MAT 4574 Vector/Complex Analysis

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[1]This courses is using both *Calculus Volume 3* from OpenStax, and *Complex Analysis with Applications* by Asmar and Grafakos as they are both freely available to students at Virginia Tech – they align very well with Stewart's *Calculus* and *Complex Analysis with Applications to Engineering* by Saff and Snider (both of which I fully recommend). The chapter/section titles in these notes will be consistent with the main course textbooks, but otherwise internal numbering of theorems, examples, etc. will likely disagree with the course text.

Contents

Preliminary Notions

0.1 Notation

Most introductory vector calculus books use the same vector notation for referencing both points in space and tangent vectors emanating from a point. While there are perfectly valid reasons to abuse notation in this way (some formulas are considerably more tractible), I remember having a particularly tough time learning this material and keeping track of what each object was *supposed to mean in context.* For this reason, I'm going to use some nonstandard notation at the beginning to differentiate between points in n-dimensional space $\mathbb{X}^n$, and vectors in n-dimensional space, $\mathbb{R}^n$; we will revert to the usual formulas only after building the necessary intuition.

- $\mathbb{R}$ – short-hand for the real numbers/scalars
- $\mathbb{C}$ – short-hand for the complex numbers/scalars
- $\mathbb{X}^n$ – these are n-tuples of real numbers $p = (x_1, \ldots, x_n)$, representing points in space.

  This is nonstandard notation.
- $\mathbb{R}^n$ – these are n-element arrays of real numbers, $\mathbf{v} = \begin{bmatrix} x_1, \ldots, x_n \end{bmatrix}$ or $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, representing vectors. Square brackets replace the angle bracket notation used in the book. We will not draw a disctinction between row vectors and column vectors, and simply use whichever is visually better.

  Square bracket notation is consistent with our treatment of vectors in Math 2114. As well, the math majors will more commonly reserve angle brackets for inner products/quadratic forms and physics majors will reserve them for bra-ket notation.
- $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ – these are the standard basis vectors for $\mathbb{R}^n$. These replace the $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$ notation found in the book.

  Part of this class involves the imaginary unit i and I'm worried that $\hat{\mathbf{i}}$ will be hard to distinguish when handwritten.

0.2 Background Deficiencies

Many of the students who take this class have not had a mutlivariable calculus class in multiple semesters. As a result, there are some concepts that are forgotten and are particularly useful for us. I will do my best to record them here as I remember them.

0.2.1 Parameterizing Curves

A **parameterization** of a curve $\mathscr{C}$ in the plane is a function $\gamma : [t_0, t_1] \to \mathbb{X}$ which traces out the curve where $[t_0, t_1]$ is some interval of real numbers. The particular t-value in this interval is called the **parameter**. Whether or not the endpoints t_0 and t_1 are included in the domain interval is a choice one makes in context (for example, if $\gamma(t_0) = \gamma(t_1)$, one might choose to omit the endpoint so that the curve doesn't cross itself). Two common (and arguably, the most important) parameterizations are circles and line segments.

Parameterizing Circles

A circle C of radius r, centered at (x_0, y_0) is usually parameterized by

$$\begin{aligned}\gamma :&[0, 2\pi] \to C \\ \gamma(t) =& (x_0, y_0) + (r\cos(t), r\sin(t))\end{aligned}$$

where here we're using $(x_0, y_0) + (r\cos(t), r\sin(t))$ to mean $\big(x_0 + r\cos(t), y_0 + r\sin(t)\big)$

Parameterizing Line Segments

A line segment ℓ from a point $p_0(x_0, y_0)$ to a point $p_1(x_1, y_1)$ is usually parameterized by

$$\begin{aligned}\gamma :&[0, 1] \to \ell \\ \gamma(t) =& (1-t)p_0 + tp_1\end{aligned}$$

wnere here we're using $(1-t)p_0 + tp_1$ to mean $\big((1-t)x_0 + tx_1,\ (1-t)y_0 + ty_1\big)$.

0.2.2 Reparameterizations

Given a parameterization γ of a curve $\mathscr{C}$, a **reparameterization** of $\mathscr{C}$ is a precomposition of γ by an invertible function (usually smooth or differentiable)

$$\varphi : [s_0, s_1] \to [t_0, t_1]$$

so that

$$\gamma \circ \varphi : [s_0, s_1] \to \mathscr{C}$$

is the reparameterization of $\mathscr{C}$.
Discuss common paramterizations – endpoint-only, unit speed, etc.

0.2.3 Power Series, Intervals of Convergence, Etc.

It's probably best to discuss this in the context of some simple real-valued functions first.

