

DISCRETE HYPERBOLIC TRANSFORMATIONS AS UNIQUE PRODUCT GROUPS

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ABSTRACT

Much is known about transformations in hyperbolic space; however, many of the properties of groups of these transformations are still unknown. This paper examines the behavior of these groups acting on hyperbolic 2- and 3- space; more specifically, the conditions under which the torsion-free discrete group of hyperbolic transformations is a unique product group.

1. Introduction

A group G is a *unique product group* if, given two nonempty finite subsets X, Y of G , there is a $g \in G$ which can be expressed exactly one way in the form xy with $x \in X$ and $y \in Y$. In a geometric system with a metric d such that $d(p, q)$ is the distance between the points p and q , and given a group of transformations G acting on these points, the following theorem holds:

Theorem 1.1. If G is a group of transformations such that, for all $g \in G$, $d(gp, q) > d(p, q)$ or $d(g^{-1}p, q) > d(p, q)$, then G is a unique product group.

Proof. Let G act on the graph X . Suppose for all points p, q in X and nonidentity elements g of G , either $d(gp, q) > d(p, q)$, or $d(g^{-1}p, q) > d(p, q)$. Now let a be a point in X and $A, B \subseteq G$; choose $h \in AB$ such that $d(hp, p)$ is maximal. Suppose h can be expressed two different ways: as ab and $a'b'$, with $a, a' \in A$ and $b, b' \in B$. Let $k = aa'^{-1}$; then $k^{-1} = a'a^{-1}$. Thus $kh = (aa'^{-1})(a'b') = ab'$ and $k^{-1}h = (a'a^{-1})(ab) = a'b$. Note that $kh, k^{-1}h$ are members of AB , and since $k, k^{-1} \in G$, it follows that $d(khp, p) > d(hp, p)$, or $d(k^{-1}hp, p) > d(hp, p)$, violating the maximality of $d(hp, p)$. Therefore, h must be expressed uniquely as ab , with $a \in A$ and $b \in B$, so G is a unique product group.

This paper intends to examine the conditions under which groups of transformations in hyperbolic space are unique product groups; more specifically, discrete, *torsion-free* transformation groups, that is, discrete groups in which every non-identity element has infinite order.

2. Discrete hyperbolic transformations in \mathbb{H}^2

The groups of transformations examined in this paper act on the hyperbolic 2- and 3- spaces \mathbb{H}^2 and \mathbb{H}^3 , represented by the Poincaré half-plane and half-space models,

respectively. In the half-plane model of \mathbb{H}^2 , a point z is represented as a complex number in the form $z = a + bi \in \mathbb{C}$, $b > 0$. This model resembles an upper half-plane of Euclidean space in which no points lie on $\partial\mathbb{H}^2$, the axis corresponding to $b = 0$. The metric $d(w, z)$ between points w and z in this model is defined in the following manner:

Definition 2.1. $\text{Cosh } d(w, z) = 1 + \frac{|w - z|^2}{2\text{Im}[w]\text{Im}[z]}$ [2, p. 80]

For the purposes of this paper, a modified version of this metric, called $\|w - z\|$, will be used, defined as:

Definition 2.2. $\|w - z\| = \frac{|w - z|^2}{\text{Im}[w]\text{Im}[z]}$

Since the function $\text{Cosh}^{-1}(x)$ is strictly increasing over its entire domain, it follows that for all points $p, q, r \in \mathbb{H}^2$, $\|p - q\| > \|r - q\|$ implies that

$$1 + \frac{|p - q|^2}{2\text{Im}[p]\text{Im}[q]} > 1 + \frac{|r - q|^2}{2\text{Im}[r]\text{Im}[q]} ;$$

Therefore $d(p, q) > d(r, q)$. This condition allows $\|p - q\|$ to be used in place of $d(p, q)$ in Theorem 1.1 to determine whether groups of hyperbolic transformations are unique product groups.

Groups of transformations in \mathbb{H}^2 can be represented as matrices in $SL_2(\mathbb{R})$ of the form $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $g^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$; $a, b, c, d \in \mathbb{R}$; $ad - bc = 1$

and which act on points in the following manner:

Definition 2.3. If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ and $z \in \mathbb{C}$, then $gz = \frac{az + b}{cz + d}$ [2, p. 80]

Groups of these transformations are subgroups of the more general group of *Möbius Transformations*. Every such transformation in \mathbb{H}^2 can be placed into one of three distinct classifications, depending on the number of points it fixes in the half-plane or, alternately, the square of the trace of the matrix which represents it:

Definition 2.4. Let $g \in SL_2(\mathbb{R})$ be a transformation acting on \mathbb{H}^2 . Then:

- 1) g is *elliptic* if and only if $\text{tr}^2(g) < 4$. The transformation g fixes a single point in \mathbb{H}^2 .
- 2) g is *hyperbolic* if and only if $\text{tr}^2(g) > 4$. The transformation g has two distinct fixed points on $\partial\mathbb{H}^2$

- 3) g is *parabolic* if and only if $\text{tr}^2(g) = 4$. The transformation g has a unique fixed point on $\partial\mathbb{H}^2$.

