y^2}}^{\sqrt{1-y^2}} xe^{(x^4+2x^2y^2+y^4)}\,dxdy$.

2. For each rational number r, define $f(r)$ to be the smallest positive integer n such that $r = m/n$ for some integer m, and denote by $P(r)$ the point in the (x,y) plane with coordinates $P(r) = (r, 1/f(r))$. Find a necessary and sufficient condition that, given two rational numbers r_1 and r_2 such that $0 < r_1 < r_2 < 1$,
$$P\Big(\frac{r_1 f(r_1) + r_2 f(r_2)}{f(r_1) + f(r_2)}\Big)$$
will be the point of intersection of the line joining $(r_1, 0)$ and $P(r_2)$ with the line joining $P(r_1)$ and $(r_2, 0)$.

3. Solve the differential equation $y^y = e^{dy/dx}$ with the initial condition $y = e$ when $x = 1$.

4. Let $f(x)$ be a twice continuously differentiable in the interval $(0,\infty)$. If
$$\lim_{x\to\infty}(x^2 f''(x) + 4xf'(x) + 2f(x)) = 1,$$
find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} xf'(x)$. Do **not** assume any special form of $f(x)$. Hint: use l'Hôpital's rule.

5. Let a_i, $i = 1,2,3,4$, be real numbers such that $a_1 + a_2 + a_3 + a_4 = 0$. Show that for arbitrary real numbers b_i, $i = 1,2,3$, the equation
$$a_1 + b_1x + 3a_2x^2 + b_2x^3 + 5a_3x^4 + b_3x^5 + 7a_4x^6 = 0$$
has at least one real root which is on the interval $-1 \le x \le 1$.

6. There are $2n$ balls in the plane such that no three balls are on the same line and such that no two balls touch each other. n balls are red and the other n balls are green. Show that there is at least one way to draw n line segments by connecting each ball to a unique different colored ball so that no two line segments intersect.

7. Let us define

$$f_{n,0}(x) = x + \frac{\sqrt{x}}{n} \qquad \text{for } x > 0,\ n \geq 1,$$
$$f_{n,j+1}(x) = f_{n,0}(f_{n,j}(x)), \qquad j = 0, 1, \ldots, n-1.$$

Find $\lim_{n\to\infty} f_{n,n}(x)$ for $x > 0$.

19th Annual
Virginia Tech Regional Mathematics Contest
From 8:30 a.m. to 11:00 a.m., November 1, 1997

Fill out the individual registration form

1. Evaluate $\iint_D \frac{x^3}{x^2+y^2}\,dA$, where D is the half disk given by $(x-1)^2+y^2 \leq 1$, $y \geq 0$.

2. Suppose that $r_1 \neq r_2$ and $r_1r_2 = 2$. If r_1 and r_2 are roots of
$$x^4 - x^3 + ax^2 - 8x - 8 = 0,$$
find r_1, r_2 and a. (Do not assume that they are real numbers.)

3. Suppose that you are in charge of taking ice cream orders for a class of 100 students. If each student orders exactly one flavor from Vanilla, Strawberry, Chocolate and Pecan, how many different combinations of flavors are possible for the 100 orders you are taking. Here are some examples of possible combinations. You do not distinguish between individual students.
   (i) $V = 30, S = 20, C = 40, P = 10$.
   (ii) $V = 80, S = 0, C = 20, P = 0$.
   (iii) $V = 0, S = 0, C = 0, P = 100$.

4. A business man works in New York and Los Angeles. If he is in New York, each day he has four options; to remain in New York, or to fly to Los Angeles by either the 8:00 a.m., 1:00 p.m. or 6:00 p.m. flight. On the other hand if he is in Los Angeles, he has only two options; to remain in Los Angeles, or to fly to New York by the 8:00 a.m. flight. In a 100 day period he has to be in New York both at the beginning of the first day of the period, and at the end of the last day of the period. How many different possible itineraries does the business man have for the 100 day period (for example if it was for a 2 day period rather than a 100 day period, the answer would be 4)?

5. The VTRC bus company serves cities in the USA. A subset S of the cities is called well-served if it has at least three cities and from every city A in S,

one can take a nonstop VTRC bus to at least two different other cities B and C in S (though there is not necessarily a nonstop VTRC bus from B to A or from C to A). Suppose there is a well-served subset S. Prove that there is a well-served subset T such that for any two cities A, B in T, one can travel by VTRC bus from A to B, stopping only at cities in T.

6. A disk of radius 1 cm. has a small hole at a point half way between the center and the circumference. The disk is lying inside a circle of radius 2 cm. A pen is put through the hole in the disk, and then the disk is moved once round the inside of the circle, keeping the disk in contact with the circle without slipping, so the pen draws a curve. What is the area enclosed by the curve?

7. Let J be the set of all sequences of real numbers, and let A, L and P be three mappings from J to J defined as follows. If $x = \{x_n\} = \{x_0, x_1, x_2, \dots\} \in J$, then

$$\begin{aligned} Ax &= \{x_n + 1\} = \{x_0 + 1,\ x_1 + 1,\ x_2 + 1, \dots\}, \\ Lx &= \{1, x_0, x_1, x_2, \dots\}, \\ Px &= \{\sum_{k=0}^{n} x_k\}. \end{aligned}$$

Finally, define the composite mapping T on J by $Tx = L \circ A \circ Px$. In the following, let $y = \{1, 1, 1, \dots\}$.

(a) Write down $T^2 y$, giving the first eight terms of the sequence and a closed formula for the n-th term.

(b) Assuming that $z = \{z_n\} = \lim_{i \to \infty} T^i y$ exists, conjecture the general form for z_n, and prove your conjecture.

20th Annual
Virginia Tech Regional Mathematics Contest

From 8:30 a.m. to 11:00 a.m., October 31, 1998

Fill out the individual registration form

1. Let $f(x,y) = \ln(1-x^2-y^2) - \dfrac{1}{(y-x)^2}$ with domain $D = \{(x,y) \mid x \neq y,\ x^2 + y^2 < 1\}$. Find the maximum value M of $f(x,y)$ over D. You have to show that $M \geq f(x,y)$ for every $(x,y) \in D$. Here $\ln(\cdot)$ is the natural logarithm function.

2. The radius of the base of a right circular cone is 1. The vertex of the cone is V, and P is a point on the circumference of the base. The length of PV is 6 and the midpoint of PV is M. A piece of string is attached to M and wound tightly twice round the cone finishing at P. What is the length of the string?

3. Find the volume of the region which is common to the interiors of the three circular cylinders $y^2 + z^2 = 1$, $z^2 + x^2 = 1$ and $x^2 + y^2 = 1$.

4. Let ABC be a triangle and let P be a point on AB. Suppose $\angle BAC = 70^\circ$, $\angle APC = 60^\circ$, $AC = \sqrt{3}$ and $PB = 1$. Prove that ABC is an isosceles triangle.

5. Let a_n be sequence of positive numbers ($n = 1, 2, \ldots, a_n \neq 0$ for all n), and let $b_n = (a_1 + \cdots + a_n)/n$, the average of the first n numbers of the sequence. Suppose $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a_n}$ is a convergent series. Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{b_n}$ is also a convergent series.

6. Ten cats are sitting on ten fence posts, numbered 1 through 10 in clockwise order and encircling a pumpkin patch. The cat on post #1 is white and the other nine cats are black. At 9:45 p.m. the cats begin a strange sort of dance. They jump from post to post according to the following two rules, applied in alternation at one second intervals. Rule 1: each cat jumps clockwise to the next post. Rule 2: all pairs of cats whose post numbers have a product that is 1 greater than a multiple of 11 exchange places. At 10 p.m., just as the Great Pumpkin rises out of the pumpkin patch, the dance stops abruptly and the cats look on in awe. If the first jump takes place according to Rule 1 at 9:45:01, and the last jump occurs at 10:00:00, on which post is the white

cat sitting when the dance stops? (The first few jumps take the white cat from post 1 to posts 2,6,7, …)

21st Annual
Virginia Tech Regional Mathematics Contest
From 8:30 a.m. to 11:00 a.m., October 30, 1999

Fill out the individual registration form

1. Let G be the set of all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$, satisfying the following properties.

   (i) $f(x) = f(x+1)$ for all x,

   (ii) $\int_0^1 f(x)\,dx = 1999$.

   Show that there is a number α such that $\alpha = \displaystyle\int_0^1 \int_0^x f(x+y)\,dydx$ for all $f \in G$.

2. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is infinitely differentiable and satisfies both of the following properties.

   (i) $f(1) = 2$,

   (ii) If α, β are real numbers satisfying $\alpha^2 + \beta^2 = 1$, then $f(\alpha x) f(\beta x) = f(x)$ for all x.