Part I

Vector Analysis (from OpenStax)

Chapter 6

Vector Calculus

6.1 Vector Fields

Definition: vector field, component functions

Let U be some region in $\mathbb{X}^n$. A **vector field on U** is a function $\mathbf{F} : U \to \mathbb{R}^n$. The **component functions** of $\mathbf{F}$, denoted $F_1, \ldots, F_n$, are just the functions in each of the entries of the output vector. That is, we can write

$$\begin{aligned}\mathbf{F}(x_1, \ldots, x_n) &= F_1(x_1, \ldots, x_n)\mathbf{e}_1 + \cdots + F_n(x_1, \ldots, x_n)\mathbf{e}_n \\ &= \begin{bmatrix} F_1(x_1, \ldots, x_n) \\ F_2(x_1, \ldots, x_n) \\ \vdots \\ F_n(x_1, \ldots, x_n) \end{bmatrix}\end{aligned}$$

Remark. Since this course is taking place exclusively in dimensions $n = 2$ and $n = 3$, in practice we will follow your book's convention and use P, Q, R for the component functions instead of F_1, F_2, F_3.

One can visualize a vector field by placing a vector in the direction $\mathbf{F}(x_1, \ldots, x_n)$ emanating from the point $(x_1, \ldots, x_n)$

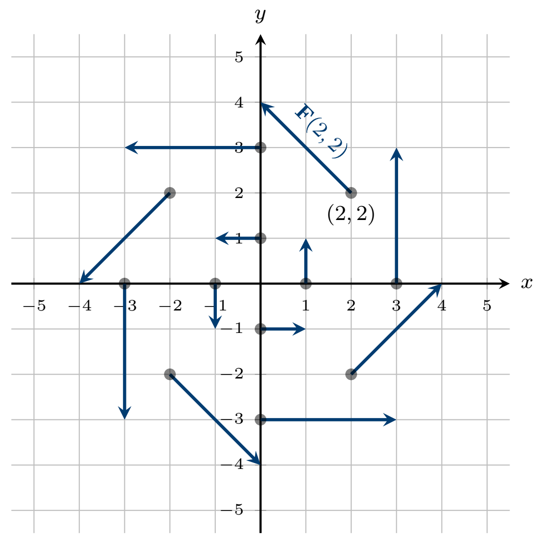
Example 6.1.1

Find the component functions of the vector field below and plot it.

$$\mathbf{F} : \mathbb{X}^2 \to \mathbb{R}^2$$
$$\mathbf{F}(x, y) = -y\mathbf{e}_1 + x\mathbf{e}_2$$

The component functions are $P(x, y) = -y$ and $Q(x, y) = x$. To visualize this vector field, we compute some values of $\mathbf{F}$.

(x, y)	$\mathbf{F}(x, y)$	(x, y)	$\mathbf{F}(x, y)$
(1,0)	[0,1]	(-2,-2)	[2,-2]
(0,1)	[-1,0]	(2,-2)	[2,2]
(-1,0)	[0,-1]	(3,0)	[0,3]
(0,-1)	[1,0]	(0,3)	[-3,0]
(2,2)	[-2,2]	(-3,0)	[0,-3]
(-2,2)	[-2,-2]	(0,-3)	[3,0]




```python
# Import required modules
import numpy as np
import matplotlib.pyplot as plt

# Meshgrid
x, y = np.meshgrid(
    np.linspace(-5,5,10),  #min value, max value, number of sample points
    np.linspace(-5,5,10))

# Directional vectors
P = -y
Q = x

# Plotting Vector Field with QUIVER
plt.quiver(x, y, P, Q)
plt.title('Vector Field')
plt.grid()
plt.show()
```

Listing 6.1: Python Code

Example 6.1.2

Find the component functions of the vector field below and plot it in your favorite software.

$$\mathbf{F} : \mathbb{X}^3 \to \mathbb{R}^3$$
$$F(x, y, z) = y\mathbf{e}_1 + z\mathbf{e}_2 + x\mathbf{e}_3$$

The component functions are

$$P(x, y, z) = y,$$
$$Q(x, y, z) = z, \quad \text{and}$$
$$R(x, y, z) = x.$$

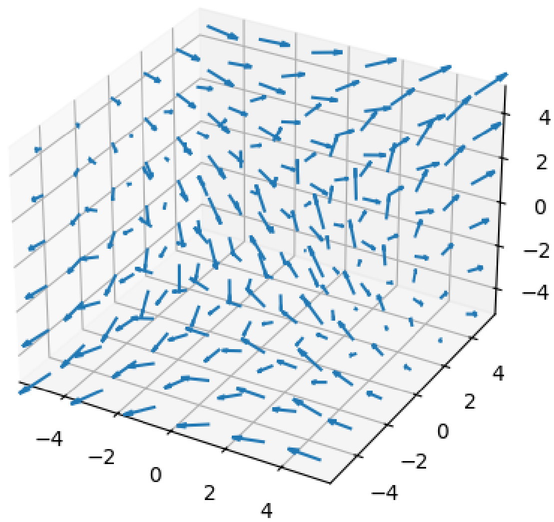


Figure 6.1: The vector field **F** output using the Python code below.

```python
# Import required modules
import matplotlib.pyplot as plt
import numpy as np
from mpl_toolkits.mplot3d import axes3d

# Creating instance of the figure and setting the axes to 3D
fig = plt.figure()
ax = plt.axes(projection="3d")

# Mesh grid
x, y, z = np.meshgrid(
    np.linspace(-5,5,6), #min value, max value, number of sample points
    np.linspace(-5,5,6),
    np.linspace(-5,5,6))

# Component Functions
P = y
Q = z
R = x

# Plotting Vector Field with Quiver
ax.quiver(x, y, z, P, Q, R, length=0.15) #length=... scales the vector lengths
ax.set_title('Vector Field')
plt.show()
```