With these theorems and definitions established, the following theorem can now be proven:

Theorem 2.5. A torsion-free group of discrete transformations acting on \mathbb{H}^2 (also known as a *torsion-free Fuchsian group*) is a unique product group.

Proof. Let p, q be points in \mathbb{H}^2 and let $g \in \text{SL}_2(\mathbb{R})$. First, it is assumed that there exists an isometry $\alpha: \mathbb{H}^2 \rightarrow \mathbb{H}^2$ such that $\alpha(p) = i$ and $\alpha(q) = i/n$ for some $n \in \mathbb{R}_{>0}$, where $d(p, q) = d(i, i/n)$. (This is essentially a shift which moves p and q to the vertical axis.) Also, since $d(i, i/n) =$

$$\text{Cosh}^{-1}\left[1 + \frac{\left|i - \frac{i}{n}\right|^2}{2n}\right] = \text{Cosh}^{-1}\left[1 + \frac{\left(\frac{n-1}{n}\right)^2}{2}\right] = \text{Cosh}^{-1}\left[1 + \frac{(n-1)^2}{2n}\right] = d(i, ni)$$

it follows that n can always be chosen such that $0 < n < 1$.¹

Since α acts on g as $\alpha g \alpha^{-1}$, the trace of the transformation remains unchanged, since $\text{tr}(\alpha g \alpha^{-1}) = \text{tr}(\alpha \alpha^{-1} g) = \text{tr}(g)$; thus the fact that g is elliptic, hyperbolic, or parabolic is unaffected by α . To show that the group of transformations is a unique product group, it is now sufficient to show that $\|g i - i/n\|$ or $\|g^{-1} i - i/n\|$ is greater than $\|i - i/n\|$. By 2.2.:

$$\frac{1}{n} \|i - i/n\| = \left(\frac{n-1}{n}\right)^2; \text{ And}$$

$$\begin{aligned} \frac{1}{n} \|g i - i/n\| &= \frac{\left|\frac{ai+b}{ci+d} - \frac{i}{n}\right|^2}{\text{Im}\left[\frac{ai+b}{ci+d}\right]} \\ &= \frac{\left|\frac{ai+b}{ci+d} \left(\frac{d-ci}{d-ci}\right) - \frac{i}{n}\right|^2}{\text{Im}\left[\frac{ai+b}{ci+d} \left(\frac{d-ci}{d-ci}\right)\right]} \\ &= \frac{\left|\frac{(bd+ac) + (ad-bc)i}{c^2+d^2} - \frac{i}{n}\right|^2}{\text{Im}\left[\frac{(bd+ac) + (ad-bc)i}{c^2+d^2}\right]} \\ &= \frac{\left(\frac{bd+ac}{c^2+d^2}\right)^2 + \left(\frac{1}{c^2+d^2} - \frac{1}{n}\right)^2}{\frac{1}{c^2+d^2}} \\ &= \frac{(bd+ac)^2}{c^2+d^2} + \left(\frac{n - (c^2+d^2)}{n}\right)^2 \frac{1}{c^2+d^2} \end{aligned}$$

$$\geq \frac{(n - (c^2 + d^2))^2}{n^2 (c^2 + d^2)}.$$

Now assume that $0 < n < 1$ and $c^2 + d^2 > 1$. Then $n^2 < c^2 + d^2$, which implies:

$$\begin{aligned} n^2(1 - (c^2 + d^2)) &> (c^2 + d^2)(1 - (c^2 + d^2)) \\ \Rightarrow n^2 + (c^2 + d^2)^2 &> (c^2 + d^2) + n^2(c^2 + d^2) \\ \Rightarrow \frac{n^2 - 2(c^2 + d^2)n + (c^2 + d^2)^2}{n^2} &> \frac{n^2(c^2 + d^2) - 2(c^2 + d^2)n + (c^2 + d^2)}{n^2} \\ \Rightarrow \frac{(n - (c^2 + d^2))^2}{n^2} &> \frac{(c^2 + d^2)(n - 1)^2}{n^2} \\ \Rightarrow \frac{(n - (c^2 + d^2))^2}{(c^2 + d^2)n^2} &> \frac{(n - 1)^2}{n^2} \\ \Rightarrow \frac{1}{n} \|gi - i/n\| &> \frac{1}{n} \|i - i/n\| \end{aligned}$$