   Find $f(x)$. Guesswork will not be accepted.

3. Let ε, M be positive real numbers, and let $A_1, A_2, \ldots$ be a sequence of matrices such that for all n,

   (i) A_n is an $n \times n$ matrix with integer entries,

   (ii) The sum of the absolute values of the entries in each row of A_n is at most M.

   If δ is a positive real number, let $e_n(\delta)$ denote the number of nonzero eigenvalues of A_n which have absolute value less that δ. (Some eigenvalues can be complex numbers.) Prove that one can choose $\delta > 0$ so that $e_n(\delta)/n < \varepsilon$ for all n.

4. A rectangular box has sides of length 3, 4, 5. Find the volume of the region consisting of all points that are within distance 1 of at least one of the sides.

5. Let $f: \mathbb{R}_+ \to \mathbb{R}_+$ be a function from the set of positive real numbers to the same set satisfying $f(f(x)) = x$ for all positive x. Suppose that f is infinitely differentiable for all positive X, and that $f(a) \neq a$ for some positive a. Prove that $\lim\limits_{x \to \infty} f(x) = 0$.

6. A set S of distinct positive integers has property **ND** if no element x of S divides the sum of the integers in any subset of $S \setminus \{x\}$. Here $S \setminus \{x\}$ means the set that remains after x is removed from S.

   (i) Find the smallest positive integer n such that $\{3,4,n\}$ has property **ND**.

   (ii) If n is the number found in (i), prove that no set S with property **ND** has $\{3,4,n\}$ as a proper subset.

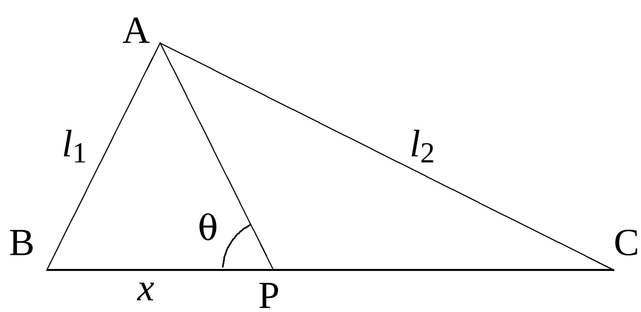
22nd Annual
Virginia Tech Regional Mathematics Contest
From 8:30 a.m. to 11:00 a.m., October 28, 2000

Fill out the individual registration form

1. Evaluate $\displaystyle\int_0^\alpha \frac{d\theta}{5-4\cos\theta}$ (your answer will involve inverse trig functions; you may assume that $0 \le \alpha < \pi$). Use your answer to show that $\displaystyle\int_0^{\pi/3} \frac{d\theta}{5-4\cos\theta} = \frac{2\pi}{9}$.

2. Let n be a positive integer and let A be an $n \times n$ matrix with real numbers as entries. Suppose $4A^4 + I = 0$, where I denotes the identity matrix. Prove that the trace of A (i.e. the sum of the entries on the main diagonal) is an integer.

3. Consider the initial value problem $y' = y^2 - t^2$; $y(0) = 0$ (where $y' = dy/dt$). Prove that $\lim_{t\to\infty} y'(t)$ exists, and determine its value.

4. 

In the above diagram, $l_1 = \overline{AB}$, $l_2 = \overline{AC}$, $x = \overline{BP}$, and $l = \overline{BC}$, where $\overline{AB}$ indicates the length of AB. Prove that $l_2 - l_1 = \displaystyle\int_0^l \cos(\theta(x))\,dx$.

5. Two diametrically opposite points P, Q lie on an infinitely long cylinder which has radius $2/\pi$. A piece of string with length 8 has its ends joined to P, is wrapped once round the outside of the cylinder, and then has its midpoint joined to Q (so there is length 4 of the string on each side of the cylinder). A paint brush is attached to the string so that it can slide along the full length the string. Find the area of the outside surface of the cylinder which can be painted by the brush.

6. Let a_n $(n \geq 1)$ be the sequence of numbers defined by the recurrence relation

$$a_1 = 1, \quad a_n = a_{n-1}a_1 + a_{n-2}a_2 + \cdots + a_2 a_{n-2} + a_1 a_{n-1}$$

(so $a_2 = a_1^2 = 1$, $a_3 = 2a_1a_2 = 2$ etc.). Prove that $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{9}\right)^n a_n = \frac{1}{3}$.

7. Two types of domino-like rectangular tiles, $[\,{}^{\bullet}{}_{\bullet} \mid \bullet\,]$ and $[\,{}^{\bullet}{}_{\bullet} \mid {}^{\bullet}{}_{\bullet}\,]$, are available. The first type may be rotated end-to-end to produce a tile of type $[\,\bullet \mid {}^{\bullet}{}_{\bullet}\,]$. Let $A(n)$ be the number of distinct chains of n tiles, placed end-to-end, that may be constructed if abutting ends are required to have the same number of dots.

   Example $A(2) = 5$, since the following five chains of length two, and no others, are allowed.

$$[\,\bullet \mid {}^{\bullet}{}_{\bullet}\,][\,{}^{\bullet}{}_{\bullet} \mid \bullet\,], \quad [\,\bullet \mid {}^{\bullet}{}_{\bullet}\,][\,{}^{\bullet}{}_{\bullet} \mid {}^{\bullet}{}_{\bullet}\,], \quad [\,{}^{\bullet}{}_{\bullet} \mid \bullet\,][\,\bullet \mid {}^{\bullet}{}_{\bullet}\,],$$
$$[\,{}^{\bullet}{}_{\bullet} \mid {}^{\bullet}{}_{\bullet}\,][\,{}^{\bullet}{}_{\bullet} \mid \bullet\,], \quad [\,{}^{\bullet}{}_{\bullet} \mid {}^{\bullet}{}_{\bullet}\,][\,{}^{\bullet}{}_{\bullet} \mid {}^{\bullet}{}_{\bullet}\,].$$

   (a) Find $A(3)$ and $A(4)$.

   (b) Find, with proof, a three-term recurrence formula for $A(n+2)$ in terms of $A(n+1)$ and $A(n)$, for $n = 1, 2, \ldots$, and use it to find $A(10)$.

23rd Annual

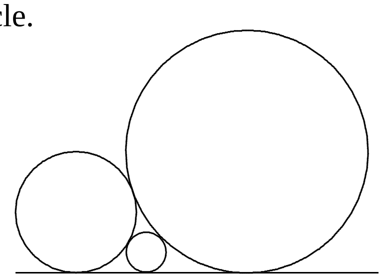
Virginia Tech Regional Mathematics Contest

From 8:30 a.m. to 11:00 a.m., November 3, 2001

Fill out the individual registration form

1. Three infinitely long circular cylinders each with unit radius have their axes along the x, y and z-axes. Determine the volume of the region common to all three cylinders. (Thus one needs the volume common to $\{y^2+z^2 \le 1\}$, $\{z^2+x^2 \le 1\}$, $\{x^2+y^2 \le 1\}$.)

2. Two circles with radii 1 and 2 are placed so that they are tangent to each other and a straight line. A third circle is nestled between them so that it is tangent to the first two circles and the line. Find the radius of the third circle.



3. For each positive integer n, let S_n denote the total number of squares in an $n \times n$ square grid. Thus $S_1 = 1$ and $S_2 = 5$, because a 2×2 square grid has four 1×1 squares and one 2×2 square. Find a recurrence relation for S_n, and use it to calculate the total number of squares on a chess board (i.e. determine S_8).

4. Let a_n be the nth positive integer k such that the greatest integer not exceeding $\sqrt{k}$ divides k, so the first few terms of $\{a_n\}$ are $\{1, 2, 3, 4, 6, 8, 9, 12, \dots\}$. Find a_{10000} and give reasons to substantiate your answer.

5. Determine the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{n^n x^n}{n!}$. (That is, determine the real numbers x for which the above power series converges; you must determine correctly whether the series is convergent at the end points of the interval.)

6. Find a function $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(f(x)) = \dfrac{3x+1}{x+3}$ for all positive real numbers x (here $\mathbb{R}^+$ denotes the positive (nonzero) real numbers).

7. Let G denote a set of invertible 2×2 matrices (matrices with complex numbers as entries and determinant nonzero) with the property that if a, b are in G, then so are ab and a^{-1}. Suppose there exists a function $f\colon G \to \mathbb{R}$ with the property that either $f(ga) > f(a)$ or $f(g^{-1}a) > f(a)$ for all a, g in G with $g \neq I$ (here I denotes the identity matrix, $\mathbb{R}$ denotes the real numbers, and the inequality signs are strict inequality). Prove that given finite nonempty subsets A, B of G, there is a matrix in G which can be written in exactly one way in the form xy with x in A and y in B.