Listing 6.2: Python Code

The arrows in the vector field plots kind of look like they follow along curves.

Definition: integral curve, flow line, streamline

If $\mathbf{F} : \mathbb{X}^n \to \mathbb{R}^n$ is a vector field on $\mathbb{X}^n$ and $\sigma(t) = (x_1(t), \ldots, x_n(t))$ is a smooth parameterized curve in $\mathbb{X}^n$, then σ is called an **integral curve** (or a **flow line** or a **streamline**) for $\mathbf{F}$ if, for every parameter value t, we have that

$$\mathbf{F}(\sigma(t)) = \left[\frac{dx_1(t)}{dt} \quad \ldots, \quad \frac{dx_n(t)}{dt}\right].$$

In other words, σ is an integral curve if the vector field arrows along the curve are precisely the tangent vectors of σ. Note that finding σ requires solving a system of first-order differential equations. We will not be covering differential equations in this course, so all of our solutions will be handled 'by inspection'; the reader is encouraged to consult standard materials on differential equations to solve such systems.

Example 6.1.3

Consider the vector field from Example 6.1.1.

$$\mathbf{F} : \mathbb{X}^2 \to \mathbb{R}^2$$
$$\mathbf{F}(x, y) = -y\mathbf{e}_1 + x\mathbf{e}_2$$

(a) Show that the circle $\sigma(t) = (\cos(t), \sin(t))$ is an integral curve for $\mathbf{F}$.

(b) Find all possible integral curves for $\mathbf{F}$.

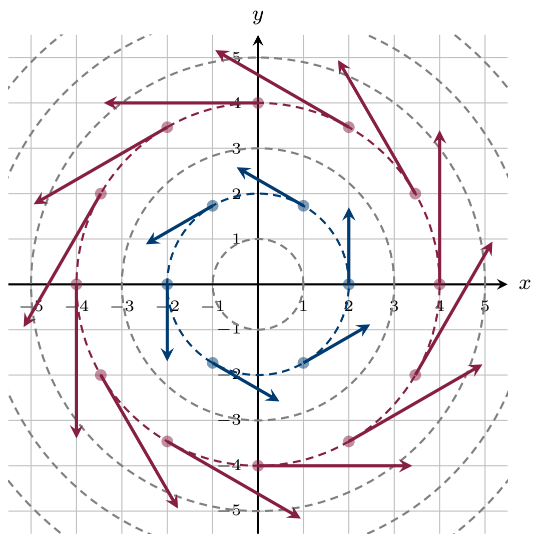


Figure 6.2: The vector field **F**. Vectors have been scaled down by 15% to make them appear nicer in the image. Integral curves have been drawn for integer radii.

(a) Given $\sigma(t) = (x(t), y(t)) = (\cos(t), \sin(t))$ we simply verify that

$$\frac{dx}{dt} = -y(t) \qquad \text{and} \qquad \frac{dy}{dt} = x(t).$$

Indeed,

$$\frac{dx}{dt} = -\sin(t) = -y(t) \qquad \text{and} \qquad \frac{dy}{dt} = \cos(t) = x(t)$$

and therefore $\sigma(t)$ is an integral curve for $\mathbf{F}$.

(b) To find all possible integral curves, we need to solve the following system of differential equations:

$$\begin{cases} x'(t) &= -y(t) \\ y'(t) &= x(t) \end{cases}$$

This can be solved in your favorite way. What we observe is that, for any real number r,

$$\sigma(t) = (x(t), y(t)) = (r\cos(t), r\sin(t))$$

is a solution (and standard theory of differential equations implies that all solutions have this form). Thus the integral curves for $\mathbf{F}$ are circles centered at $(0, 0)$.