Therefore, $d(gi, i/n) > d(i, i/n)$. Now the assumptions made in proving this property – namely, that $0 < n < 1$ and $c^2 + d^2 > 1$ – must be addressed. As stated previously in ¹, n can always be chosen such that $0 < n < 1$, since $d(i, i/n) = d(i, n)$. Furthermore, if g is hyperbolic or parabolic, then $\text{tr}^2(g) \geq 4$ requires either $a^2 + c^2 \geq 1$ or $d^2 + c^2 \geq 1$. If $d^2 + c^2 > 1$, then $d(gi, i/n) > d(i, i/n)$; if $a^2 + c^2 > 1$, then the above proof can be applied using g^{-1} , with the result that $d(g^{-1}i, i/n) > d(i, i/n)$. Finally, if $a^2 = d^2 = 1$ and $c = 0$, then $b \neq 0$ for non-identity transformations; thus

$$\frac{(bd + ac)^2}{c^2 + d^2} + \left(\frac{n - (c^2 + d^2)}{n} \right)^2 \frac{1}{c^2 + d^2} = b^2 + \left(\frac{n - 1}{n} \right)^2 > \left(\frac{n - 1}{n} \right)^2$$

Therefore, we can surmise that $d(gp, q) > d(p, q)$. or $d(g^{-1}p, q) > d(p, q)$. As a result, the property holds for all hyperbolic and parabolic transformations.

To show that the torsion-free group of discrete transformations in \mathbb{H}^2 contains only hyperbolic and parabolic elements, the following lemma will be used:

Lemma 2.6. A group G which fixes a point is discrete if and only if G is finite.
[2, p. 122]

Since a discrete group of transformations containing an elliptic element fixes a point in \mathbb{H}^2 , the group must be finite. Therefore, the elements in this group must have finite order,

thus it cannot be torsion-free. By combining this result with the previous proof, one can verify that a torsion-free Fuchsian group is a unique product group.

3. Discrete hyperbolic transformations in \mathbb{H}^3

The results from the previous section will now be extended into the hyperbolic 3-space \mathbb{H}^3 . In the Poincaré half-space model of \mathbb{H}^3 , a point w is represented by the quaternion $w = a + bi + cj \in \mathbb{H}$, $c > 0$. This model is similar the earlier half-plane model of \mathbb{H}^2 , in that it resembles an upper half-plane of Euclidean 3-space in which no points lie on the plane $c = 0$. The metric $\rho(w, z)$ between points w and z in \mathbb{H}^3 is also similar, and is defined as:

Definition 3.1. If $w = a + bi + cj$ and $z = q + ri + sj$ are points in \mathbb{H}^3 , then

$$\text{Cosh } \rho(w, z) = 1 + \frac{|w - z|^2}{2cs} \quad [2, \text{p. 271}]$$

Once again, the modified metric $\|w - z\|$ will be used, defined as:

$$\text{Definition 3.2. } \|w - z\| = \frac{|w - z|^2}{cs}$$

The same argument as in Definition 2.2 can be employed to demonstrate that $\|w - z\| > \|x - z\|$ implies that $\rho(w, z) > \rho(x, z)$ for all w, x, z in \mathbb{H}^3 .

Groups of transformations in hyperbolic 3-space can be represented as matrices in $SL_2(\mathbb{C})$ in the form $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $g^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$; $a, b, c, d \in \mathbb{C}$, $ad - bc = 1$. These

transformations are defined as acting on points z in \mathbb{H}^3 as in Definition 2.3, except that g is now an element of $SL_2(\mathbb{C})$ and z is represented by a quaternion. This action can also be represented in an alternate, but equivalent form, called the *Poincaré Extension*:

Definition 3.3 (Poincaré Extension). If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C})$ and $z = s + tj$, $a \in \mathbb{C}$,

$$gz = \frac{(as + b)(\overline{cs + d}) + a\bar{c}t^2 + |ad - bc|tj}{|cs + d|^2 + |c|^2t^2} \quad [1, \text{p. 58}]$$