24th Annual
Virginia Tech Regional Mathematics Contest

From 8:30 a.m. to 11:00 a.m., October 26, 2002

Fill out the individual registration form

1. Let a,b be positive constants. Find the volume (in the first octant) which lies above the region in the xy-plane bounded by $x=0$, $x=\pi/2$, $y=0$, $y\sqrt{b^2\cos^2 x+a^2\sin^2 x}=1$, and below the plane $z=y$.

2. Find rational numbers a,b,c,d,e such that

$$\sqrt{7+\sqrt{40}}=a+b\sqrt{2}+c\sqrt{5}+d\sqrt{7}+e\sqrt{10}.$$

3. Let A and B be nonempty subsets of $S=\{1,2,\ldots,99\}$ (integers from 1 to 99 inclusive). Let a and b denote the number of elements in A and B respectively, and suppose $a+b=100$. Prove that for each integer s in S, there are integers x in A and y in B such that $x+y=s$ or $s+99$.

4. Let $\{1,2,3,4\}$ be a set of abstract symbols on which the associative binary operation $*$ is defined by the following operation table (associative means $(a*b)*c=a*(b*c)$):

$*$	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

If the operation $*$ is represented by juxtaposition, e.g., $2*3$ is written as 23 etc., then it is easy to see from the table that of the four possible “words” of length two that can be formed using only 2 and 3, i.e., 22, 23, 32 and 33, exactly two, 22 and 33, are equal to 1. Find a formula for the number $A(n)$ of words of length n, formed by using only 2 and 3, that equal 1. From the table and the example just given for words of length two, it is clear that $A(1)=0$ and $A(2)=2$. Use the formula to find $A(12)$.

(Please turn over)

5. Let n be a positive integer. A bit string of length n is a sequence of n numbers consisting of 0's and 1's. Let $f(n)$ denote the number of bit strings of length n in which every 0 is surrounded by 1's. (Thus for $n = 5$, 11101 is allowed, but 10011 and 10110 are not allowed, and we have $f(3) = 2$, $f(4) = 3$.) Prove that $f(n) < (1.7)^n$ for all n.

6. Let S be a set of 2×2 matrices with complex numbers as entries, and let T be the subset of S consisting of matrices whose eigenvalues are ± 1 (so the eigenvalues for each matrix in T are $\{1,1\}$ or $\{1,-1\}$ or $\{-1,-1\}$). Suppose there are exactly three matrices in T. Prove that there are matrices A, B in S such that AB is not a matrix in S ($A = B$ is allowed).

7. Let $\{a_n\}_{n\geq 1}$ be an infinite sequence with $a_n \geq 0$ for all n. For $n \geq 1$, let b_n denote the geometric mean of $a_1, \ldots, a_n$, that is $(a_1 \ldots a_n)^{1/n}$. Suppose $\sum_{n=1}^{\infty} a_n$ is convergent. Prove that $\sum_{n=1}^{\infty} b_n^2$ is also convergent.

25th Annual
Virginia Tech Regional Mathematics Contest

From 8:30 a.m. to 11:00 a.m., November 1, 2003

Fill out the individual registration form

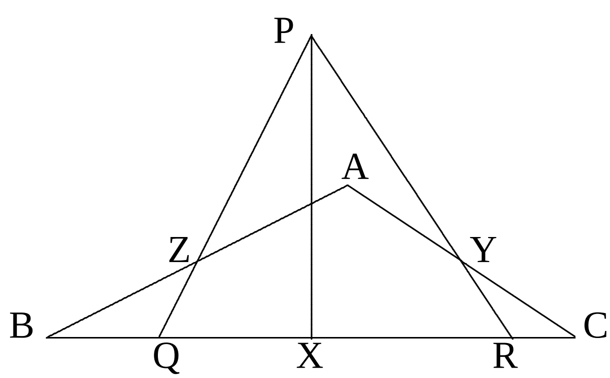
1. An investor buys stock worth \$10,000 and holds it for n business days. Each day he has an equal chance of either gaining 20% or losing 10%. However in the case he gains every day (i.e. n gains of 20%), he is deemed to have lost all his money, because he must have been involved with insider trading. Find a (simple) formula, with proof, of the amount of money he will have on average at the end of the n days.

2. Find $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)} = \frac{x}{1\cdot 2} + \frac{x^2}{2\cdot 3} + \frac{x^3}{3\cdot 4} + \cdots$ for $|x| < 1$.

3. Determine all invertible 2 by 2 matrices A with complex numbers as entries satisfying $A = A^{-1} = A'$, where A' denotes the transpose of A.

4. It is known that $2\cos^3\frac{\pi}{7} - \cos^2\frac{\pi}{7} - \cos\frac{\pi}{7}$ is a rational number. Write this rational number in the form p/q, where p and q are integers with q positive.

5. In the diagram below, X is the midpoint of BC, Y is the midpoint of AC, and Z is the midpoint of AB. Also $\angle ABC + \angle PQC = \angle ACB + \angle PRB = 90°$. Prove that $\angle PXC = 90°$.



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6. Let $f\colon [0,1] \longrightarrow [0,1]$ be a continuous function such that $f(f(f(x))) = x$ for all $x \in [0,1]$. Prove that $f(x) = x$ for all $x \in [0,1]$. Here $[0,1]$ denotes the closed interval of all real numbers between 0 and 1, including 0 and 1.

7. Let T be a solid tetrahedron whose edges all have length 1. Determine the volume of the region consisting of points which are at distance at most 1 from some point in T (your answer should involve $\sqrt{2}, \sqrt{3}, \pi$).

26th Annual
Virginia Tech Regional Mathematics Contest

From 8:30 a.m. to 11:00 a.m., October 23, 2004

Fill out the individual registration form

1. Let I denote the 2×2 identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and let

$$M = \begin{pmatrix} I & A \\ B & C \end{pmatrix}, \quad N = \begin{pmatrix} I & B \\ A & C \end{pmatrix}.$$

where A, B, C are arbitrary 2×2 matrices which entries in $\mathbb{R}$, the real numbers. Thus M and N are 4×4 matrices with entries in $\mathbb{R}$. Is it true that M is invertible (i.e. there is a 4×4 matrix X such that $MX = XM =$ the identity matrix) implies N is invertible? Justify your answer.

2. A sequence of integers $\{f(n)\}$ for $n = 0, 1, 2, \ldots$ is defined as follows: $f(0) = 0$ and for $n > 0$,

$$f(n) = \begin{cases} f(n-1) + 3, & \text{if } n = 0 \text{ or } 1 \text{ (mod 6)}, \\ f(n-1) + 1, & \text{if } n = 2 \text{ or } 5 \text{ (mod 6)}, \\ f(n-1) + 2, & \text{if } n = 3 \text{ or } 4 \text{ (mod 6)}. \end{cases}$$

Derive an explicit formula for $f(n)$ when $n = 0$ (mod 6), showing all necessary details in your derivation.

3. A computer is programmed to randomly generate a string of six symbols using only the letters A, B, C. What is the probability that the string will not contain three consecutive A's?

4. A 9×9 chess board has two squares from opposite corners and its central square removed (so 3 squares on the same diagonal are removed, leaving 78 squares). Is it possible to cover the remaining squares using dominoes, where each domino covers two adjacent squares? Justify your answer.

5. Let $f(x) = \int_0^x \sin(t^2 - t + x)\, dt$. Compute $f''(x) + f(x)$ and deduce that $f^{(12)}(0) + f^{(10)}(0) = 0$ ($f^{(10)}$ indicates 10 th derivative).

(Please turn over)

6. An enormous party has an infinite number of people. Each two people either know or don't know each other. Given a positive integer n, prove there are n people in the party such that either they all know each other, or nobody knows each other (so the first possibility means that if A and B are any two of the n people, then A knows B, whereas the second possibility means that if A and B are any two of the n people, then A does not know B).

7. Let $\{a_n\}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} a_n = 0$. Prove that $\sum_{n=1}^{\infty} \left| 1 - \frac{a_{n+1}}{a_n} \right|$ is divergent.

27th Annual

Virginia Tech Regional Mathematics Contest

From 8:30 a.m. to 11:00 a.m., October 29, 2005

Fill out the individual registration form

1. Find the largest positive integer n with the property that $n+6(p^3+1)$ is prime whenever p is a prime number such that $2 \le p < n$. Justify your answer.

2. Find, and write out explicitly, a permutation $(p(1), p(2), \ldots, p(20))$ of $(1,2,\ldots,20)$ such that $k+p(k)$ is a power of 2 for $k=1,2,\ldots,20$, and prove that only one such permutation exists. (To illustrate, a permutation of $(1,2,3,4,5)$ such that $k+p(k)$ is a power of 2 for $k=1,2,\ldots,5$ is clearly $(1,2,5,4,3)$, because $1+1=2$, $2+2=4$, $3+5=8$, $4+4=8$, and $5+3=8$.)