Example 6.1.4

Let $\mathbf{F} : \mathbb{X}^2 \to \mathbb{R}^2$ be the vector field $\mathbf{F}(x, y) = [-x, y]$. Find all possible parameterized integral curves for this vector field.
Hint: It may be useful to recall that $\frac{d}{dt} e^{kt} = ke^t$*.*

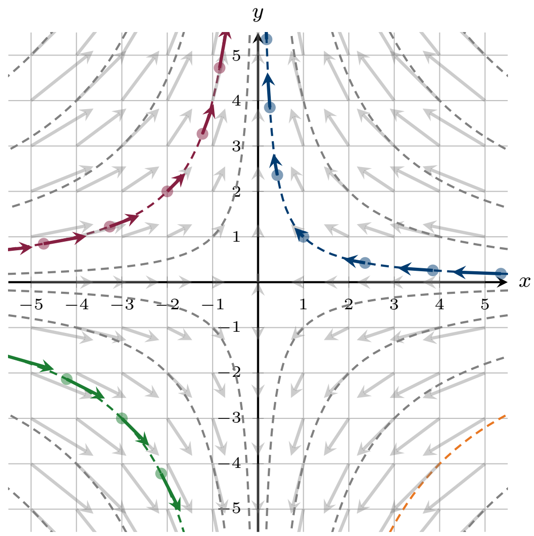


Figure 6.3: The vector field **F**. Vectors have been scaled down by about 75% to make them appear nicer in the image.

Let $\sigma(t) = (x(t), y(t))$ be a parameterized curve. We solve the system of differential equations:

$$\begin{cases} \dfrac{dx}{dt} &= -x(t) \\ \dfrac{dy}{dt} &= y(t) \end{cases}$$

This can be solved in your favorite way. What we observe is that, for any real numbers r_1 and r_2, we have

$$\sigma(t) = (x(t), y(t)) = \left(r_1 e^{-t}, r_2 e^t\right)$$

is a solution. Thus the integral curves for **F** are hyperbolas.

This can be seen more easily by observing that, for $r_1 \neq 0$ *and letting* $k = r_1 r_2$*, we exactly have that* $y(t) = k\dfrac{1}{x(t)}$*, which is the familiar equation of a hyperbola.*

6.1.1 Gradient Fields

Recall that a **scalar function** is a function $f : \mathbb{X}^n \to \mathbb{R}$ and the **gradient** of f at the point $(x_1, \ldots, x_n)$ is the vector (denoted either ∇f or $\operatorname{grad} f$) in $\mathbb{R}^n$ given by

$$\begin{aligned}\nabla f(x_1, x_2, \ldots, x_n) &= \frac{\partial}{\partial x_1} f(x_1, \ldots, x_n)\mathbf{e}_1 + \cdots + \frac{\partial}{\partial x_n} f(x_1, \ldots, x_n)\mathbf{e}_n \\ &= \begin{bmatrix} \frac{\partial}{\partial x_1} f(x_1, \ldots, x_n) \\ \vdots \\ \frac{\partial}{\partial x_n} f(x_1, \ldots, x_n) \end{bmatrix}\end{aligned}$$

This means that ∇f is a function whose inputs are in $\mathbb{X}^n$ and whose outputs are in $\mathbb{R}^n$, i.e., is a vector field!

Definition: gradient field

Given a scalar function $f : \mathbb{X}^n \to \mathbb{R}$, the corresponding **gradient field** is

$$\nabla f : \mathbb{X}^n \to \mathbb{R}^n.$$

The component functions are the partial derivatives

$$\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_k}.$$

Example 6.1.5

Let f be the following scalar function

$$f : \mathbb{X}^2 \to \mathbb{R}$$
$$f(x, y) = x^2 - y^2.$$

Find the gradient field ∇f, and plot a few contour lines for f (that is, for a few values of c, plot all points (x, y) so that $f(x, y) = c$.)

$$\nabla f(x, y) = \frac{\partial f}{\partial x}\mathbf{e}_1 + \frac{\partial f}{\partial y}\mathbf{e}_2 = 2x\mathbf{e}_1 - 2y\mathbf{e}_2$$

When plotting the gradient field and the contour lines, one sees that the vectors are perpendicular to the contour lines, and the longer arrows occur when contour lines are close together. This makes sense as close contour lines imply that the corresponding surface is steeper at that point.

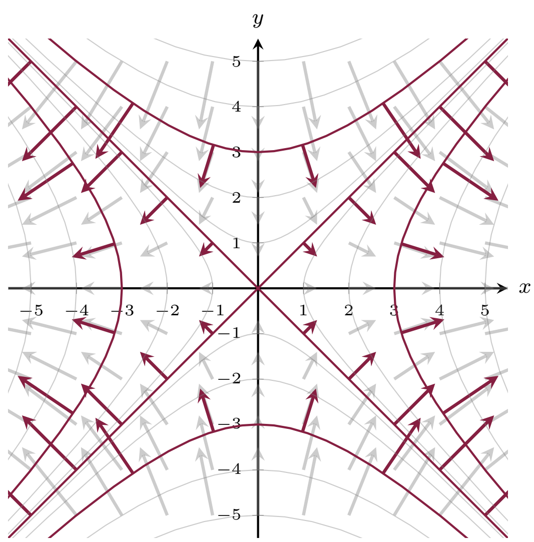


Figure 6.4: The vector field $\mathbf{F} = \nabla f$. Vectors have been scaled down by about 85% to make them appear nicer in the image.