There are four classifications for transformations in \mathbb{H}^3 : as before, elliptic, hyperbolic, and parabolic, classified by the square of the trace of its matrix representation as in Definition 2.4. However, an elliptic transformation will now fix a single point in \mathbb{H}^3 while parabolic and hyperbolic transformations will fix one and two distinct points in the horizontal plane $c = 0$ respectively. In addition, there is now a fourth classification for transformations:

Definition 3.4. Let $g \in \text{SL}_2(\mathbb{C})$ be a transformation acting on \mathbb{H}^3 . Then g is *strictly loxodromic* if and only if $\text{tr}^2(g) \notin [0, +\infty)$; that is, if $\text{Im}[\text{tr}^2(g)] \neq 0$. [1, p. 67]

The following theorem, a more general version of Theorem 2.5, can now be proven:

Theorem 3.5. A torsion-free group G of discrete transformations acting on \mathbb{H}^3 is a unique product group provided, for all strictly loxodromic transformations

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G, \quad |a|, |c|, \text{ or } |d| \geq 1.$$

Proof. Let p, q be points in \mathbb{H}^3 and let $g \in \text{SL}_2(\mathbb{C})$. As in the proof of Theorem 2.5, assume there exists an isometry $\alpha: \mathbb{H}^3 \rightarrow \mathbb{H}^3$ such that $\alpha(p) = j$ and $\alpha(q) = j/n$, $0 < n < 1$, where $\rho(p, q) = \rho(j, j/n)$; note that α again preserves the trace of g . It is now sufficient to show that either $\|gj - j/n\| > \|j - j/n\|$ or $\|g^{-1}j - j/n\| > \|j - j/n\|$. By 3.3:

$$\begin{aligned} \frac{1}{n} \|j - j/n\| &= \left(\frac{n-1}{n} \right)^2 \\ g j &= \frac{(b\bar{d} + a\bar{c}) + j}{|c|^2 + |d|^2} \\ \frac{1}{n} \|g j - j/n\| &= \frac{\left| \frac{(b\bar{d} + a\bar{c})}{|c|^2 + |d|^2} + \frac{j}{|c|^2 + |d|^2} - \frac{j}{n} \right|^2}{\frac{1}{|c|^2 + |d|^2}} \\ &= \frac{\left| \frac{(b\bar{d} + a\bar{c})}{|c|^2 + |d|^2} \right|^2 + \left(\frac{n - |c|^2 - |d|^2}{n(|c|^2 + |d|^2)} \right)^2}{\frac{1}{|c|^2 + |d|^2}} \\ &\geq \frac{(n - (|c|^2 + |d|^2))^2}{n^2 (|c|^2 + |d|^2)}. \end{aligned}$$

The remainder of the proof that either $\|gj - j/n\| > \|j - j/n\|$ or $\|g^{-1}j - j/n\| > \|j - j/n\|$ is now identical to that of Theorem 2.5 for hyperbolic and parabolic transformations. In the case of strictly loxodromic transformations, $|a|$, $|c|$, or $|d| \geq 1$ is required for the condition to hold. Transformations of this form include *Picard's Group* [1, p. 96]: that is, matrices in which a , b , c , and d are Gaussian integers (of the form $m + ni$ where $m, n \in \mathbb{Z}$).

All that remains is to now show that a torsion-free discrete transformation group in \mathbb{H}^3 contains no elliptic elements. To prove this, the following lemma will be used, a variation of Lemma 2.6:

Lemma 3.6. A subgroup G of \mathcal{M} , the group of Möbius transformations, such that G acts on \mathbb{H}^3 , is discrete if and only if for every compact subset K of \mathbb{H}^3 , $g(K) \cap K = \emptyset$, except for a finite number of g in G [1, p. 94, 95].

Let G be a torsion-free group of transformations and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an elliptic element of G . Then $g^2 = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix}$; therefore $\text{tr}(g) = a^2 + 2bc + d^2 = a^2 + 2(ad - 1) + d^2 = (a + d)^2 - 2 = \text{tr}^2(g) - 2$. Since g is elliptic, $0 < \text{tr}^2(g) < 4$, so $-2 < \text{tr}(g^2) < 2$. Therefore, $0 < \text{tr}^2(g) < 4$ and so g^2 is elliptic. As a result, g will generate an infinite number of elliptic elements in G , namely $\{g^{2n} : n \in \mathbb{Z}^+\}$. Since each of these elements fixes a point in \mathbb{H}^3 , for an infinite number of h in G there exists a compact subset K of \mathbb{H}^3 such that $h(K) \cap K \neq \emptyset$; namely, the compact subset containing the fixed point of h . Therefore, if G contains an elliptic element, G is not discrete.

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