3. We wish to tile a strip of n 1-inch by 1-inch squares. We can use dominos which are made up of two tiles which cover two adjacent squares, or 1-inch square tiles which cover one square. We may cover each square with one or two tiles and a tile can be above or below a domino on a square, but no part of a domino can be placed on any part of a different domino. We do not distinguish whether a domino is above or below a tile on a given square. Let $t(n)$ denote the number of ways the strip can be tiled according to the above rules. Thus for example, $t(1)=2$ and $t(2)=8$. Find a recurrence relation for $t(n)$, and use it to compute $t(6)$.

4. A cubical box with sides of length 7 has vertices at $(0,0,0)$, $(7,0,0)$, $(0,7,0)$, $(7,7,0)$, $(0,0,7)$, $(7,0,7)$, $(0,7,7)$, $(7,7,7)$. The inside of the box is lined with mirrors and from the point $(0,1,2)$, a beam of light is directed to the point $(1,3,4)$. The light then reflects repeatedly off the mirrors on the inside of the box. Determine how far the beam of light travels before it first returns to its starting point at $(0,1,2)$.

5. Define $f(x,y) = \dfrac{xy}{x^2+(y\ln(x^2))^2}$ if $x \neq 0$, and $f(0,y)=0$ if $y \neq 0$. Determine whether $\lim_{(x,y)\to(0,0)} f(x,y)$ exists, and what its value is if the limit does exist.

(Please turn over)

6. Compute $\int_0^1 \left((e-1)\sqrt{\ln(1+ex-x)}+e^{(x^2)}\right)dx$.

7. Let A be a 5×10 matrix with real entries, and let A' denote its transpose (so A' is a 10×5 matrix, and the ijth entry of A' is the jith entry of A). Suppose every 5×1 matrix with real entries (i.e. column vector in 5 dimensions) can be written in the form $A\mathbf{u}$ where $\mathbf{u}$ is a 10×1 matrix with real entries. Prove that every 5×1 matrix with real entries can be written in the form $AA'\mathbf{v}$ where $\mathbf{v}$ is a 5×1 matrix with real entries.

28th Annual Virginia Tech Regional Mathematics Contest

From 9:00 a.m. to 11:30 a.m., October 28, 2006

Fill out the individual registration form

1. Find, and give a proof of your answer, all positive integers n such that neither n nor n^2 contain a 1 when written in base 3.

2. Let $S(n)$ denote the number of sequences of length n formed by the three letters A,B,C with the restriction that the C's (if any) all occur in a single block immediately following the first B (if any). For example ABCCAA, AAABAA, and ABCCCC are counted in, but ACACCB and CAAAAA are not. Derive a simple formula for $S(n)$ and use it to calculate $S(10)$.

3. Recall that the Fibonacci numbers $F(n)$ are defined by $F(0) = 0$, $F(1) = 1$, and $F(n) = F(n-1) + F(n-2)$ for $n \geq 2$. Determine the last digit of $F(2006)$ (e.g. the last digit of 2006 is 6).

4. We want to find functions $p(t), q(t), f(t)$ such that

   (a) p and q are continuous functions on the open interval $(0, \pi)$.

   (b) f is an infinitely differentiable nonzero function on the whole real line $(-\infty, \infty)$ such that $f(0) = f'(0) = f''(0)$.

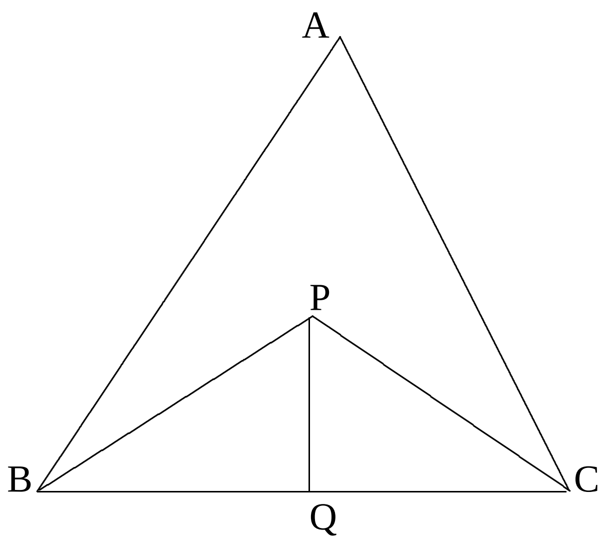
   (c) $y = \sin t$ and $y = f(t)$ are solutions of the differential equation $y'' + p(t)y' + q(t)y = 0$ on $(0, \pi)$.

   Is this possible? Either prove this is not possible, or show this is possible by providing an explicit example of such f, p, q.

5. Let $\{a_n\}$ be a monotonic decreasing sequence of positive real numbers with limit 0 (so $a_1 \geq a_2 \geq \cdots \geq 0$). Let $\{b_n\}$ be a rearrangement of the sequence such that for every non-negative integer m, the terms $b_{3m+1}, b_{3m+2}, b_{3m+3}$ are a rearrangement of the terms $a_{3m+1}, a_{3m+2}, a_{3m+3}$ (thus, for example, the first 6 terms of the sequence $\{b_n\}$ could be $a_3, a_2, a_1, a_4, a_6, a_5$). Prove or give a counterexample to the following statement: the series $\sum_{n=1}^{\infty} (-1)^n b_n$ is convergent.

(Please turn over)

6. In the diagram below BP bisects $\angle ABC$, CP bisects $\angle BCA$, and PQ is perpendicular to BC. If $BQ.QC = 2PQ^2$, prove that $AB + AC = 3BC$.



7. Three spheres each of unit radius have centers P, Q, R with the property that the center of each sphere lies on the surface of the other two spheres. Let C denote the cylinder with cross-section PQR (the triangular lamina with vertices P, Q, R) and axis perpendicular to PQR. Let M denote the space which is common to the three spheres and the cylinder C, and suppose the mass density of M at a given point is the distance of the point from PQR. Determine the mass of M.

29th Annual Virginia Tech Regional Mathematics Contest

From 9:00 a.m. to 11:30 a.m., October 27, 2007

Fill out the individual registration form

1. Evaluate $\displaystyle\int_0^x \frac{d\theta}{2+\tan\theta}$, where $0 \le x \le \pi/2$. Use your result to show that $\displaystyle\int_0^{\pi/4} \frac{d\theta}{2+\tan\theta} = \frac{\pi+\ln(9/8)}{10}$.

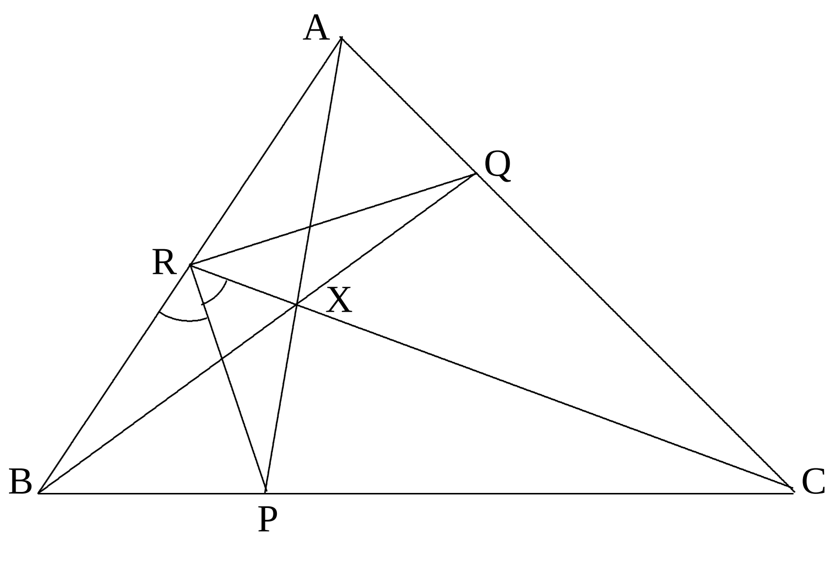
2. Given that $e^x = 1/0! + x/1! + x^2/2! + \cdots + x^n/n! + \cdots$ find, in terms of e, the exact values of

   (a) $\displaystyle\frac{1}{1!}+\frac{2}{3!}+\frac{3}{5!}+\cdots+\frac{n}{(2n-1)!}+\cdots$ and

   (b) $\displaystyle\frac{1}{3!}+\frac{2}{5!}+\frac{3}{7!}+\cdots+\frac{n}{(2n+1)!}+\cdots$

3. Solve the initial value problem $\displaystyle\frac{dy}{dx} = y\ln y + ye^x$, $y(0)=1$ (i.e. find y in terms of x).

4. In the diagram below, P, Q, R are points on BC, CA, AB respectively such that the lines AP, BQ, CR are concurrent at X. Also PR bisects $\angle BRC$, i.e. $\angle BRP = \angle PRC$. Prove that $\angle PRQ = 90^\circ$.