Definition: conservative vector field, potential function

A vector field $\mathbf{F}$ is **conservative** if we can write $\mathbf{F} = \nabla f$ for some scalar function f. In this case, f is called the **potential function** for $\mathbf{V}$.

Remark. Not every vector field is conservative.

If a vector field is conservative, we can find the potential function.

Example 6.1.6

The vector field $\mathbf{F}$ given by $\mathbf{F}(x,y) = [2x, 2y]$ is conservative. Find a potential function f.

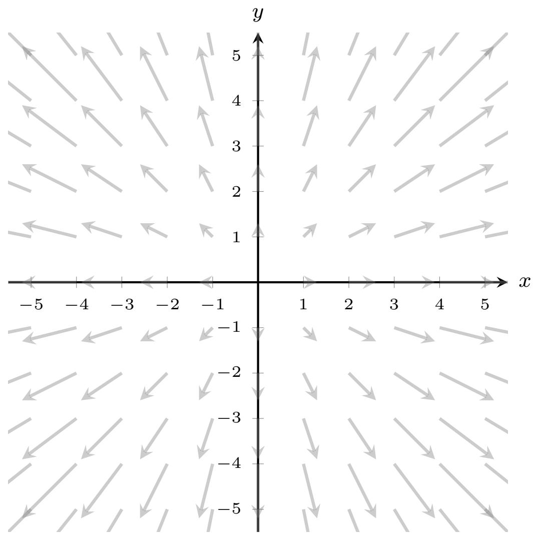


Figure 6.5: The vector field $\mathbf{F}$. Vectors have been scaled down by about 85% to make them appear nicer in the image.

By definition of a conservative vector field, we should have that

$$\big[2x, 2y\big] = \mathbf{F}(x,y) = \nabla f(x,y) = \left[\tfrac{\partial f}{\partial x}, \tfrac{\partial f}{\partial y}\right]$$

So

$$\frac{\partial f}{\partial x} = 2x \qquad \Longrightarrow \qquad f(x,y) = x^2 + K(y)$$

where K is a single-variable function solely in terms of y. From the other partial derivative, we get

$$2y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\big(x^2 + K(y)\big) = K'(y) \qquad \Longrightarrow \qquad K(y) = y^2 + Const.$$

and therefore any function $f(x,y) = x^2 + y^2 + Const.$ is a potential function for $\mathbf{F}$.

Example 6.1.7

Using the technique outlined in Example 6.1.6, show that the vector field in Example 6.1.1 is not conservative.

$$\mathbf{F} : \mathbb{X}^2 \to \mathbb{R}^2$$
$$\mathbf{F}(x, y) = -y\mathbf{e}_1 + x\mathbf{e}_2$$

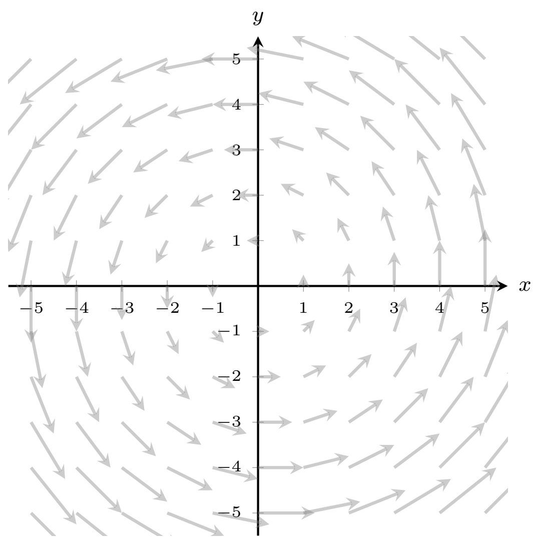


Figure 6.6: The vector field **F**. Vectors have been scaled down by about 75% to make them appear nicer in the image.

Assume that f is some potential function for **F**. Then we must have that

$$\left[-y, x\right] = \mathbf{F}(x, y) = \nabla f(x, y) = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right]$$

Then

$$\frac{\partial f}{\partial x} = -y \qquad \Longrightarrow \qquad f(x, y) = -xy + K(y)$$

where $K(y)$ is a single-variable function solely in terms of y. From the other partial derivative, we get

$$x = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(-xy + K(y)) = -x + K'(y) \qquad \Longrightarrow \qquad K'(y) = 2x$$

and this is clearly problematic as $K(y)$ (and thus $K'(y)$ as well) is *not* a function of x.

We'll come back to find that there are other (sometimes easier) ways to test whether a vector field is conservative.