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5. Find the third digit after the decimal point of
$$(2+\sqrt{5})^{100}\big((1+\sqrt{2})^{100}+(1+\sqrt{2})^{-100}\big).$$
For example, the third digit after the decimal point of $\pi = 3.14159\ldots$ is 1.

6. Let n be a positive integer, let A, B be square symmetric $n \times n$ matrices with real entries (so if a_{ij} are the entries of A, the a_{ij} are real numbers and $a_{ij} = a_{ji}$). Suppose there are $n \times n$ matrices X, Y (with complex entries) such that $\det(AX + BY) \neq 0$. Prove that $\det(A^2 + B^2) \neq 0$ (det indicates the determinant).

7. Determine whether the series $\displaystyle\sum_{n=2}^{\infty} n^{-(1+(\ln(\ln n))^{-2})}$ is convergent or divergent (ln denotes natural log).

30th Annual Virginia Tech Regional Mathematics Contest

From 9:00 a.m. to 11:30 a.m., November 1, 2008

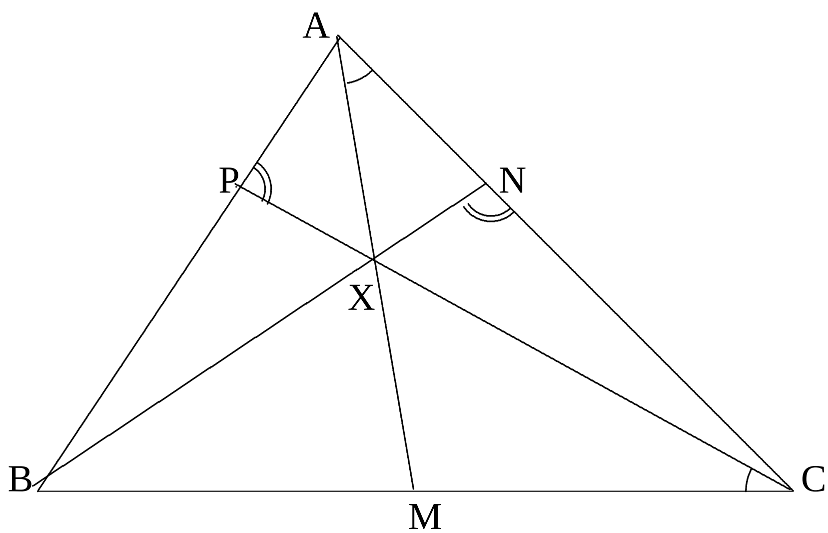
Fill out the individual registration form

1. Find the maximum value of $xy^3+yz^3+zx^3-x^3y-y^3z-z^3x$ where $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$.

2. How many sequences of 1's and 3's sum to 16? (Examples of such sequences are $\{1,3,3,3,3,3\}$ and $\{1,3,1,3,1,3,1,3\}$.)

3. Find the area of the region of points (x,y) in the xy-plane such that $x^4+y^4 \le x^2-x^2y^2+y^2$.

4. Let ABC be a triangle, let M be the midpoint of BC, and let X be a point on AM. Let BX meet AC at N, and let CX meet AB at P. If $\angle MAC = \angle BCP$, prove that $\angle BNC = \angle CPA$.



5. Let $a_1, a_2, \ldots$ be a sequence of nonnegative real numbers and let π, ρ be permutations of the positive integers $\mathbb{N}$ (thus $\pi, \rho \colon \mathbb{N} \to \mathbb{N}$ are one-to-one and onto maps). Suppose that $\sum_{n=1}^{\infty} a_n = 1$ and ε is a real number such that $\sum_{n=1}^{\infty} |a_n - a_{\pi n}| + \sum_{n=1}^{\infty} |a_n - a_{\rho n}| < \varepsilon$. Prove that there exists a finite subset X of $\mathbb{N}$ such that $|X \cap \pi X|, |X \cap \rho X| > (1-\varepsilon)|X|$ (here $|X|$ indicates the number of elements in X; also the inequalities $<, >$ are strict).

(Please turn over)

6. Find all pairs of positive (nonzero) integers a, b such that $ab - 1$ divides $a^4 - 3a^2 + 1$.

7. Let $f_1(x) = x$ and $f_{n+1}(x) = x^{f_n(x)}$ for n a positive integer. Thus $f_2(x) = x^x$ and $f_3(x) = x^{(x^x)}$. Now define $g(x) = \lim_{n\to\infty} 1/f_n(x)$ for $x > 1$. Is g continuous on the open interval $(1, \infty)$? Justify your answer.

31st Annual Virginia Tech Regional Mathematics Contest

From 9:00 a.m. to 11:30 a.m., October 24, 2009

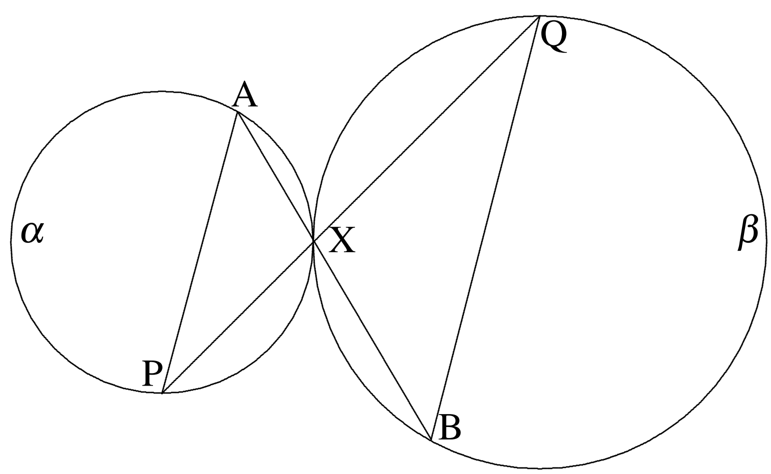
Fill out the individual registration form

1. A walker and a jogger travel along the same straight line in the same direction. The walker walks at one meter per second, while the jogger runs at two meters per second. The jogger starts one meter in front of the walker. A dog starts with the walker, and then runs back and forth between the walker and the jogger with constant speed of three meters per second. Let $f(n)$ meters denote the total distance travelled by the dog when it has returned to the walker for the n th time (so $f(0) = 0$). Find a formula for $f(n)$.

2. Given that 40! =
   abc def 283 247 897 734 345 611 269 596 115 894 272 *pqr stu vwx*
   find p,q,r,s,t,u,v,w,x, and then find a,b,c,d,e,f.

3. Define $f(x) = \int_0^x \int_0^x e^{u^2v^2}\,du\,dv$. Calculate $2f''(2) + f'(2)$ (here $f'(x) = df/dx$).

4. Two circles α, β touch externally at the point X. Let A, P be two distinct points on α different from X, and let AX and PX meet β again in the points B and Q respectively. Prove that AP is parallel to QB.



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5. Let $\mathbb{C}$ denote the complex numbers and let $\mathrm{M}_3(\mathbb{C})$ denote the 3 by 3 matrices with entries in $\mathbb{C}$. Suppose $A, B \in \mathrm{M}_3(\mathbb{C})$, $B \neq 0$, and $AB = 0$ (where 0 denotes the 3 by 3 matrix with all entries zero). Prove that there exists $0 \neq D \in \mathrm{M}_3(\mathbb{C})$ such that $AD = DA = 0$.

6. Let n be a nonzero integer. Prove that $n^4 - 7n^2 + 1$ can never be a perfect square (i.e. of the form m^2 for some integer m).

7. Does there exist a twice differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f'(x) = f(x+1) - f(x)$ for all x and $f''(0) \neq 0$? Justify your answer. (Here $\mathbb{R}$ denotes the real numbers and f' denotes the derivative of f.)

32nd Annual Virginia Tech Regional Mathematics Contest

From 9:00 a.m. to 11:30 a.m., October 30, 2010

Fill out the individual registration form

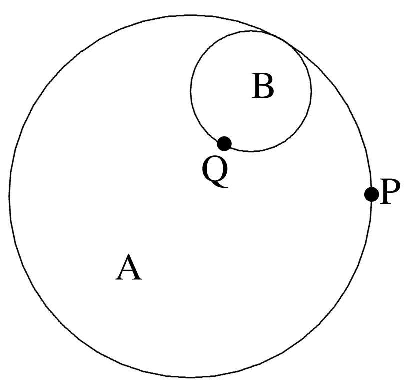
1. Let d be a positive integer and let A be a $d \times d$ matrix with integer entries. Suppose $I + A + A^2 + \cdots + A^{100} = 0$ (where I denotes the identity $d \times d$ matrix, so I has 1's on the main diagonal, and 0 denotes the zero matrix, which has all entries 0). Determine the positive integers $n \leq 100$ for which $A^n + A^{n+1} + \cdots + A^{100}$ has determinant ± 1.

2. For n a positive integer, define $f_1(n) = n$ and then for i a positive integer, define $f_{i+1}(n) = f_i(n)^{f_i(n)}$. Determine $f_{100}(75) \mod 17$ (i.e. determine the remainder after dividing $f_{100}(75)$ by 17, an integer between 0 and 16). Justify your answer.

3. Prove that $\cos(\pi/7)$ is a root of the equation $8x^3 - 4x^2 - 4x + 1 = 0$, and find the other two roots.

4. Let $\triangle ABC$ be a triangle with sides a, b, c and corresponding angles A, B, C (so $a = BC$ and $A = \angle BAC$ etc.). Suppose that $4A + 3C = 540°$. Prove that $(a-b)^2(a+b) = bc^2$.

5. Let A, B be two circles in the plane with B inside A. Assume that A has radius 3, B has radius 1, P is a point on A, Q is a point on B, and A and B touch so that P and Q are the same point. Suppose that A is kept fixed and B is rolled once round the inside of A so that Q traces out a curve starting and finishing at P. What is the area enclosed by this curve?



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6. Define a sequence by $a_1 = 1$, $a_2 = 1/2$, and $a_{n+2} = a_{n+1} - \dfrac{a_n a_{n+1}}{2}$ for n a positive integer. Find $\lim\limits_{n\to\infty} n a_n$.

7. Let $\sum_{n=1}^{\infty} a_n$ be a convergent series of positive terms (so $a_i > 0$ for all i) and set $b_n = \dfrac{1}{n a_n^2}$ for $n \geq 1$. Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{n}{b_1 + b_2 + \cdots + b_n}$ is convergent.

33rd Annual Virginia Tech Regional Mathematics Contest

From 9:00 a.m. to 11:30 a.m., October 29, 2011

Fill out the individual registration form

1. Evaluate $\displaystyle\int_1^4 \frac{x-2}{(x^2+4)\sqrt{x}}\,dx$.

2. A sequence (a_n) is defined by $a_0 = -1$, $a_1 = 0$, and
$$a_{n+1} = a_n^2 - (n+1)^2 a_{n-1} - 1$$
for all positive integers n. Find a_{100}.

3. Find $\displaystyle\sum_{k=1}^{\infty} \frac{k^2-2}{(k+2)!}$.

4. Let m,n be positive integers and let $[a]$ denote the residue class mod mn of the integer a (thus $\{[r] \mid r \text{ is an integer}\}$ has exactly mn elements). Suppose the set $\{[ar] \mid r \text{ is an integer}\}$ has exactly m elements. Prove that there is a positive integer q such that q is prime to mn and $[nq] = [a]$.

5. Find $\displaystyle\lim_{x\to\infty} (2x)^{1+\frac{1}{2x}} - x^{1+\frac{1}{x}} - x$.

6. Let S be a set with an asymmetric relation $<$; this means that if $a,b \in S$ and $a < b$, then we do not have $b < a$. Prove that there exists a set T containing S with an asymmetric relation $\prec$ with the property that if $a,b \in S$, then $a < b$ if and only if $a \prec b$, and if $x,y \in T$ with $x \prec y$, then there exists $t \in T$ such that $x \prec t \prec y$ ($t \in T$ means "t is an element of T").

7. Let $P(x) = x^{100} + 20x^{99} + 198x^{98} + a_{97}x^{97} + \cdots + a_1 x + 1$ be a polynomial where the a_i ($1 \le i \le 97$) are real numbers. Prove that the equation $P(x) = 0$ has at least one complex root (i.e. a root of the form $a + bi$ with a,b real numbers and $b \ne 0$).

34th Annual Virginia Tech Regional Mathematics Contest

From 9:00 a.m. to 11:30 a.m., October 27, 2012

Fill out the individual registration form

1. Evaluate
$$\int_0^{\pi/2} \frac{\cos^4 x + \sin x \, \cos^3 x + \sin^2 x \, \cos^2 x + \sin^3 x \, \cos x}{\sin^4 x + \cos^4 x + 2\sin x \, \cos^3 x + 2\sin^2 x \, \cos^2 x + 2\sin^3 x \, \cos x}\, dx.$$

2. Solve in real numbers the equation $3x - x^3 = \sqrt{x+2}$.

3. Find nonzero complex numbers a, b, c, d, e such that
$$\begin{aligned} a+b+c+d+e &= -1 \\ a^2+b^2+c^2+d^2+e^2 &= 15 \\ \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e} &= -1 \\ \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}+\frac{1}{e^2} &= 15 \\ abcde &= -1 \end{aligned}$$

4. Define $f(n)$ for n a positive integer by $f(1) = 3$ and $f(n+1) = 3^{f(n)}$. What are the last two digits of $f(2012)$?

5. Determine whether the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n} - \left(\frac{1}{\ln n}\right)^{(n+1)/n}$ is convergent.

6. Define a sequence (a_n) for n a positive integer inductively by $a_1 = 1$ and $a_n = \dfrac{n}{\prod_{\substack{d|n \\ 1\le d<n}} a_d}$. Thus $a_2 = 2$, $a_3 = 3$, $a_4 = 2$ etc. Find a_{999000}.

7. Let A_1, A_2, A_3 be 2×2 matrices with entries in $\mathbb{C}$ (the complex numbers). Let tr denote the trace of a matrix (so $\operatorname{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d$). Suppose $\{A_1, A_2, A_3\}$ is closed under matrix multiplication (i.e. given i, j, there exists k such that $A_iA_j = A_k$), and $\operatorname{tr}(A_1 + A_2 + A_3) \neq 3$. Prove that there exists i such that $A_iA_j = A_jA_i$ for all j (here i, j are 1, 2 or 3).

35th Annual Virginia Tech Regional Mathematics Contest

From 9:00 a.m. to 11:30 a.m., October 26, 2013

Fill out the individual registration form

1. Let $I = 3\sqrt{2}\int_0^x \frac{\sqrt{1+\cos t}}{17-8\cos t}dt$. If $0 < x < \pi$ and $\tan I = \frac{2}{\sqrt{3}}$, what is x?

2. Let ABC be a right-angled triangle with $\angle ABC = 90°$, and let D on AB such that $AD = 2DB$. What is the maximum possible value of $\angle ACD$?

3. Define a sequence (a_n) for $n \geq 1$ by $a_1 = 2$ and $a_{n+1} = a_n^{1+n^{-3/2}}$. Is (a_n) convergent (i.e. $\lim_{n\to\infty} a_n < \infty$)?

4. A positive integer n is called *special* if it can be represented in the form $n = \frac{x^2+y^2}{u^2+v^2}$, for some positive integers x, y, u, v. Prove that
   (a) 25 is special;
   (b) 2013 is not special;
   (c) 2014 is not special.

5. Prove that $\frac{x}{\sqrt{1+x^2}} + \frac{y}{\sqrt{1+y^2}} + \frac{z}{\sqrt{1+z^2}} \leq \frac{3\sqrt{3}}{2}$ for any positive real numbers x, y, z such that $x + y + z = xyz$.

6. Let $X = \begin{pmatrix} 7 & 8 & 9 \\ 8 & -9 & -7 \\ -7 & -7 & 9 \end{pmatrix}$, $Y = \begin{pmatrix} 9 & 8 & -9 \\ 8 & -7 & 7 \\ 7 & 9 & 8 \end{pmatrix}$, let $A = Y^{-1} - X$ and let B be the inverse of $X^{-1} + A^{-1}$. Find a matrix M such that $M^2 = XY - BY$ (you may assume that A and $X^{-1} + A^{-1}$ are invertible).

7. Find $\sum_{n=1}^{\infty} \frac{n}{(2^n+2^{-n})^2} + \frac{(-1)^n n}{(2^n-2^{-n})^2}$.

36th Annual Virginia Tech Regional Mathematics Contest

From 9:00 a.m. to 11:30 a.m., October 25, 2014

Fill out the individual registration form

1. Find $\displaystyle\sum_{n=2}^{n=\infty} \frac{n^2-2n-4}{n^4+4n^2+16}$.

2. Evaluate $\displaystyle\int_0^2 \frac{(16-x^2)x}{16-x^2+\sqrt{(4-x)(4+x)(12+x^2)}}\,dx$.

3. Find the least positive integer n such that 2^{2014} divides 19^n-1.

4. Suppose we are given a 19×19 chessboard (a table with 19^2 squares) and remove the central square. Is it possible to tile the remaining $19^2-1=360$ squares with 4×1 and 1×4 rectangles? (So each of the 360 squares is covered by exactly one rectangle.) Justify your answer.