6.2 Line Integrals

For simplicity, suppose that f is a scalar function on the xy-plane and Σ is a surface in $\mathbb{X}^3$ given by $z = f(x, y)$. Let $\mathscr{C}$ be a (smooth) curve in the xy-plane. Then $z = f(\mathscr{C})$ is simply a curve in the surface lying above $\mathscr{C}$. Visually,

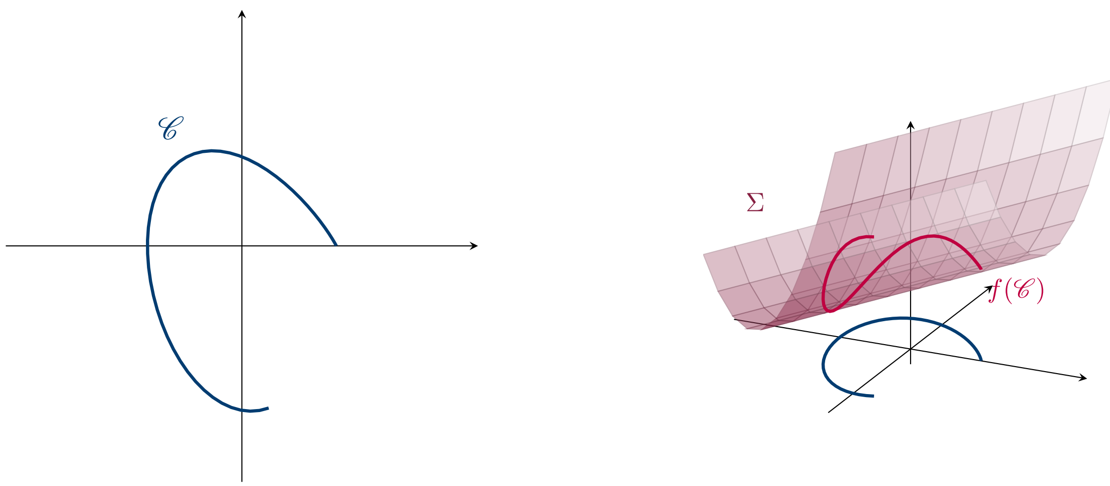
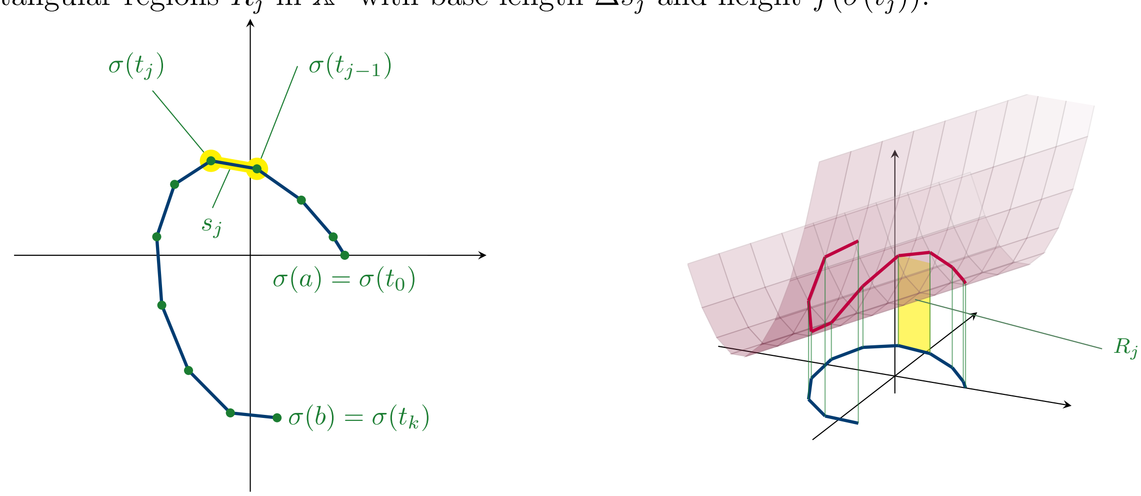


Figure 6.7: On the left, a curve $\mathscr{C}$ in the plane. On the right, a curve $f(\mathscr{C})$ in the surface Σ, lying above $\mathscr{C}$ in the plane.

We imagine that there is a “curtain” between $\mathscr{C}$ and $f(\mathscr{C})$. This “curtain” should have some area, and so it is natural to try to compute this area.

To do so, we implement some notation. Suppose that σ is the parametric function defining $\mathscr{C}$; that is, suppose $\mathscr{C} = \sigma(t)$ for $a \leq t \leq b$. By partitioning $[a, b]$ into k subintervals, $[t_0, t_1], \ldots, [t_{k-1}, t_k]$, then we can approximate $\mathscr{C}$ by using line segments s_j between each pair of points $\sigma(t_{j-1})$ and $\sigma(t_j)$ along the curve. *(We note that the lengths of each line sigment s_j will almost certainly be different, as the lengths depend on the particular parameterization of $\mathscr{C}$).* In turn, these line segments give rise to rectangular regions R_j in $\mathbb{X}^3$ with base length Δs_j and height $f(\sigma(t_j))$.



Taken together, these rectangular regions approximate the area of this “curtain”:

$$\sum_{j=1}^{k} f(\sigma(t_j))\, \Delta s_j.$$

Or, if we prefer to write $(x_j, y_j) = \sigma(t_j)$, the area is approximately

$$\sum_{j=1}^{k} f(x_j, y_j)\,\Delta s_j.$$

We have constructed a Riemann sum, and thus we define the following:

Definition: Line integral

Suppose $\mathscr{C}$ is a smooth curve in $\mathbb{X}^n$ and f is a scalar function defined on $\mathscr{C}$. Let $p_0, \ldots, p_k$ be a collection of points along $\mathscr{C}$ and let s_j be the line segment from p_{j-1} to p_j. Then the **line integral** (or **path integral** or **contour integral**) **of** f **along** $\mathscr{C}$ is given by

$$\int_{\mathscr{C}} f\,ds = \lim_{k\to\infty} \sum_{j=1}^{k} f(p_j)\Delta s_j,$$

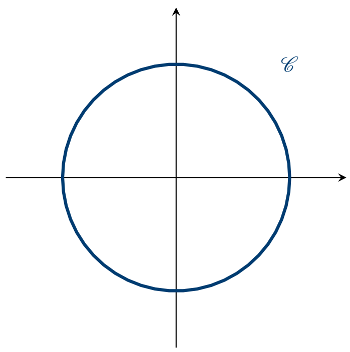
provided this limit exists.

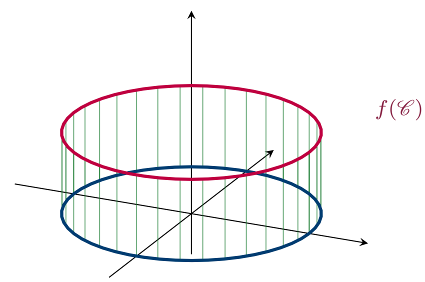
Remark. If f and $\mathscr{C}$ are sufficiently nice (which they always will be in this class), then then limit above always exists.

Example 6.2.1

Let $f(x,y)=1$ and let $\mathscr{C}$ be the unit circle. Evaluate $\int_{\mathscr{C}} f\,ds$.

In this case, $f(\mathscr{C})$ is also the unit circle, but sitting at height $z=1$, so the area under this curve is just the surface area of a cylinder of height 1. Thus it should be that $\int_{\mathscr{C}} f\,ds = 2\pi rh = 2\pi(1)(1) = 2\pi$.





We first parameterize $\mathscr{C}$ via $\sigma(t) = (\cos(t), \sin(t))$, which traverses the unit circle one time counterclockwise when $0 \le t \le 2\pi$. We partition this interval into k subintervals, each of length $\frac{2\pi}{k}$. For each $j = 1, \dots, k$, the line segment s_j will thusly have endpoints $\sigma\left(\frac{2\pi(j-1)}{k}\right)$ and $\sigma\left(\frac{2\pi j}{k}\right)$, and has length determined by the usual distance formula:

$$\begin{aligned}\Delta s_j &= \sqrt{\left(\cos\left(\frac{2\pi j}{k}\right) - \cos\left(\frac{2\pi(j-1)}{k}\right)\right)^2 + \left(\sin\left(\frac{2\pi j}{k}\right) - \sin\left(\frac{2\pi(j-1)}{k}\right)\right)^2} \\ &= 2\left|\sin\left(\frac{\pi}{k}\right)\right|,\end{aligned}$$

where the second step follows from applications of various trigonometric identities. Therefore

$$\begin{aligned}\int_{\mathscr{C}} f\,ds &= \lim_{k\to\infty} \sum_{j=1}^{k} f\left(\sigma\left(\frac{2\pi j}{k}\right)\right)\,\Delta s_j \\ &= \lim_{k\to\infty} \sum_{j=1}^{k} (1)\,2\left|\sin\left(\frac{\pi}{k}\right)\right| \\ &= \lim_{k\to\infty} 2k\left|\sin\left(\frac{\pi}{k}\right)\right| \\ &= 2\pi\end{aligned}$$

(where the limit calculation follows quickly from a familiar calculus 1 limit: $\lim_{x\to\infty} x\sin(1/x) = 1$).