5. Let $n\geq 1$ and $r\geq 2$ be positive integers. Prove that there is no integer m such that $n(n+1)(n+2)=m^r$.

6. Let S denote the set of 2 by 2 matrices with integer entries and determinant 1, and let T denote those matrices of S which are congruent to the identity matrix I mod 3 (so $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in T$ means that $a,b,c,d\in\mathbb{Z}$, $ad-bc=1$, and 3 divides b, c, $a-1$, $d-1$; "$\in$" means "is in").

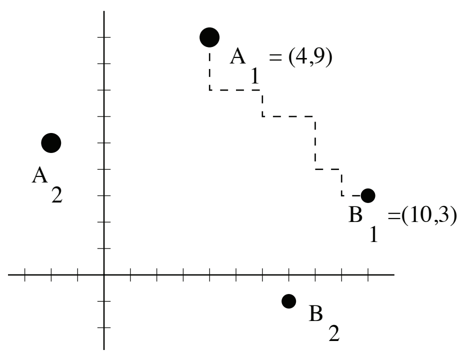
   (a) Let $f\colon T\to\mathbb{R}$ (the real numbers) be a function such that for every $X,Y\in T$ with $Y\neq I$, either $f(XY)>f(X)$ or $f(XY^{-1})>f(X)$ (or both). Show that given two finite nonempty subsets A,B of T, there are matrices $a\in A$ and $b\in B$ such that if $a'\in A$, $b'\in B$ and $a'b'=ab$, then $a'=a$ and $b'=b$.

   (b) Show that there is no $f\colon S\to\mathbb{R}$ such that for every $X,Y\in S$ with $Y\neq\pm I$, either $f(XY)>f(X)$ or $f(XY^{-1})>f(X)$.

(Please turn over)

7. Let A, B be two points in the plane with integer coordinates $A = (x_1, y_1)$ and $B = (x_2, y_2)$. (Thus $x_i, y_i \in \mathbb{Z}$, for $i = 1, 2$.) A path $\pi\colon A \to B$ is a sequence of **down** and **right** steps, where each step has an integer length, and the initial step starts from A, the last step ending at B. In the figure below, we indicated a path from $A_1 = (4,9)$ to $B_1 = (10,3)$. The distance $d(A,B)$ between A and B is the number of such paths. For example, the distance between $A = (0,2)$ and $B = (2,0)$ equals 6. Consider now two pairs of points in the plane $A_i = (x_i, y_i)$ and $B_i = (u_i, z_i)$ for $i = 1, 2$, with integer coordinates, and in the configuration shown in the picture (but with arbitrary coordinates):

   - $x_2 < x_1$ and $y_1 > y_2$, which means that A_1 is North-East of A_2; $u_2 < u_1$ and $z_1 > z_2$, which means that B_1 is North-East of B_2.
   - Each of the points A_i is North-West of the points B_j, for $1 \leq i, j \leq 2$. In terms of inequalities, this means that $x_i < \min\{u_1, u_2\}$ and $y_i > \max\{z_1, z_2\}$ for $i = 1, 2$.



   (a) Find the distance between two points A and B as before, as a function of the coordinates of A and B. Assume that A is North-West of B.

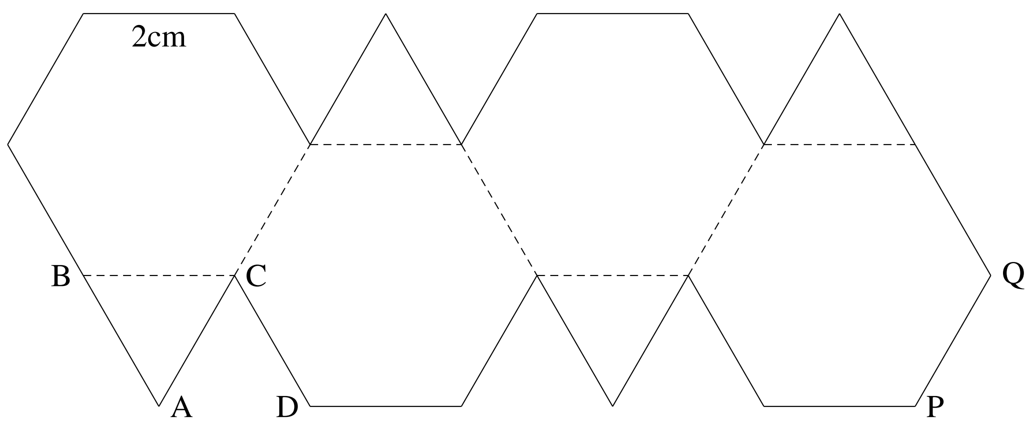
   (b) Consider the 2×2 matrix $M = \begin{pmatrix} d(A_1,B_1) & d(A_1,B_2) \\ d(A_2,B_1) & d(A_2,B_2) \end{pmatrix}$. Prove that for any configuration of points A_1, A_2, B_1, B_2 as described before, $\det M > 0$.

37th Annual Virginia Tech Regional Mathematics Contest
From 9:00 a.m. to 11:30 a.m., October 24, 2015

Fill out the individual registration form

1. Find all integers n for which $n^4 + 6n^3 + 11n^2 + 3n + 31$ is a perfect square.

2. The planar diagram below, with equilateral triangles and regular hexagons, sides length 2cm., is folded along the dashed edges of the polygons, to create a closed surface in three dimensional Euclidean spaces. Edges on the periphery of the planar diagram are identified (or glued) with precisely one other edge on the periphery in a natural way. Thus for example, BA will be joined to QP and AC will be joined to DC. Find the volume of the three-dimensional region enclosed by the resulting surface.



3. Let $(a_i)_{1 \le i \le 2015}$ be a sequence consisting of 2015 integers, and let $(k_i)_{1 \le i \le 2015}$ be a sequence of 2015 positive integers (positive integer excludes 0). Let

$$A = \begin{pmatrix} a_1^{k_1} & a_1^{k_2} & \cdots & a_1^{k_{2015}} \\ a_2^{k_1} & a_2^{k_2} & \cdots & a_2^{k_{2015}} \\ \vdots & \vdots & \cdots & \vdots \\ a_{2015}^{k_1} & a_{2015}^{k_2} & \cdots & a_{2015}^{k_{2015}} \end{pmatrix}.$$

Prove that 2015! divides $\det A$.

(Please turn over)

4. Consider the harmonic series $\sum_{n\geq 1} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} \cdots$. Prove that every positive rational number can be obtained as an *unordered* partial sum of this series. (An unordered partial sum may skip some of the terms $\frac{1}{k}$.)

5. Evaluate $\displaystyle\int_0^\infty \frac{\arctan(\pi x) - \arctan(x)}{x}\, dx$ (where $0 \leq \arctan(x) < \pi/2$ for $0 \leq x < \infty$).

6. Let $(a_1, b_1), \ldots, (a_n, b_n)$ be n points in $\mathbb{R}^2$ (where $\mathbb{R}$ denotes the real numbers), and let $\varepsilon > 0$ be a positive number. Can we find a real-valued function $f(x,y)$ that satisfies the following three conditions?

   (a) $f(0,0) = 1$;

   (b) $f(x,y) \neq 0$ for only finitely many $(x,y) \in \mathbb{R}^2$;

   (c) $\displaystyle\sum_{r=1}^{r=n} |f(x+a_r, y+b_r) - f(x,y)| < \varepsilon$ for every $(x,y) \in \mathbb{R}^2$.

   Justify your answer.

7. Let n be a positive integer and let $x_1, \ldots, x_n$ be n nonzero points in $\mathbb{R}^2$. Suppose $\langle x_i, x_j \rangle$ (scalar or dot product) is a rational number for all i, j ($1 \leq i, j \leq n$). Let S denote all points of $\mathbb{R}^2$ of the form $\sum_{i=1}^{i=n} a_i x_i$ where the a_i are integers. A closed disk of radius R and center P is the set of points at distance at most R from P (includes the points distance R from P). Prove that there exists a positive number R and closed disks $D_1, D_2, \ldots$ of radius R such that

   (a) Each disk contains exactly two points of S;

   (b) Every point of S lies in at least one disk;

   (c) Two distinct disks intersect in at most one point.