Theorem 6.2.2

Suppose $\mathscr{C}$ is a smooth curve in $\mathbb{X}^n$ parameterized by $\sigma : [a,b] \to \mathbb{X}^n$, and let $f : \mathbb{X}^n \to \mathbb{R}$ be a scalar function that is continuous on σ. Then

$$\int_{\mathscr{C}} f\,ds = \int_a^b f(x_1(t),\ldots,x_n(t))\sqrt{\left(\frac{dx_1}{dt}\right)^2 + \cdots + \left(\frac{dx_n}{dt}\right)^2}\,dt.$$

6.3 Conservative Vector Fields

6.4 Green's Theorem

6.5 Divergence and Curl

6.6 Surface Integrals

6.7 Stokes' Theorem

6.8 The Divergence Theorem

Part II

Complex Analysis (from Asmar–Grafakos)

Chapter 1

Complex Numbers and Functions

1.1 Complex Numbers

Definition: ciomplex number

A **complex number** is a symbol $x + iy$ or $x + yi$, where x, y are real numbers and i satisfies $i^2 = -1$. The collection of complex numbers is denoted $\mathbb{C}$. Writing $z = x + yi$, we say that x is the **real part** of z, denoted $\mathrm{Re}(z)$, and we say that y is the **imaginary part** of z, denoted $\mathrm{Im}(z)$. A nonzero complex number z is said to be **(purely) real** if $\mathrm{Im}(z) = 0$, and **(purely) imaginary** if $\mathrm{Re}(z) = 0$.

Remark. Some authors use $\Re(z)$ and $\Im(z)$ to denote the real and imaginary parts of z, respectively.

Definition

Two complex numbers $a + bi$ and $c + di$ are said to be **equal** if and only if both $a = c$ and $b = d$.

1.1.1 Algebraic Properties of Complex numbers

If we take the perspective that the complex numbers are polynomials with indeterminate i with an additional simplification of $i^2 = -1$, then there are natural arithmetic operations one can define.

Definition

Suppose $z = a + bi$ and $w = c + di$ are two complex numbers. We have the following algebraic operations:

- **Addition**: $z + w = (a + bi) + (c + di) = (a + c) + (b + d)i$
- **Addition**: $z - w = (a + bi) - (c + di) = (a - c) + (b - d)i$
- **Multiplication**: $zw = (a + bi)(c + di) = (ac - bd) + (ad + bc)i$

Proposition 1.1.1: Properties of Complex Arithmetic

Complex arithmetic has the following familiar properties from arithmetic of the real numbers. For all $u, v, w \in \mathbb{C}$, we have

- Associative addition: $u + (v + w) = (u + v) + w$
- Commutative addition: $u + v = v + u$
- Associative multiplication: $u(vw) = (uv)w$
- Commutative multiplication: $uv = vu$
- Distributive law: $u(v + w) = uv + uw$
- $w + 0 = w$

- $1w = w$

Exercise 1.1.2

Prove Proposition 1.1.1.

Definition: complex conjugation

Given a complex number $z = x + yi$, the **complex conjugate** is the complex number $\overline{z} = x - yi$.

Remark. Complex conjugation changes the sign of the imaginary part and the real part is unaffected. That is,

$$\operatorname{Re}(\overline{z}) = \operatorname{Re}(z) \qquad \text{and} \qquad \operatorname{Im}(\overline{z}) = -\operatorname{Im}(z).$$

Observe that $z\overline{z}$ is *always* a real number:

$$(x + yi)(x - yi) = x^2 + xyi - xyi - (yi)^2 = x^2 - (-1)y^2 = x^2 + y^2$$

and since $x^2, y^2 \geq 0$, this number is zero precisely when $x = y = 0$.
MULTIPLICATIVE INVERSE, DIVISION, EXAMPLES, PERIODICITY OF i, INTERPLAY OF ALGEBRAIC OPERATIONS AND CONJUGATION

1.2 The Complex Plane

1.3 Polar Form

1.4 Complex Functions

1.5 Sequences and Series of Complex numbers

1.6 The Complex Exponential

1.7 Trigonometric and Hyperbolic Functions

1.8 Logarithms and Powers

Chapter 2

Analytic Functions

2.1 Regions of the Complex Plane

2.2 Limits and Continuity

2.3 Analytic Functions

2.4 Differentiation of Functions of Two Real Variables

2.5 The Cuachy–Riemann Equations

Chapter 3

Complex Integration

3.1 Paths (Contours) in the Complex Plane

3.2 Complex Integration

3.3 Independence of Path

3.4 Cauchy's Integral Theorem for Simple Paths

3.5 The Cauchy–Goursat Theorem

3.6 Cauchy's Integral Theorem for Simply Connected Regions

3.7 Cauchy's Integral Theorem for Multiply Connected Regions

3.8 Cauchy's Integral Formula

Chapter 4

Series of Analytic Functions and Singularities

4.1 Sequences and Series of Functions

4.2 Power Series

4.3 Taylor Series

4.4 Laurent Series

4.5 Zeros and Singularities

Chapter 5

Residue Theory

5.1 Cauchy's Residue Theorem

5.2 Definite Integrals of Trigonometric Functions

5.3 Improper Integrals Involving Rational and Exponential Functions

5.4 Products of Rational and Trigonometric Functions

5.5 Summing Series by Residues

Chapter 6

Harmonic Functions and Applications

6.1 Harmonic Functions

6.2 Dirichlet Problems

Chapter 7

Conformal Mappings

7.1 Basic Properties

7.2 Fractional Linear Transformations

7.4 The Schwarz–Christoffel Transformation

Index