38th Annual Virginia Tech Regional Mathematics Contest

From 9:00 a.m. to 11:30 a.m., October 22, 2016

Fill out the individual registration form

1. Evaluate $\displaystyle\int_1^2 \frac{\ln x}{2-2x+x^2}\,dx$.

2. Determine the real numbers k such that $\displaystyle\sum_{n=1}^{\infty}\left(\frac{(2n)!}{4^n n!n!}\right)^k$ is convergent.

3. Let n be a positive integer and let $\mathrm{M}_n(\mathbb{Z}_2)$ denote the n by n matrices with entries from the integers mod 2. If $n \geq 2$, prove that the number of matrices A in $\mathrm{M}_n(\mathbb{Z}_2)$ satisfying $A^2 = 0$ (the matrix with all entries zero) is an even positive integer.

4. For a positive integer a, let $P(a)$ denote the largest prime divisor of a^2+1. Prove that there exist infinitely many triples (a,b,c) of distinct positive integers such that $P(a) = P(b) = P(c)$.

5. Suppose that m,n,r are positive integers such that
$$1+m+n\sqrt{3} = (2+\sqrt{3})^{2r-1}.$$
Prove that m is a perfect square.

6. Let A,B,P,Q,X,Y be square matrices of the same size. Suppose that
$$\begin{aligned} A+B+AB &= XY & \qquad AX &= XQ \\ P+Q+PQ &= YX & \qquad PY &= YB. \end{aligned}$$
Prove that $AB = BA$.

7. Let q be a real number with $|q| \neq 1$ and let k be a positive integer. Define a Laurent polynomial $f_k(X)$ in the variable X, depending on q and k, by $f_k(X) = \displaystyle\prod_{i=0}^{k-1}(1-q^iX)(1-q^{i+1}X^{-1})$. (Here $\prod$ denotes product.) Show that the constant term of $f_k(X)$, i.e. the coefficient of X^0 in $f_k(X)$, is equal to
$$\frac{(1-q^{k+1})(1-q^{k+2})\cdots(1-q^{2k})}{(1-q)(1-q^2)\cdots(1-q^k)}.$$

39th Annual Virginia Tech Regional Mathematics Contest

From 9:00 a.m. to 11:30 a.m., October 21, 2017

Fill out the individual registration form

1. Determine the number of real solutions to the equation $\sqrt{2-x^2} = \sqrt[3]{3-x^3}$.

2. Evaluate $\displaystyle\int_0^a \frac{dx}{1+\cos x+\sin x}$ for $-\pi/2 < a < \pi$. Use your answer to show that $\displaystyle\int_0^{\pi/2} \frac{dx}{1+\cos x+\sin x} = \ln 2$.

3. Let ABC be a triangle and let P be a point in its interior. Suppose $\angle BAP = 10^\circ$, $\angle ABP = 20^\circ$, $\angle PCA = 30^\circ$ and $\angle PAC = 40^\circ$. Find $\angle PBC$.

4. Let P be an interior point of a triangle of area T. Through the point P, draw lines parallel to the three sides, partitioning the triangle into three triangles and three parallelograms. Let a, b and c be the areas of the three triangles. Prove that $\sqrt{T} = \sqrt{a}+\sqrt{b}+\sqrt{c}$.

5. Let $f(x,y) = \dfrac{x+y}{2}$, $g(x,y) = \sqrt{xy}$, $h(x,y) = \dfrac{2xy}{x+y}$, and let
$$S = \{(a,b) \in \mathbb{N}\times\mathbb{N} \mid a \neq b \text{ and } f(a,b), g(a,b), h(a,b) \in \mathbb{N}\},$$
where $\mathbb{N}$ denotes the positive integers. Find the minimum of f over S.

6. Let $f(x) \in \mathbb{Z}[x]$ be a polynomial with integer coefficients such that $f(1) = -1$, $f(4) = 2$ and $f(8) = 34$. Suppose $n \in \mathbb{Z}$ is an integer such that $f(n) = n^2 - 4n - 18$. Determine all possible values for n.

7. Find all pairs (m,n) of nonnegative integers for which
$m^2 + 2\cdot 3^n = m(2^{n+1}-1)$.

40th Annual Virginia Tech Regional Mathematics Contest

From 9:00 a.m. to 11:30 a.m., October 27, 2018

Fill out the individual registration form

1. It is known that $\int_1^2 \frac{\arctan(1+x)}{x}\,dx = q\pi\ln(2)$ for some rational number q. Determine q. Here $0 \le \arctan(x) < \pi/2$ for $0 \le x < \infty$.

2. Let $A, B \in \mathrm{M}_6(\mathbb{Z})$ such that $A \equiv I \equiv B \bmod 3$ and $A^3B^3A^3 = B^3$. Prove that $A = I$. Here $\mathrm{M}_6(\mathbb{Z})$ indicates the 6 by 6 matrices with integer entries, I is the identity matrix, and $X \equiv Y \bmod 3$ means all entries of $X - Y$ are divisible by 3.

3. Prove that there is no function $f\colon \mathbb{N} \to \mathbb{N}$ such that $f(f(n)) = n+1$. Here $\mathbb{N}$ is the positive integers $\{1,2,3,\dots\}$.

4. Let m, n be integers such that $n \ge m \ge 1$. Prove that $\frac{\gcd(m,n)}{n}\binom{n}{m}$ is an integer. Here gcd denotes greatest common divisor and $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ denotes the binomial coefficient.

5. For $n \in \mathbb{N}$, let $a_n = \int_0^{1/\sqrt{n}} |1 + e^{it} + e^{2it} + \cdots + e^{nit}|\,dt$. Determine whether the sequence $(a_n) = a_1, a_2, \dots$ is bounded.

6. For $n \in \mathbb{N}$, define $a_n = \frac{1 + 1/3 + 1/5 + \cdots + 1/(2n-1)}{n+1}$ and $b_n = \frac{1/2 + 1/4 + 1/6 + \cdots + 1/(2n)}{n}$. Find the maximum and minimum of $a_n - b_n$ for $1 \le n \le 999$.

7. A continuous function $f\colon [a,b] \to [a,b]$ is called piecewise monotone if $[a,b]$ can be subdivided into finitely many subintervals

$$I_1 = [c_0, c_1],\ I_2 = [c_1, c_2],\ \dots,\ I_\ell = [c_{\ell-1}, c_\ell]$$

such that f restricted to each interval I_j is strictly monotone, either increasing or decreasing. Here we are assuming that $a = c_0 < c_1 < \cdots < c_{\ell-1} < c_\ell = b$. We are also assuming that each I_j is a maximal interval on which f is strictly monotone. Such a maximal interval is called a lap of the function

f, and the number $\ell = \ell(f)$ of distinct laps is called the lap number of f. If $f\colon [a,b] \to [a,b]$ is a continuous piecewise-monotone function, show that the sequence $(\sqrt[n]{\ell(f^n)})$ converges; here f^n means f composed with itself n-times, so $f^2(x) = f(f(x))$ etc.

41st Annual Virginia Tech Regional Mathematics Contest

From 9:00 a.m. to 11:30 a.m., October 26, 2019

Fill out the individual registration form

1. For each positive integer n, define $f(n)$ to be the sum of the digits of 2771^n (so $f(1) = 17$). Find the minimum value of $f(n)$ (where $n \geq 1$). Justify your answer.

2. Let X be the point on the side AB of the triangle ABC such that $BX/XA = 9$. Let M be the midpoint of BX and let Y be the point on BC such that $\angle BMY = 90^\circ$. Suppose AC has length 20 and that the area of the triangle XYC is 9/100 of the area of the triangle ABC. Find the length of BC.

3. Let n be a nonnegative integer and let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{R}[x]$ be a polynomial with real coefficients a_i. Suppose that

$$\frac{a_n}{(n+1)(n+2)} + \frac{a_{n-1}}{n(n+1)} + \cdots + \frac{a_1}{6} + \frac{a_0}{2} = 0.$$

   Prove that $f(x)$ has a real zero.

4. Compute $\displaystyle\int_0^1 \frac{x^2}{x+\sqrt{1-x^2}}\,dx$ (the answer is a rational number).

5. Find the general solution of the differential equation

$$x^4 \frac{d^2y}{dx^2} + 2x^2 \frac{dy}{dx} + (1-2x)y = 0$$

   valid for $0 < x < \infty$.

6. Let S be a subset of $\mathbb{R}$ with the property that for every $s \in S$, there exists $\varepsilon > 0$ such that $(s-\varepsilon, s+\varepsilon) \cap S = \{s\}$. Prove there exists a function $f\colon S \to \mathbb{N}$, the positive integers, such that for all $s, t \in S$, if $s \neq t$ then $f(s) \neq f(t)$.

7. Let S denote the positive integers that have no 0 in their decimal expansion. Determine whether $\displaystyle\sum_{n\in S} n^{-99/100}$ is convergent.